Tableau Correspondences and Representation Theory

Amritanshu Prasad[1]

The Institute of Mathematical Sciences, Chennai
Hombi Bhabha National Institute, Mumbai

ICADMA 2017, Aurangabad

[1]Based on joint work with Digjoy Paul and Arghya Sadhukhan

Semistandard Tableaux

1. A left-justified array of boxes with rows of weakly decreasing length filled with numbers
2. Numbers increase weakly from left to right along rows
3. Numbers increase strictly from top to bottom along columns

Semistandard Tableaux

1. A left-justified array of boxes with rows of weakly decreasing length filled with numbers
2. Numbers increase weakly from left to right along rows
3. Numbers increase strictly from top to bottom along columns

2	2	3	5	5
4	4	4	6	
5	7			

Semistandard Tableaux

1. A left-justified array of boxes with rows of weakly decreasing length filled with numbers
2. Numbers increase weakly from left to right along rows
3. Numbers increase strictly from top to bottom along columns

2	2	3	5	5
4	4	4	6	
5	7			

- Shape: $(5, 4, 2)$
- Weight: $(0, 2, 1, 3, 3, 1, 1)$, or $x_2^2 x_3 x_4^3 x_5^3 x_6 x_7$

Semistandard Tableaux

1. A left-justified array of boxes with rows of weakly decreasing length filled with numbers
2. Numbers increase weakly from left to right along rows
3. Numbers increase strictly from top to bottom along columns

2	2	3	5	5
4	4	4	6	
5	7			

- Shape: $(5, 4, 2)$
- Weight: $(0, 2, 1, 3, 3, 1, 1)$, or $x_2^2 x_3 x_4^3 x_5^3 x_6 x_7$

$\mathrm{Tab}(\lambda, \mu)$: set of semistandard tableaux of shape λ and type μ.

Semistandard Tableaux

1. A left-justified array of boxes with rows of weakly decreasing length filled with numbers
2. Numbers increase weakly from left to right along rows
3. Numbers increase strictly from top to bottom along columns

2	2	3	5	5
4	4	4	6	
5	7			

- Shape: $(5, 4, 2)$
- Weight: $(0, 2, 1, 3, 3, 1, 1)$, or $x_2^2 x_3 x_4^3 x_5^3 x_6 x_7$

$\mathrm{Tab}(\lambda, \mu)$: set of semistandard tableaux of shape λ and type μ.
Kostka Number: $K_{\lambda\mu} = |\mathrm{Tab}(\lambda, \mu)|$.

Representation Theory

A **homogeneous polynomial representation** of $GL_n(K)$ of degree d is a homomorphism:

$$\rho : GL_n(K) \to GL(V)$$

such that the matrix coefficients $g \mapsto \langle \xi, \rho(g)v \rangle$ are homogeneous polynomials of degree d in the entries of g.

Representation Theory

A **homogeneous polynomial representation** of $GL_n(K)$ of degree d is a homomorphism:

$$\rho : GL_n(K) \to GL(V)$$

such that the matrix coefficients $g \mapsto \langle \xi, \rho(g)v \rangle$ are homogeneous polynomials of degree d in the entries of g.

Examples

1. $V = K^n$, $\rho : GL_n(K) \to GL(K^n)$ is the identity map.

Representation Theory

A **homogeneous polynomial representation** of $GL_n(K)$ of degree d is a homomorphism:

$$\rho : GL_n(K) \to GL(V)$$

such that the matrix coefficients $g \mapsto \langle \xi, \rho(g) v \rangle$ are homogeneous polynomials of degree d in the entries of g.

Examples

1. $V = K^n$, $\rho : GL_n(K) \to GL(K^n)$ is the identity map.
2. $V = K^n \otimes K^n$, $\rho(g) = g \otimes g$ (Kronecker tensor).

Representation Theory

A **homogeneous polynomial representation** of $GL_n(K)$ of degree d is a homomorphism:

$$\rho : GL_n(K) \to GL(V)$$

such that the matrix coefficients $g \mapsto \langle \xi, \rho(g)v \rangle$ are homogeneous polynomials of degree d in the entries of g.

Examples

1. $V = K^n$, $\rho : GL_n(K) \to GL(K^n)$ is the identity map.
2. $V = K^n \otimes K^n$, $\rho(g) = g \otimes g$ (Kronecker tensor).
3. $V = K$, $\rho(g)(v) = \det(g)v$.

Representation Theory

A **homogeneous polynomial representation** of $GL_n(K)$ of degree d is a homomorphism:

$$\rho : GL_n(K) \to GL(V)$$

such that the matrix coefficients $g \mapsto \langle \xi, \rho(g)v \rangle$ are homogeneous polynomials of degree d in the entries of g.

Examples

1. $V = K^n$, $\rho : GL_n(K) \to GL(K^n)$ is the identity map.
2. $V = K^n \otimes K^n$, $\rho(g) = g \otimes g$ (Kronecker tensor).
3. $V = K$, $\rho(g)(v) = \det(g)v$.
4. $V = \mathrm{Sym}^d(K^n)$, $\rho(g)(v_1 \cdots v_d) = g(v_1) \cdots g(v_d)$.

Representation Theory

A **homogeneous polynomial representation** of $GL_n(K)$ of degree d is a homomorphism:

$$\rho : GL_n(K) \to GL(V)$$

such that the matrix coefficients $g \mapsto \langle \xi, \rho(g)v \rangle$ are homogeneous polynomials of degree d in the entries of g.

Examples

1. $V = K^n$, $\rho : GL_n(K) \to GL(K^n)$ is the identity map.
2. $V = K^n \otimes K^n$, $\rho(g) = g \otimes g$ (Kronecker tensor).
3. $V = K$, $\rho(g)(v) = \det(g)v$.
4. $V = \mathrm{Sym}^d(K^n)$, $\rho(g)(v_1 \cdots v_d) = g(v_1) \cdots g(v_d)$.
5. $V = \mathrm{Alt}^d(K^n)$, $\rho(g)(v_1 \wedge \cdots \wedge v_d) = g(v_1) \wedge \cdots \wedge g(v_d)$.

Weight spaces

V decomposes into eigenspaces for the diagonal torus T:

$$V(\mu) = \{v \in V \mid \rho(\Delta(x_1, \dots, x_n))v = x_1^{\mu_1} \cdots x_n^{\mu_n} v\}.$$

Weight spaces

V decomposes into eigenspaces for the diagonal torus T:

$$V(\mu) = \{v \in V \mid \rho(\Delta(x_1, \ldots, x_n))v = x_1^{\mu_1} \cdots x_n^{\mu_n} v\}.$$

Theorem (Schur, Kostka)

When $\mathrm{char}(K) = 0$, irreducible polynomial representations of degree d correspond to partitions of d with at most n parts.

Weight spaces

V decomposes into eigenspaces for the diagonal torus T:

$$V(\mu) = \{v \in V \mid \rho(\Delta(x_1, \dots, x_n))v = x_1^{\mu_1} \cdots x_n^{\mu_n} v\}.$$

Theorem (Schur, Kostka)

When $\mathrm{char}(K) = 0$, irreducible polynomial representations of degree d correspond to partitions of d with at most n parts.

$$\dim V_\lambda(\mu) = K_{\lambda\mu} \text{ for all } \mu.$$

Weight spaces

V decomposes into eigenspaces for the diagonal torus T:

$$V(\mu) = \{v \in V \mid \rho(\Delta(x_1, \ldots, x_n))v = x_1^{\mu_1} \cdots x_n^{\mu_n} v\}.$$

Theorem (Schur, Kostka)

When $\mathrm{char}(K) = 0$, irreducible polynomial representations of degree d correspond to partitions of d with at most n parts.

$$\dim V_\lambda(\mu) = K_{\lambda\mu} \text{ for all } \mu.$$

Representations with the same weight multiplicities are isomorphic.

Symmetric Polynomials

If V is a representation of $GL_n(K)$ with weight multiplicities:

$$V(\mu) = \alpha_\mu,$$

it is easy to see that:

$$\mathrm{char}(V) := \sum_\mu \alpha_\mu x_1^{\mu_1} \cdots x_n^{\mu_n}$$

is a symmetric polynomial in n variables, called the *character of* V.

Symmetric Polynomials

If V is a representation of $GL_n(K)$ with weight multiplicities:

$$V(\mu) = \alpha_\mu,$$

it is easy to see that:

$$\mathrm{char}(V) := \sum_\mu \alpha_\mu x_1^{\mu_1} \cdots x_n^{\mu_n}$$

is a symmetric polynomial in n variables, called the *character of* V.

Restatement of Schur's theorem

$$\mathrm{char}(V_\lambda) = s_\lambda(x_1, \dots, x_n),$$

and representations with the same character are isomorphic.

Symmetric Polynomials

If V is a representation of $GL_n(K)$ with weight multiplicities:

$$V(\mu) = \alpha_\mu,$$

it is easy to see that:

$$\mathrm{char}(V) := \sum_\mu \alpha_\mu x_1^{\mu_1} \cdots x_n^{\mu_n}$$

is a symmetric polynomial in n variables, called the *character of V*.

Restatement of Schur's theorem

$$\mathrm{char}(V_\lambda) = s_\lambda(x_1, \ldots, x_n),$$

and representations with the same character are isomorphic.

Note

We have $\mathrm{char}(V \otimes W) = \mathrm{char}(V)\mathrm{char}(W)$.

Robinson-Schensted-Knuth Correspondence

Robinson-Schensted-Knuth Correspondence

Let $\mu = (\mu_1, \ldots, \mu_m)$, $\nu = (\nu_1, \ldots, \nu_n)$.

Robinson-Schensted-Knuth Correspondence

Let $\mu = (\mu_1, \ldots, \mu_m)$, $\nu = (\nu_1, \ldots, \nu_n)$.

$$M_{\mu\nu} = \{(a_{ij}) \in M_{m \times n}(\mathbf{Z}_{\geq 0}) \mid \sum_j a_{ij} = \mu_i, \sum_i a_{ij} = \nu_j\}$$

Robinson-Schensted-Knuth Correspondence

Let $\mu = (\mu_1, \dots, \mu_m)$, $\nu = (\nu_1, \dots, \nu_n)$.

$$M_{\mu\nu} = \{(a_{ij}) \in M_{m\times n}(\mathbf{Z}_{\geq 0}) \mid \sum_j a_{ij} = \mu_i, \sum_i a_{ij} = \nu_j\}$$

RSK Correspondence

$$\mathrm{RSK} : M_{\mu\nu} \tilde{\to} \coprod_{\lambda} \mathrm{Tab}(\lambda, \nu) \times \mathrm{Tab}(\lambda, \mu).$$

Robinson-Schensted-Knuth Correspondence

Let $\mu = (\mu_1, \ldots, \mu_m)$, $\nu = (\nu_1, \ldots, \nu_n)$.

$$M_{\mu\nu} = \{(a_{ij}) \in M_{m \times n}(\mathbf{Z}_{\geq 0}) \mid \sum_j a_{ij} = \mu_i, \sum_i a_{ij} = \nu_j\}$$

RSK Correspondence

$$\mathrm{RSK} : M_{\mu\nu} \xrightarrow{\sim} \coprod_\lambda \mathrm{Tab}(\lambda, \nu) \times \mathrm{Tab}(\lambda, \mu).$$

Example

$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 1 & 1 & 3 & 1 \\ 3 & 2 & 2 & 1 \end{pmatrix} \mapsto \left(\begin{array}{cccccccccc} 1&1&1&1&1&2&2&3&3&4 \\ 2&3&3&3&3&4 \\ 3&4&4 \end{array}, \begin{array}{cccccccccc} 1&1&1&1&1&2&2&3&3&3 \\ 2&2&2&2&3&3 \\ 3&3&3 \end{array} \right)$$

RSK correspondence in representation theory

RSK correspondence in representation theory

Consider $V = \mathrm{Sym}^d(K^m \otimes K^n)$, a representation of $GL_m(K) \times GL_n(K)$.

RSK correspondence in representation theory

Consider $V = \mathrm{Sym}^d(K^m \otimes K^n)$, a representation of $GL_m(K) \times GL_n(K)$.

The symmetric tensors:

$$v_A = \prod (e_i \otimes f_j)^{a_{ij}},$$

as $A = (a_{ij})$ runs over matrices in $M_{m\times n}(\mathbf{Z}_{\geq 0})$ with entries summing to d form a basis for V.

RSK correspondence in representation theory

Consider $V = \mathrm{Sym}^d(K^m \otimes K^n)$, a representation of $GL_m(K) \times GL_n(K)$.

The symmetric tensors:

$$v_A = \prod (e_i \otimes f_j)^{a_{ij}},$$

as $A = (a_{ij})$ runs over matrices in $M_{m\times n}(\mathbf{Z}_{\geq 0})$ with entries summing to d form a basis for V.

The vectors v_A is an eigenvector for $T_m(K) \times T_n(K)$ with weight (μ, ν) if $A \in M_{\mu\nu}$.

$$\begin{aligned} \mathrm{char}(V) &= \sum_{\mu,\nu} |M_{\mu\nu}| x^\mu y^\nu \\ &= \sum_{\lambda} (K_{\lambda\mu} x^\mu)(K_{\lambda\nu} x^\nu) \qquad \text{[by RSK].} \end{aligned}$$

RSK correspondence in representation theory

Consider $V = \mathrm{Sym}^d(K^m \otimes K^n)$, a representation of $GL_m(K) \times GL_n(K)$.

The symmetric tensors:

$$v_A = \prod (e_i \otimes f_j)^{a_{ij}},$$

as $A = (a_{ij})$ runs over matrices in $M_{m\times n}(\mathbf{Z}_{\geq 0})$ with entries summing to d form a basis for V.

The vectors v_A is an eigenvector for $T_m(K) \times T_n(K)$ with weight (μ, ν) if $A \in M_{\mu\nu}$.

$$\begin{aligned} \mathrm{char}(V) &= \sum_{\mu,\nu} |M_{\mu\nu}| x^\mu y^\nu \\ &= \sum_\lambda (K_{\lambda\mu} x^\mu)(K_{\lambda\nu} x^\nu) \qquad \text{[by RSK].} \end{aligned}$$

We get the so-called (by Roger Howe) $GL(m)$-$GL(n)$-duality theorem:

$$\mathrm{Sym}^d(K^m \otimes K^n) = \bigoplus_\lambda V_\lambda^m \otimes V_\lambda^n.$$

Dual Robinson-Schensted-Knuth Correspondence

Dual Robinson-Schensted-Knuth Correspondence

Let $\mu = (\mu_1, \ldots, \mu_m)$, $\nu = (\nu_1, \ldots, \nu_n)$.

Dual Robinson-Schensted-Knuth Correspondence

Let $\mu = (\mu_1, \ldots, \mu_m)$, $\nu = (\nu_1, \ldots, \nu_n)$.

$$N_{\mu\nu} = \{(a_{ij}) \in M_{m \times n}(\{0,1\}) \mid \sum_j a_{ij} = \mu_i, \sum_i a_{ij} = \nu_j\}$$

Dual Robinson-Schensted-Knuth Correspondence

Let $\mu = (\mu_1, \ldots, \mu_m)$, $\nu = (\nu_1, \ldots, \nu_n)$.

$$N_{\mu\nu} = \{(a_{ij}) \in M_{m\times n}(\{0,1\}) \mid \sum_j a_{ij} = \mu_i, \sum_i a_{ij} = \nu_j\}$$

Dual RSK Correspondence

$$\mathrm{RSK}^* : N_{\mu\nu} \xrightarrow{\sim} \coprod_\lambda \mathrm{Tab}(\lambda', \nu) \times \mathrm{Tab}(\lambda, \mu).$$

Dual Robinson-Schensted-Knuth Correspondence

Let $\mu = (\mu_1, \ldots, \mu_m)$, $\nu = (\nu_1, \ldots, \nu_n)$.

$$N_{\mu\nu} = \{(a_{ij}) \in M_{m\times n}(\{0,1\}) \mid \sum_j a_{ij} = \mu_i, \sum_i a_{ij} = \nu_j\}$$

Dual RSK Correspondence

$$\mathrm{RSK}^* : N_{\mu\nu} \xrightarrow{\sim} \coprod_\lambda \mathrm{Tab}(\lambda', \nu) \times \mathrm{Tab}(\lambda, \mu).$$

Example

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \mapsto \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 2 & 3 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array} \right)$$

Dual RSK correspondence in representation theory

Dual RSK correspondence in representation theory

Consider $V = \mathrm{Alt}^d(K^m \otimes K^n)$, a representation of $GL_m(K) \times GL_n(K)$.

Dual RSK correspondence in representation theory

Consider $V = \mathrm{Alt}^d(K^m \otimes K^n)$, a representation of $GL_m(K) \times GL_n(K)$.

The alternating tensors:

$$v_A = \wedge(e_i \otimes f_j)^{a_{ij}},$$

as $A = (a_{ij})$ runs over matrices in $N_{m\times n}(\{0,1\})$ with entries summing to d form a basis for V.

Dual RSK correspondence in representation theory

Consider $V = \mathrm{Alt}^d(K^m \otimes K^n)$, a representation of $GL_m(K) \times GL_n(K)$.

The alternating tensors:

$$v_A = \wedge(e_i \otimes f_j)^{a_{ij}},$$

as $A = (a_{ij})$ runs over matrices in $N_{m\times n}(\{0,1\})$ with entries summing to d form a basis for V.

The vectors v_A is an eigenvector for $T_m(K) \times T_n(K)$ with weight (μ, ν) if $A \in M_{\mu\nu}$.

$$\begin{aligned} \mathrm{char}(V) &= \sum_{\mu,\nu} |N_{\mu\nu}| x^\mu y^\nu \\ &= \sum_\lambda (K_{\lambda\mu} x^\mu)(K_{\lambda'\nu} x^\nu) \qquad \text{[by dual RSK].} \end{aligned}$$

We get the so-called "dual" $GL(m)$-$GL(n)$-duality theorem:

$$\mathrm{Alt}^d(K^m \otimes K^n) = \bigoplus_\lambda V_\lambda^m \otimes V_{\lambda'}^n.$$

Symmetry Property of RSK

Theorem

If $RSK(A) = (P, Q)$, then $RSK(A^T) = (Q, P)$.

Symmetry Property of RSK

Theorem

If $RSK(A) = (P, Q)$, then $RSK(A^T) = (Q, P)$.

Corollary

$$RSK : M_{\mu\mu}^{\text{symmetric}} \xrightarrow{\sim} \coprod_{\lambda} \mathrm{Tab}(\lambda, \mu).$$

Symmetry Property of RSK

Theorem

If $RSK(A) = (P, Q)$, then $RSK(A^T) = (Q, P)$.

Corollary

$$RSK : M_{\mu\mu}^{\text{symmetric}} \xrightarrow{\sim} \coprod_{\lambda} \mathrm{Tab}(\lambda, \mu).$$

Refinement (Schützenberger)

If A is symmetric with trace k, and $RSK(A) = (P, P)$, where P has shape λ, then the number of odd columns in λ is k.

Symmetry in representation theory

Symmetry in representation theory

$$V = \mathrm{Sym}^d(K^n \oplus \wedge^2 K^n) = \bigoplus_{k+2l=d} \mathrm{Sym}^k(K^n) \otimes \mathrm{Sym}^l(\wedge^2 K^n).$$

Symmetry in representation theory

$$V = \mathrm{Sym}^d(K^n \oplus \wedge^2 K^n) = \bigoplus_{k+2l=d} \mathrm{Sym}^k(K^n) \otimes \mathrm{Sym}^l(\wedge^2 K^n).$$

$$\left\{ v_A := \prod e_i^{a_{ii}} \prod_{i<j} (e_i \wedge e_j)^{a_{ij}} \mid A = (a_{ij}) \in M_{\mu\mu}^{\mathrm{symmetric}} \right\}$$

is a basis for $V(\mu)$.

Symmetry in representation theory

$$V = \mathrm{Sym}^d(K^n \oplus \wedge^2 K^n) = \bigoplus_{k+2l=d} \mathrm{Sym}^k(K^n) \otimes \mathrm{Sym}^l(\wedge^2 K^n).$$

$$\left\{ v_A := \prod e_i^{a_{ii}} \prod_{i<j} (e_i \wedge e_j)^{a_{ij}} \mid A = (a_{ij}) \in M_{\mu\mu}^{\mathrm{symmetric}} \right\}$$

is a basis for $V(\mu)$.

Gelfand Model

$$\mathrm{Sym}^d(K^n \oplus \wedge^2 K^n) = \bigoplus_{\lambda} V_\lambda,$$

the sum being over all partitions of d with at most n parts.

Symmetry in representation theory

$$V = \mathrm{Sym}^d(K^n \oplus \wedge^2 K^n) = \bigoplus_{k+2l=d} \mathrm{Sym}^k(K^n) \otimes \mathrm{Sym}^l(\wedge^2 K^n).$$

$$\left\{ v_A := \prod e_i^{a_{ii}} \prod_{i<j} (e_i \wedge e_j)^{a_{ij}} \mid A = (a_{ij}) \in M_{\mu\mu}^{\mathrm{symmetric}} \right\}$$

is a basis for $V(\mu)$.

Gelfand Model

$$\mathrm{Sym}^d(K^n \oplus \wedge^2 K^n) = \bigoplus_{\lambda} V_\lambda,$$

the sum being over all partitions of d with at most n parts.

Refinement

$$\bigoplus_{k+2l=d} \mathrm{Sym}^k(K^n) \otimes \mathrm{Sym}^l(\wedge^2 K^n) = \bigoplus_{\lambda \text{ has } k \text{ odd cols.}} V_\lambda,$$

The Burge Correspondence

Similar to the RSK, is the Burge correspondence:

$$\mathrm{BUR} : M_{\mu\nu} \tilde{\to} \coprod_{\lambda} \mathrm{Tab}(\lambda, \nu) \times \mathrm{Tab}(\lambda, mu),$$

The Burge Correspondence

Similar to the RSK, is the Burge correspondence:

$$\mathrm{BUR} : M_{\mu\nu} \tilde{\to} \coprod_{\lambda} \mathrm{Tab}(\lambda, \nu) \times \mathrm{Tab}(\lambda, mu),$$

which also satisfies the symmetry condition:

$$\text{if } \mathrm{BUR}(A) = (P, Q), \text{ then } \mathrm{BUR}(A^T) = (Q, P).$$

The Burge Correspondence

Similar to the RSK, is the Burge correspondence:

$$\mathrm{BUR} : M_{\mu\nu} \tilde{\to} \coprod_{\lambda} \mathrm{Tab}(\lambda, \nu) \times \mathrm{Tab}(\lambda, mu),$$

which also satisfies the symmetry condition:

$$\text{if } \mathrm{BUR}(A) = (P, Q), \text{ then } \mathrm{BUR}(A^T) = (Q, P).$$

But the analog of Schützenberger's result is:

The Burge Correspondence

Similar to the RSK, is the Burge correspondence:

$$\mathrm{BUR} : M_{\mu\nu} \tilde{\to} \coprod_{\lambda} \mathrm{Tab}(\lambda, \nu) \times \mathrm{Tab}(\lambda, mu),$$

which also satisfies the symmetry condition:

$$\text{if } \mathrm{BUR}(A) = (P, Q), \text{ then } \mathrm{BUR}(A^T) = (Q, P).$$

But the analog of Schützenberger's result is:

Schützenberger-type lemma

If A is symmetric with k odd entries along its diagonal, and $\mathrm{BUR}(A) = (P, P)$, with P of shape λ, then k is the number of odd rows in λ.

Burge correspondence in representation theory

$$V_{k,l} = \bigoplus_{k+2l=d} \mathrm{Alt}^k(K^n) \otimes \mathrm{Sym}^l(\mathrm{Sym}^2 K^n).$$

Burge correspondence in representation theory

$$V_{k,l} = \bigoplus_{k+2l=d} \mathrm{Alt}^k(K^n) \otimes \mathrm{Sym}^l(\mathrm{Sym}^2 K^n).$$

Given a symmetric matrix A, write A as $\Delta + B + B^T$, where Δ is a 0-1 diagonal matrix, and B is upper-triangular.

Burge correspondence in representation theory

$$V_{k,l} = \bigoplus_{k+2l=d} \mathrm{Alt}^k(K^n) \otimes \mathrm{Sym}^l(\mathrm{Sym}^2 K^n).$$

Given a symmetric matrix A, write A as $\Delta + B + B^T$, where Δ is a 0-1 diagonal matrix, and B is upper-triangular.

$$\left\{ v_A := \wedge e_i^{\Delta_{ii}} \otimes \prod_{i \leq j} (e_i e_j)^{b_{ij}} \right\}$$

where A runs over matrices in $M_{\mu\mu}^{\mathrm{symmetric}}$ with k odd elements along the diagonal, forms a basis of $V_{k,l}(\mu)$.

Burge correspondence in representation theory

$$V_{k,l} = \bigoplus_{k+2l=d} \mathrm{Alt}^k(K^n) \otimes \mathrm{Sym}^l(\mathrm{Sym}^2 K^n).$$

Given a symmetric matrix A, write A as $\Delta + B + B^T$, where Δ is a 0-1 diagonal matrix, and B is upper-triangular.

$$\left\{ v_A := \wedge e_i^{\Delta_{ii}} \otimes \prod_{i \leq j} (e_i e_j)^{b_{ij}} \right\}$$

where A runs over matrices in $M_{\mu\mu}^{\mathrm{symmetric}}$ with k odd elements along the diagonal, forms a basis of $V_{k,l}(\mu)$.

Gelfand Model

$$\bigoplus_{k+2l=d} V_{k,l} = \bigoplus_{\lambda} V_\lambda.$$

Burge correspondence in representation theory

$$V_{k,l} = \bigoplus_{k+2l=d} \mathrm{Alt}^k(K^n) \otimes \mathrm{Sym}^l(\mathrm{Sym}^2 K^n).$$

Given a symmetric matrix A, write A as $\Delta + B + B^T$, where Δ is a 0-1 diagonal matrix, and B is upper-triangular.

$$\left\{ v_A := \wedge e_i^{\Delta_{ii}} \otimes \prod_{i \leq j} (e_i e_j)^{b_{ij}} \right\}$$

where A runs over matrices in $M_{\mu\mu}^{\mathrm{symmetric}}$ with k odd elements along the diagonal, forms a basis of $V_{k,l}(\mu)$.

Refinement

$$V_{k,l} = \bigoplus_{\lambda \text{ has } k \text{ odd rows}} V_\lambda.$$

More Correspondences from Burge

A partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is determined by its Young diagram, the shape:

$$Y(\lambda) = \{(i,j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

More Correspondences from Burge

A partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is determined by its Young diagram, the shape:

$$Y(\lambda) = \{(i,j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

Define

$$\sigma(i,j) = \begin{cases} (j, i-1) & \text{if} i \leq j, \\ (j+1, i) & \text{if} i > j. \end{cases}$$

More Correspondences from Burge

A partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is determined by its Young diagram, the shape:

$$Y(\lambda) = \{(i,j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

Define

$$\sigma(i,j) = \begin{cases} (j, i-1) & \text{if} i \leq j, \\ (j+1, i) & \text{if} i > j. \end{cases}$$

Define $\mathcal{B}$ to be the class of partitions whose Young diagrams are invariant under σ.

More Correspondences from Burge

A partition $\lambda = (\lambda_1, \dots, \lambda_l)$ is determined by its Young diagram, the shape:

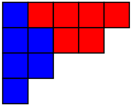
$$Y(\lambda) = \{(i,j) \mid 1 \le i \le l,\ 1 \le j \le \lambda_i\}.$$

Define

$$\sigma(i,j) = \begin{cases} (j, i-1) & \text{if} i \le j, \\ (j+1, i) & \text{if} i > j. \end{cases}$$

Define $\mathcal{B}$ to be the class of partitions whose Young diagrams are invariant under σ.

Example

If $\lambda = (5,4,2,1)$, then $Y(\lambda) =$  , so $\lambda \in \mathcal{B}$.

More Correspondences of Burge

Burge defined two more bijective correspondences.

More Correspondences of Burge

Burge defined two more bijective correspondences.

$$N^0_{\mu\mu} = \{A \in N^{\mathrm{symmetric}}_{\mu\mu} \mid \mathrm{trace} A = 0\}$$

More Correspondences of Burge

Burge defined two more bijective correspondences.

$$N^0_{\mu\mu} = \{A \in N^{\text{symmetric}}_{\mu\mu} \mid \text{trace} A = 0\}$$

BUR1

$$\text{BUR1} : N^0_{\mu\mu} \tilde{\to} \coprod_{\lambda \in \mathcal{B}} \text{Tab}(\lambda, \mu).$$

More Correspondences of Burge

Burge defined two more bijective correspondences.

$$N^0_{\mu\mu} = \{A \in N^{\text{symmetric}}_{\mu\mu} \mid \text{trace} A = 0\}$$

BUR1

$$\text{BUR1} : N^0_{\mu\mu} \tilde{\to} \coprod_{\lambda \in \mathcal{B}} \text{Tab}(\lambda, \mu).$$

$$N^1_{\mu\mu} = \{A \in M^{\text{symmetric}}_{\mu\mu} \mid \text{off-diag entries 0 or 1, diag entries 0 or 2}\}$$

More Correspondences of Burge

Burge defined two more bijective correspondences.

$$N^0_{\mu\mu} = \{A \in N^{\text{symmetric}}_{\mu\mu} \mid \text{trace} A = 0\}$$

BUR1

$$\text{BUR1} : N^0_{\mu\mu} \tilde{\to} \coprod_{\lambda \in \mathcal{B}} \text{Tab}(\lambda, \mu).$$

$$N^1_{\mu\mu} = \{A \in M^{\text{symmetric}}_{\mu\mu} \mid \text{off-diag entries 0 or 1, diag entries 0 or 2}\}$$

BUR2

$$\text{BUR2} : N^1_{\mu\mu} \tilde{\to} \coprod_{\lambda' \in \mathcal{B}} \text{Tab}(\lambda, \mu).$$

Representation-theoretic interpretation

$$\mathrm{Alt}^d(\mathrm{Alt}^2 K^n) = \bigoplus_{\lambda \in \mathcal{B}} V_\lambda.$$

$$\mathrm{Alt}^d(\mathrm{Sym}^2 K^n) = \bigoplus_{\lambda' \in \mathcal{B}} V_\lambda.$$

Representations of symmetric groups

Representations of symmetric groups

Closely related to polynomial representation theory of $GL_n(K)$:

Representations of symmetric groups

Closely related to polynomial representation theory of $GL_n(K)$:

Observation

The symmetric group S_n is a subgroup of $GL_n(K)$ via permutation matrices. Given a representation V of $GL_n(K)$, the subpsace

$$\bar{V} := V(1, \ldots, 1) = \{v \in V \mid \rho(\Delta(x_1, \ldots, x_n))v = x_1 \cdots x_n v\}$$

is a representation of S_n.

Representations of symmetric groups

Closely related to polynomial representation theory of $GL_n(K)$:

Observation

The symmetric group S_n is a subgroup of $GL_n(K)$ via permutation matrices. Given a representation V of $GL_n(K)$, the subpsace

$$\bar{V} := V(1,\dots,1) = \{v \in V \mid \rho(\Delta(x_1,\dots,x_n))v = x_1 \cdots x_n v\}$$

is a representation of S_n.

- Applying this to $GL(n)$-$GL(n)$-duality gives a decomposition of the regular representation of S_n, showing that $\bar{V}_\lambda$, as λ runs over all integer partitions of n gives a complete set of irreducible representations of S_n.

Representations of symmetric groups

Closely related to polynomial representation theory of $GL_n(K)$:

Observation

The symmetric group S_n is a subgroup of $GL_n(K)$ via permutation matrices. Given a representation V of $GL_n(K)$, the subpsace

$$\bar{V} := V(1, \ldots, 1) = \{v \in V \mid \rho(\Delta(x_1, \ldots, x_n))v = x_1 \cdots x_n v\}$$

is a representation of S_n.

- Applying this to $GL(n)$-$GL(n)$-duality gives a decomposition of the regular representation of S_n, showing that $\bar{V}_\lambda$, as λ runs over all integer partitions of n gives a complete set of irreducible representations of S_n.
- Involutory models due to Klyachko for the representations of S_n are obtained from the models for $GL_n(K)$.

Representations of symmetric groups

Closely related to polynomial representation theory of $GL_n(K)$:

Observation

The symmetric group S_n is a subgroup of $GL_n(K)$ via permutation matrices. Given a representation V of $GL_n(K)$, the subpsace

$$\bar{V} := V(1, \ldots, 1) = \{v \in V \mid \rho(\Delta(x_1, \ldots, x_n))v = x_1 \cdots x_n v\}$$

is a representation of S_n.

- Applying this to $GL(n)$-$GL(n)$-duality gives a decomposition of the regular representation of S_n, showing that $\bar{V}_\lambda$, as λ runs over all integer partitions of n gives a complete set of irreducible representations of S_n.
- Involutory models due to Klyachko for the representations of S_n are obtained from the models for $GL_n(K)$.
- A combinatorial proof of Schur-Weyl duality can be obtained.