END-SEMESTER EXAMINATION

ANALYSIS I

Maximum score: 100. Total points: $6 \times 20 = 120$. Time: 3 hours.

(1) Let a_1 and b_1 be two positive real numbers. Define the sequences $\{a_n\}$ and $\{b_n\}$ inductively by
$$a_{n+1} = \frac{a_n + b_n}{2}, \text{ and } b_{n+1} = \sqrt{a_n b_n}.$$
Show that the sequences $\{a_n\}$ and $\{b_n\}$ have a common limit l. The common limit l was called the *arithmetico-geometric* mean of a_1 and b_1 by Gauss.

(2) Let f be a continuous function on $[a, b]$ taking positive values. Let M denote the maximum of f on $[a, b]$. Show that
$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^n dx \right)^{\frac{1}{n}} = M.$$

(3) Let f be a continuous real-valued function defined on $[0, 1]$. The *moments* of f are the numbers $\int_0^1 f(x)x^n dx$, where $n = 0, 1, 2, \ldots$. Prove (using the Weierstrass approximation theorem, or otherwise) that two continuous real-valued functions are identical if they have the same sequence of moments.

(4) Suppose f is a convex function on $(0, \infty)$. Let g be the function defined by $g(x) = f(x)/x$. Show that either g is monotonic or the graph of g consists of two monotonic pieces.

(5) Let $\{a_n\}$ be a decreasing sequence of positive terms. Show that the series $\sum a_n \sin(nx)$ converges uniformly on $\mathbf{R}$ if and only if $na_n \to 0$ as $n \to \infty$.

(6) (a) Calculate the Fourier coefficients of the 2π-periodic functions whose values between 0 and 2π are given by x and x^2 respectively.

(b) Apply Parseval's theorem to an appropriate quadratic polynomial to calculate $\sum_{n=1}^{\infty} n^{-4}$.

Date: December 3, 2004.