HOMEWORK X

ANALYSIS I

1. Recall that
$$\omega_f(x) = \lim_{h\to 0} \sup\{|f(y) - f(y')| : y, y' \in (x-h, x+h)\}.$$
Show that f is continuous at x if and only if $\omega_f(x) = 0$.

2. Show that a countable union of sets of measure zero has measure zero.

3. Suppose $g : [a,b] \to \mathbf{R}$ is a Riemann integrable function such that $m \le g(x) \le M$ for all $x \in [a,b]$. If $f : [m,M] \to \mathbf{R}$ is continuous then show that $f \circ g$ is Riemann integrable on $[a,b]$.

4. Find functions $g : [a,b] \to \mathbf{R}$ such that $m \le g(x) \le M$ for each $x \in [a,b]$ such that g is Riemann integrable on $[a,b]$ and $f : [m,M] \to \mathbf{R}$ such that f is Riemann integrable on $[m,M]$, but $f \circ g$ is not Riemann integrable on $[a,b]$ (this sort of perverse behaviour is not seen for Lebesgue integrable functions).
5. Let C be the *Cantor set*
$$C = \left\{ x \in [0,1] \,\middle|\, x = \sum_{n=1}^{\infty} a_n 3^{-n}, \text{ where } a_n \in \{0,2\} \text{ for each } n \right\}.$$
Show that the characteristic function of C is Riemann integrable.

Date: due on 9 November 2004.