HOMEWORK IV

ANALYSIS I

(1) (a) Show that the series
$$1 + 2x + 3x^2 + 4x^3 + \cdots$$
converges uniformly on $(-k, k)$ for any $0 < k < 1$ but does not converge uniformly on $(-1, 1)$.

(b) The series
$$1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \cdots$$
converges uniformly on $(-1, 1)$.

(c) The series
$$1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$$
converges uniformly on $(-k, 1)$ for any $0 < k < 1$ but does not converge uniformly on $(-1, 1)$.

(2) (A Dirichlet type test for uniform convergence) Suppose $\{a_n(z)\}$ is a sequence of functions $X \to \mathbf{R}$ (X a subset of $\mathbf{R}$ or $\mathbf{C}$) such that $|\sum_{n=1}^{p} a_n(z)| \leq k$ where k is independent of p and z, and if $f_n(z) \geq f_{n+1}(z)$ and $f_n(z) \to 0$ *uniformly* on X as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n(z) f_n(z)$ converges uniformly on X [Hint: recall the proof of Dirichlet's test for convergence of a series].

(3) (Apostol's *Mathematical Analysis*, p.424, Ex. 13-6) Let $\{f_n\}$ be a sequence of continuous functions defined on a compact set S and assume that $\{f_n\}$ converges pointwise on S to a limit function f. Show that $f_n \to f$ uniformly on S, if and only if, the following two conditions hold:

(a) The limit function f is continuous on X.

(b) For every $\epsilon > 0$, there exists an $m > 0$ and a $\delta > 0$ such that $n > m$ and $|f_k(x) - f(x)| < \delta$ implies $|f_{k+n}(x) - f(x)| < \epsilon$ for all $x \in S$ and all $k = 1, 2, \ldots$.

Hint: To prove the sufficiency of (a) and (b), show that for each x_0 in S there exists $\theta > 0$ and an integer k (depending on x_0) such that
$$|f_k(x) - f(x)| < \delta \text{ if } |x - x_0| < \theta.$$
By compactness, a finite set of integers, say $A = \{k_1, \ldots, k_r\}$, has the property that, for each $x \in S$, some k in A satisfies $|f_k(x) - f(x)| < \delta$. Uniform convergence is an easy consequence of this fact.

(4) (Due to Dini, see loc. cit. Ex. 13-7) Use the previous exercise to prove that it $\{f_n\}$ is a sequence of real-valued functions converging pointwise to a continuous limit f on a compact set S, and if $f_n(x) \geq f_{n+1}(x)$ for each $x \in S$ and every $n \in \mathbf{N}$, then $f_n \to f$ uniformly on S.

(5) Let $f_n(x) = 1/(nx+1)$ for $x \in (0, 1)$, $n \in \mathbf{N}$. Show that $\{f_n\}$ converges pointwise but now uniformly on $(0, 1)$.
[This example demonstrates that the compactness of S is essential in the previous exercise]

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