HOMEWORK XII

ANALYSIS I

(1) Prove the two-variable Stone-Weierstrass theorem: if $f(x, y)$ is a real function defined and continuous on the closed rectangle $X = [a, b] \times [c, d]$ in the Euclidean plane $\mathbf{R}^2$, then f can be uniformly approximated on X by polynomials in x and y with real coefficients.

(2) Let X be the closed unit disc in the complex plane. Show that any continuous complex-valued function f can be uniformly approximated on X by polynomials in z and $\overline{z}$ with complex coefficients.

(3) Suppose f is a Riemann integrable function on a closed interval $[a, b]$. Show that for every $\epsilon > 0$ there exists a polynomial $P(x)$ such that

$$\int_a^b |f(x) - P(x)|^2 dx < \epsilon.$$

(4) For $x \in \mathbf{R}$ and $n = 1, 2, \ldots$, let $f_n(x) = (x^2 - 1)^n$ and define

$$\phi_0(x) = 1, \quad \phi_n(x) = \frac{1}{2^n n!} f_n^{(n)}(x).$$

It is clear that ϕ_n is a polynomial. It is called the *Legendre polynomial* of order n. Derive the following properties of Legendre polynomials:

(a) $\phi_n'(x) = x\phi_{n-1}'(x) + n\phi_{n-1}(x)$.

(b) $\phi_n(x) = x\phi_{n-1}(x) + \frac{x^2-1}{n}\phi_{n-1}'(x)$.

(c) $(n + 1)\phi_{n+1}(x) = (2n + 1)x\phi_n(x) - n\phi_{n-1}(x)$.

(d) $y = \phi_n(x)$ satisfies the differential equation $[(1 - x^2)y']' + n(n + 1)y = 0$.

(e) $[(1-x^2)\Delta(x)]' + [m(m+1) - n(n+1)]\phi_m(x)\phi_n(x) = 0$, where $\Delta = \phi_n\phi_m' + \phi_m\phi_n'$.

(f) The set $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is orthogonal on $[-1, 1]$, i.e., $\int_{-1}^1 \phi_m(x)\phi_n(x)dx = 0$ when $m \neq n$.

(g) $\int_{-1}^1 \phi_n^2(x)dx = \frac{2n-1}{2n+1} \int_{-1}^1 \phi_{n-1}(x)^2 dx$.

(h) $\|\phi_n\|^2 = \frac{2}{2n+1}$.

Date: 30th November 2004.