HOMEWORK XI

ANALYSIS I

(1) Show that if f is a periodic function on $\mathbf{R}$ with period 2π and Riemann integrable on $[-\pi, \pi]$ and g is defined by

$$g(x) = c + f(x + s)$$

where c is complex and s is real, then $c_n(g) = c_n(f)e^{nis}$ for all $n \neq 0$ and $c_0(g) = c_0(f) + c$ (here $c_n(f)$ and $c_n(g)$ denote the nth Fourier coefficient of f and g respectively).

(2) Calculate the Fourier series of the periodic functions whose values on $[-\pi, \pi]$ are given by
  (a) $f(x) = -1$ for $-\pi \leq x \leq 0$ and $f(x) = 1$ for $0 < x < \pi$.
  (b) $g(x) = x + \pi$ for $-\pi \leq x \leq 0$ and $g(x) = x - \pi$ for $0 < x < \pi$.
  (c) $h(x) = (1 - re^{ix})^{-1}$, where $0 < r < 1$.

(3) Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| < \delta$, $f(x) = 0$ if $\delta < x \leq |\pi|$ and f is 2π-periodic,
  (a) Compute the Fourier coefficients of f.
  (b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

  (c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

  (d) Let $\delta \to 0$ and prove that

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

  (e) Put $\delta = \frac{\pi}{2}$ in (3c). What do you get?

(4) Let f be the 2π-periodic function for which $f(x) = x$ for $0 \leq x < 2\pi$. Apply Parseval's theorem and conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Date: due on 23 November 2004.