HOMEWORK III

ANALYSIS I

(1) Given any real number x construct a rearrangement of the conditionally convergent series $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \cdots$ that converges to x (recall that a rearrangement of the series $\sum a_n$ is a series of the form $\sum a_{\sigma(n)}$ where $\sigma : \mathbf{N} \to \mathbf{N}$ is a bijection).

(2) Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \left\{ \frac{1 \cdot 3 \cdots 2n-1}{2 \cdot 4 \cdots 2n} \cdot \frac{4n+3}{2n+2} \right\}^2 .$$

(3) Show that when $s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \sum_{n=1}^{\infty} \left[\frac{1}{n^s} + \frac{1}{s-1} \left\{ \frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}} \right\} \right] .$$

and show that the series on the right converges when $0 < s < 1$.

(4) If

$$a_{m,n} = \frac{(m-n)}{2^{m+n}} \frac{(m+n-1)!}{m!\, n!}, \text{ for } m, n > 0,$$
$$a_{m,0} = 2^{-m},\ a_{0,n} = -2^{-n}, \text{ and } a_{0,0} = 0$$

show that

$$\sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} a_{m,n} \right) = -1,\ \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{m,n} \right) = 1.$$

(5) Given $x, y \in \mathbf{R}$ (or $\mathbf{C}$-the proof is essentially the same) show that there exist up en sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. This is known as the *Hausdorff property* in topology.

(6) Find a countable family of open sets in $\mathbf{R}$ such that every open subset of $\mathbf{R}$ can be written as a union of sets in this family. This is known as the *second countability property* in topology.

(7) Do the same for $\mathbf{C}$.

(8) True or false? Every compact subset of $\mathbf{R}$ is a finite union of sets of closed intervals $[a, b]$ with $a \leq b$ (give proof/counterexample).

(9) Does there exist an uncountable family $\{U_\alpha\}$ of subsets of $\mathbf{N}$ such that any two sets in the family
  (a) are disjoint.
  (b) have only a finite number of elements in their intersection?

(10) If K_1 and K_2 are compact subsets of $\mathbf{R}$ then

$$K = \{x + iy \mid x \in K_1 \text{ and } y \in K_2\} \subset \mathbf{C}$$

is a compact set.

Date: due on 31 August 2004.