# Crystals for stable Grothendieck polynomials

Anne Schilling, UC Davis

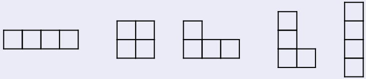
based on Jennifer Morse, AS, IMRN 2016(8) (2016) 2239
Jennifer Morse, Jianping Pan, Wencin Poh, AS, `arXiv:1911.08732`

University of Virginia
University of California, Davis

IMSc Algebraic Combinatorics Seminar, April 8, 2020

# Outline

# Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$

Indexed by partitions: 

- Tensor product multiplicities

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} c^{\nu}_{\lambda\mu} V(\nu)$$

- Symmetric function coefficients

$$s_\lambda \, s_\mu = \sum_{\nu} c^{\nu}_{\lambda\mu} \, s_\nu$$

$$s_{\nu/\mu} = \sum_{\lambda} c^{\nu}_{\lambda\mu} \, s_\lambda$$

- Intersections in the Grassmannian

$$c^{\nu}_{\lambda\mu} = X_\lambda \cap X_\mu \cap X_{\nu^\vee}$$

# Combinatorial description

Littlewood–Richardson rule

$c^{\nu}_{\lambda\mu} = \#$ skew tableaux $t$ of shape $\nu/\lambda$ and weight $\mu$ such that $\mathrm{row}(t)$ is a reverse lattice word.

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### Example

$$s_{\ydiagram{1,2}}\, s_{\ydiagram{1,2}} = \cdots + ?s_{\ydiagram{1,2,3}} + \cdots$$

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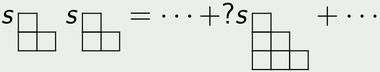
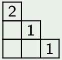
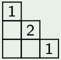
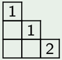


 211

 121

 112

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$$\begin{ytableau} 2 \\ \; & 1 \\ \; & \; & 1 \end{ytableau}\ 211 \qquad \begin{ytableau} 1 \\ \; & 2 \\ \; & \; & 1 \end{ytableau}\ 121 \qquad \begin{ytableau} 1 \\ \; & 1 \\ \; & \; & 2 \end{ytableau}\ 112 \qquad \Rightarrow c^{321}_{21,21} = 2$$

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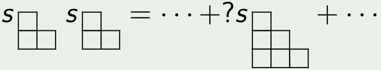


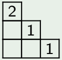


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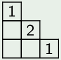


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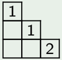


112

$\Rightarrow c^{321}_{21,21} = 2$

Gordon James (1987) on the Littlewood-Richardson rule:

> *"Unfortunately the Littlewood-Richardson rule is much harder to prove than was at first suspected. The author was once told that the Littlewood-Richardson rule helped to get men on the moon but was not proved until after they got there."*

# Crystal graph

Action of crystal operators $e_i, f_i, s_i$ on tableaux:

1. Consider letters $i$ and $i+1$ in row reading word of the tableau
2. Successively "bracket" pairs of the form $(i+1, i)$
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$$e_i(i^r(i+1)^s) = \begin{cases} i^{r+1}(i+1)^{s-1} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

$$f_i(i^r(i+1)^s) = \begin{cases} i^{r-1}(i+1)^{s+1} & \text{if } r > 0 \\ 0 & \text{else} \end{cases}$$

$$s_i(i^r(i+1)^s) = i^s(i+1)^r$$

# Crystal reformulation

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | | | | | | | |
| | 1 | 2 | 2 | 3 | | | |
| | | 1 | 1 | 2 | 3 | 3 | 3 |

# Crystal reformulation

| | | | | | | | |
|---|---|---|---|---|---|---|---|
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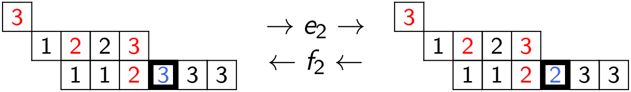


$e_2$: change leftmost unpaired 3 into 2

$f_2$: change rightmost unpaired 2 into 3

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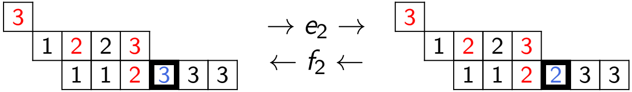


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**Theorem**

*$b$ where all $e_i(b) = 0$ (highest weight)*

*$\leftrightarrow$ connected component*

*$\leftrightarrow$ irreducible*

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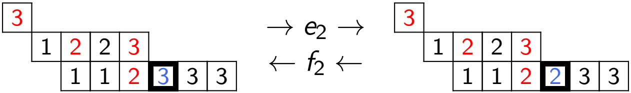


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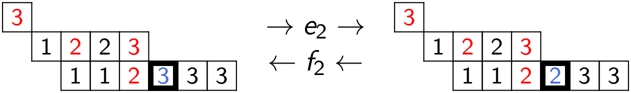
$\leftrightarrow$ *connected component*

$\leftrightarrow$ *irreducible*

## Reformulation of LR rule

$c^{\nu}_{\lambda\mu}$ counts tableaux of shape $\nu/\lambda$ and weight $\mu$ which are highest weight.

# Crystal reformulation

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*$\leftrightarrow$ connected component*
*$\leftrightarrow$ irreducible*

## Mechanism to get Schur expansion

$$s_{\nu/\lambda} = \sum_{T \in B(\nu/\lambda)} x^{weight(T)} = \sum_{\substack{T \in B(\nu/\lambda) \\ highest\ \ weight}} s_{weight(T)}$$

# Variation $c_{uv}^{w}$

Indexed by permutations: (1,2,3) (2,1,3) (3,2,1) $\cdots$

- Intersections in the set $\mathbb{F}_n$ of complete flags $0 = W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n$

$$c_{uv}^{w} = X_u \cap X_v \cap X_{w_0 w}$$

- Schubert polynomial coefficients

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{uv}^{w} \mathfrak{S}_{w_0 w}$$

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WHAT ARE THESE COUNTING?

# Outline

# Stable Schubert polynomials $F_w$

- restriction: $\mathfrak{S}_{1_m \times w} \longrightarrow$ Stanley symmetric functions $F_w$ for $w \in S_n$

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- coefficient of $x_1 x_2 \cdots x_r$ counts reduced words of $w$

$$S_n = \langle s_1, \ldots, s_{n-1} \rangle \qquad s_i s_j = s_j s_i \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad s_i^2 = id$$

$$(3,2,1,4) = s_1 s_2 s_1 = s_2 s_1 s_2 = s_3 s_3 s_1 s_2 s_1$$

# Stable Schubert polynomials

$$F_w = \sum_{v^r \cdots v^1 = w} x_1^{\ell(v^1)} \dots x_r^{\ell(v^r)}$$

Decreasing factorization of $w$

1. $w$ is the product of permutations $v^r \cdots v^1$
2. each $v^i$ has a strictly decreasing reduced word
3. $\ell(w) = \ell(v^r) + \cdots + \ell(v^1)$

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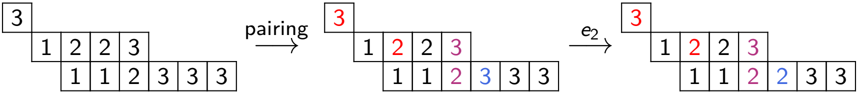
$w = (2,1,4,3) = s_1 s_3 = s_3 s_1$:

$$\begin{aligned} (s_1)(s_3) &\longrightarrow x_1 x_2 \\ (s_3)(s_1) &\longrightarrow x_1 x_2 \\ ()(s_3 s_1) &\longrightarrow x_1^2 \\ (s_3 s_1)() &\longrightarrow x_2^2 \end{aligned}$$

$$F_{(2,1,4,3)} = 2\, x_1 x_2 + x_1^2 + x_2^2$$

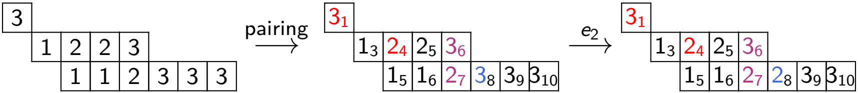
# Crystal operators on factorizations – residue map

Recall $e_i$ pairing and action:



# Crystal operators on factorizations – residue map

Label cells diagonally



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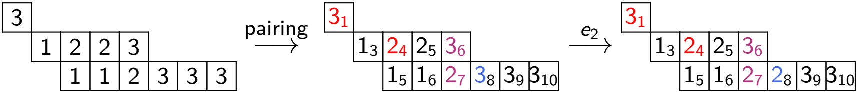


$$\underbrace{(10\ 9\ 8\ 6\ 1)}_{\text{label of 3's}}\ \underbrace{(7\ 5\ 4)}_{\text{label of 2's}}$$

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$$\begin{array}{cccccccc} 3 \\ & 1 & 2 & 2 & 3 \\ & & 1 & 1 & 2 & 3 & 3 & 3 \end{array} \xrightarrow{\text{pairing}} \begin{array}{cccccccc} 3_1 \\ & 1_3 & 2_4 & 2_5 & 3_6 \\ & & 1_5 & 1_6 & 2_7 & 3_8 & 3_9 & 3_{10} \end{array} \xrightarrow{e_2} \begin{array}{cccccccc} 3_1 \\ & 1_3 & 2_4 & 2_5 & 3_6 \\ & & 1_5 & 1_6 & 2_7 & 2_8 & 3_9 & 3_{10} \end{array}$$

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**operator $e_i$**

from big to small:
pair $x \in$ 3's with smallest $y \in$ 2's that is bigger than $x$

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$$(9\,8\,6\,5\,4\,1)(9\,6\,5\,2\,1) \to (9\,8\,5\,4\,1)(9\,6\,5\,4\,2\,1)$$

# Crystal Theorem

## Definition

Fix $w \in S_n$.

Graph $B(w)$

1. vertices are decreasing factorizations of $w$
2. edges are imposed and colored by $f_i$, $e_i$
3. highest weights are vertices with no unpaired entries

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### Proof

Checking Stembridge local axioms

# Examples

$$\begin{tikzcd}[column sep=small]
 & ()()(31) \arrow[d, "1"] & \\
 & ()(3)(1) \arrow[dl, "2"'] \arrow[dr, "1"] & \\
(3)()(1) \arrow[dr, "1"'] & & ()(31)() \arrow[dl, "2"] \\
 & (3)(1)() \arrow[d, "2"] & \\
 & (31)()() &
\end{tikzcd}
\qquad
\begin{tikzcd}
()(1)(3) \arrow[d, "2"] \\
(1)()(3) \arrow[d, "1"] \\
(1)(3)()
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & ()()(21) \arrow[d, "1"] & \\
 & ()(2)(1) \arrow[dl, "1"'] \arrow[dr, "2"] & \\
()(21)() \arrow[dr, "2"'] & & (2)()(1) \arrow[dl, "1"] \\
 & (2)(1)() \arrow[d, "2"] & \\
 & (21)()() &
\end{tikzcd}$$

# Schur expansion

Fix $w \in S_n$

**Theorem (with Morse; 2016)**

$$F_w = \sum_{\lambda} a_{w\lambda}\, s_\lambda$$

*$a_{w\lambda}$ counts highest weights $v^r \cdots v^1$ of $B(w)$ with $(\ell(v^1), \ldots, \ell(v^r)) = \lambda$*

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In $S_5$:

$$\begin{tikzcd} ()(42) \arrow[d, "1"] & (2)(4) \\ (4)(2) \arrow[d, "1"] & \\ (42)() & \end{tikzcd}$$

$$\Longrightarrow F_{s_2 s_4} = s_2 + s_{11}$$

# Edelman-Greene insertion

**Theorem (with Morse; 2016)**

*For any permutation $w \in S_n$, the crystal isomorphism*

$$B(w) \cong \bigoplus_{\lambda} B(\lambda)^{\oplus a_{w\lambda}}$$

*is explicitly given by the Edelman-Greene insertion $\varphi^{Q}_{\mathrm{EG}}(v^\ell \cdots v^1) = Q$:*

$$\varphi^{Q}_{\mathrm{EG}} \circ e_i = e_i \circ \varphi^{Q}_{\mathrm{EG}}$$


Emil
i Lönneberga

# Outline

# Motivation: Schubert Calculus

## Polynomial Representatives for Schubert Cells

| | **Grassmannian** $\mathbb{G}_{m,n}$ | **Flag Varieties** $Fl_n$ |
|---|---|---|
| **Cohomology** | $s_\lambda$ | $\mathfrak{S}_w \to F_w$ |
| $K$**-theory** | $\mathfrak{G}_\lambda$ | $\mathfrak{G}_w$ |

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Grassmannian Grothendieck polynomials: $\mathfrak{G}_\lambda$ Lascoux, Schützenberger 1982
Stable Grothendieck polynomials: $\mathfrak{G}_w$ Fomin, Kirillov 1994

## Combinatorial Approach?

Combining:

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- Crystal structure on decreasing factorizations for $F_w$ (Morse, S. 2016)

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## Combinatorial Approach?

Combining:

- Crystal structure on decreasing factorizations for $F_w$ (Morse, S. 2016)
- Crystal structure for $\mathfrak{G}_\lambda$ on set-valued tableaux (Monical & Pechenik & Scrimshaw 2018)

# 0-Hecke Monoid

## Definition

0-Hecke monoid $\mathcal{H}_0(n)$:
monoid of all finite words in $[n] := \{1, 2, \ldots, n\}$ such that

$$\begin{aligned} pp \equiv p, \quad pqp &\equiv qpq \quad \text{for all } p, q \in [n] \\ pq &\equiv qp \quad \text{if } |p - q| > 1 \end{aligned}$$

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## Examples

- 2112

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## Examples

- $2112 \equiv 212$

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0-Hecke monoid $\mathcal{H}_0(n)$:
monoid of all finite words in $[n] := \{1, 2, \dots, n\}$ such that

$$\begin{aligned} pp &\equiv p, \quad pqp \equiv qpq && \text{for all } p, q \in [n] \\ pq &\equiv qp && \text{if } |p - q| > 1 \end{aligned}$$

## Examples

- $2112 \equiv 212 \equiv 121$
- $2121$

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# Decreasing factorizations in $\mathcal{H}_0(n)$

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A decreasing factorization of $w \in \mathcal{H}_0(n)$ into $m$ factors is a product of decreasing factors

$$\mathbf{h} = h^m \dots h^2 h^1$$

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## Example

Decreasing factorizations for $132 \in \mathcal{H}_0(3)$ of length 5 with 3 factors:

$$\begin{array}{ccc} (31)(31)(2) & (31)(32)(2) & (31)(1)(32) \\ (31)(3)(32) & (1)(31)(32) & (3)(31)(32) \end{array}$$

# Stable Grothendieck polynomials for $w$

## Definition

Stable Grothendieck polynomial (or $K$-Stanley symmetric function):

$$\mathfrak{G}_w(\mathbf{x},\beta) = \sum_{h^m \ldots h^2 h^1 \in \mathcal{H}_w^m} \beta^{\ell(h^1)+\cdots+\ell(h^m)-\ell(w)} x_1^{\ell(h^1)} \ldots x_m^{\ell(h^m)}$$

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Decreasing factorizations for constant term:
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$$\beta^0 : \ (x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2) + 2x_1x_2x_3 = s_{21}$$

# Schur positivity

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$$\mathfrak{G}_w(\mathbf{x},\beta) = \sum_\lambda \beta^{|\lambda|-\ell(w)} g_w^\lambda s_\lambda(\mathbf{x})$$

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$$\mathfrak{G}_{132}(\mathbf{x}, \beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$

# 321-avoiding Hecke words (braid-free)

## Definition

$w \in \mathcal{H}_0(n)$ is 321-avoiding if none of the reduced expressions for $w$ contain a consecutive subword of the form $i\ i+1\ i$ for any $i \in [n-1]$.

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## Example

- $(\ )(1)(21) \in \mathcal{H}^3, \notin \mathcal{H}^{3,\star}$
- $(31)(2) \in \mathcal{H}^{2,\star}$
- $(2)(21)(32) \in \mathcal{H}^{3,\star}$

# $\star$-Crystal on $\mathcal{H}^{m,\star}$ (Morse, Pan, Poh, S.)

## Bracketing rule on $h^m \dots h^{i+1}\, h^i \dots h^1$

1. Start with the **largest** letter $b$ in $h^{i+1}$, pair it with the smallest $a \geqslant b$ in $h^i$. If there is no such $a$, then $b$ is unpaired.
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## Example

- (1)(32)

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- $(7532)(621)$

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- $(7532)(621) \xrightarrow{\text{bracket}} (7532)(621) \xrightarrow{f_1^\star} (75321)(61)$

## Vertices and edges

$w = 132, \ \beta^1$

$$\mathfrak{G}_{132}(\mathbf{x}, \beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$

1. $(3,1)(3,2)(\,)$
2. $(3,1)(1)(2)$
3. $(3,1)(2)(2)$
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5. $(1)(3,1)(2)$
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$$\begin{tikzcd}
 & (\,)(3,1)(3,2) \arrow[d, "2"] & & (1)(1)(3,2) \arrow[d, "1"] & (1)(3)(3,2) \arrow[d, "1"] \\
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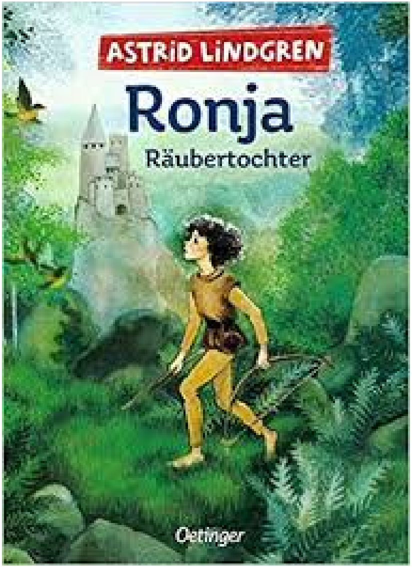
 & (3,1)(3)(2) \arrow[d, "1"] & & & \\
 & (3,1)(3,2)(\,) & & &
\end{tikzcd}$$


ASTRID LINDGREN
Ronja
Räubertochter
Oetinger

# Outline

# Grothendieck polynomials for skew shapes

$$\mathfrak{G}_{\nu/\lambda}(\mathbf{x};\beta) = \sum_{T \in \mathrm{SVT}(\nu/\lambda)} \beta^{\mathrm{ex}(T)} \mathbf{x}^{\mathrm{wt}(T)} \qquad \text{(Buch 2002)}$$

$\mathrm{SVT}(\nu/\lambda)$ = set of semistandard set-valued tableaux of shape $\nu/\lambda$

Excess in $T$ is $\mathrm{ex}(T)$

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## Semistandard set-valued tableaux $\mathrm{SVT}(\nu/\lambda)$

Fill boxes of skew shape $\nu/\lambda$ with nonempty sets. Semistandardness:

$$\begin{array}{|c|c} \hline C & \\ \hline A & \multicolumn{1}{|c|}{B} \\ \hline \end{array} \qquad \max(A) \leqslant \min(B), \ \max(A) < \min(C)$$

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## Example (Which one is a valid filling?)

$$\begin{array}{|c|c|c|} \hline 34 & 45 & \\ \hline & 12 & 25 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 34 & 35 & \\ \hline & 12 & 456 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 2 & 35 & \\ \hline & 14 & 56 \\ \hline \end{array}$$

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## Signature rule

Assign $-$ to every column of T containing an $i$ but not an $i+1$.
Assign $+$ to every column of $T$ containing an $i+1$ but not an $i$.
Successively pair each $+$ that is adjacent to a $-$.

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$$\begin{array}{|c|c|c|} \cline{1-2} 34 & 45 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & 25 \\ \cline{2-3} \end{array} \xrightarrow{f_2}$$

# Crystal structure on SVT (Monical, Pechenik, Scrimshaw)

## Signature rule

Assign $-$ to every column of T containing an $i$ but not an $i+1$.
Assign $+$ to every column of $T$ containing an $i+1$ but not an $i$.
Successively pair each $+$ that is adjacent to a $-$.

## Crystal operator $f_i$

- changes the rightmost unpaired $i$ $-$ to $i+1$, except
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$+ \quad - \qquad -$

$$\begin{array}{|c|c|c} \hline 34 & 45 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & \multicolumn{1}{c|}{25} \\ \cline{2-3} \end{array} \xrightarrow{f_2} \qquad \qquad \begin{array}{|c|c|c} \hline 34 & 45 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & \multicolumn{1}{c|}{25} \\ \cline{2-3} \end{array} \xrightarrow{f_4}$$

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$+ \quad - \qquad -$

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$- \quad\quad\quad +$

$$\begin{array}{|c|c|c} \hline 34 & 45 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & \multicolumn{1}{c|}{25} \\ \cline{2-3} \end{array} \xrightarrow{f_4}$$

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$$\begin{array}{ccc} - & & + \\ \end{array}$$

$$\begin{array}{|c|c|c} \hline 34 & 45 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & \multicolumn{1}{c|}{25} \\ \cline{2-3} \end{array} \xrightarrow{f_4} \begin{array}{|c|c|c} \hline 345 & 5 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & \multicolumn{1}{c|}{25} \\ \cline{2-3} \end{array}$$

# Residue map as a crystal isomorphism

**Theorem (Morse, Pan, Poh, S. 2019)**

*The crystal on skew semistandard set-valued tableaux and the crystal on decreasing factorizations $\mathcal{H}^{m,\star}$ intertwine under the residue map. That is, the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,\star} \\ \downarrow f_k & & \downarrow f_k^\star \\ \mathsf{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,\star}. \end{array}$$

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## Example

$$\begin{array}{|c|c} \hline 3_{1} & \\ \hline 1_{2} & 123_{3} \\ \hline \end{array} \xrightarrow{\quad \mathsf{res} \quad}$$

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**Example**

$$\begin{array}{|c|c} \hline 3_1 & \\ \hline 1_2 & \multicolumn{1}{|c|}{123_3} \\ \hline \end{array} \xrightarrow{\quad \mathrm{res} \quad} (31)(3)(32)$$

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# $\star$-Insertion

## Insert $x$ into row $R$ of a transpose of a semistandard tableau

1. Try to append $x$ to the right of $R$ (terminate and record)
2. $x \notin R$, bump the minimal $z > x$ (proceed to the next row)
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## Example

$$\mathbf{h} = (42)(42)(31) = \begin{bmatrix} 3 & 3 & 2 & 2 & 1 & 1 \\ 4 & 2 & 4 & 2 & 3 & 1 \end{bmatrix}$$

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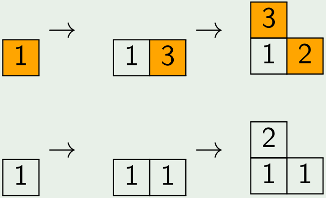

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$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \rightarrow \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} \rightarrow \begin{array}{|c|cc}\hline 3 & & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \rightarrow \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array} \rightarrow \begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

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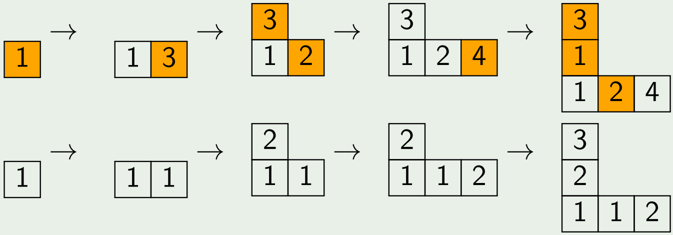

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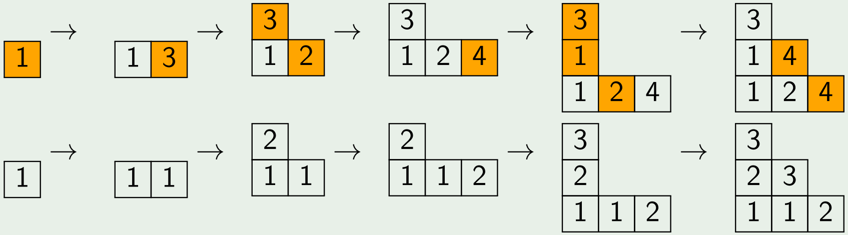

# Association with $\star$-crystal

**Theorem (Morse, Pan, Poh, S. 2019)**

*The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathrm{SSYT}^m \\ \downarrow f_i^\star & & \downarrow f_i \\ \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathrm{SSYT}^m \end{array}$$

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$$\begin{tikzcd} (42)(42)(31) \arrow[r, "\star"] \arrow[d, "f_1^\star"] & \begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & \multicolumn{1}{|c|}{4} & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} , \begin{array}{|c|cc} \hline 3 & & \\ \hline 2 & \multicolumn{1}{|c|}{3} & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \arrow[d, "f_1"] \\ (42)(421)(3) \arrow[r, "\star"] & \begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & \multicolumn{1}{|c|}{4} & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} , \begin{array}{|c|cc} \hline 3 & & \\ \hline 2 & \multicolumn{1}{|c|}{3} & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \end{tikzcd}$$

# Uncrowding SVT

Uncrowding operator Lenart 2000; Buch 2002; Bandlow, Morse 2012; Patrias 2016; Reiner, Tenner, Yong 2018

- Identify the topmost row in $T$ containing a multicell.
- Let $x$ be the largest letter in that row which lies in a multicell.
- Delete $x$ and perform RSK algorithm into the rows above. Repeat.
- Result is a single-valued skew tableau.

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| | | |
|---|---|---|
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**Example**

$$\begin{array}{|c|c|c} \hline 34 & 4{\color{red}5} & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & \multicolumn{1}{c|}{25} \\ \cline{2-3} \end{array} \to \begin{array}{|c|c|c} \cline{1-1} 5 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 3{\color{red}4} & 4 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & \multicolumn{1}{c|}{25} \\ \cline{2-3} \end{array} \to \begin{array}{|c|c|c} \cline{1-1} 5 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12 & \multicolumn{1}{c|}{2{\color{red}5}} \\ \cline{2-3} \end{array}$$

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**Example**

$$\begin{array}{ccc} 34 & 45 & \\ & 12 & 25 \end{array} \to \begin{array}{ccc} 5 & & \\ 34 & 4 & \\ & 12 & 25 \end{array} \to \begin{array}{ccc} 5 & & \\ 4 & & \\ 3 & 4 & \\ & 12 & 25 \end{array} \to \begin{array}{ccc} 5 & & \\ 4 & & \\ 3 & 4 & 5 \\ & 12 & 2 \end{array}$$

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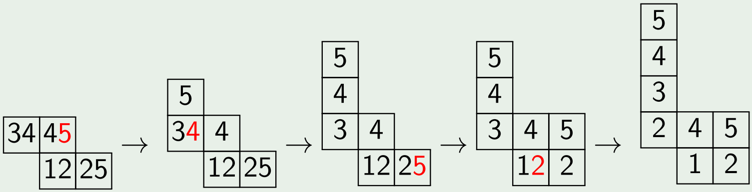

# Connection to uncrowding map

**Theorem (Morse, Pan, Poh, S. 2019)**

*Let* $T \in \mathrm{SVT}^m(\lambda)$, $(\tilde{P}, \tilde{Q}) = \mathrm{uncrowd}(T)$, *and* $(P, Q) = \star \circ \mathrm{res}(T)$. *Then* $Q = \tilde{P}$.

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$$\begin{array}{|c|c|} \hline 3_1 & \\ \cline{1-1} 1_2 & 123_3 \\ \hline \end{array}$$

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**Example**

$$\begin{array}{|c|c|} \hline 3 & \\ \cline{1-1} 2 & 3 \\ \hline 1 & 1 \\ \hline \end{array} \xleftarrow{\text{uncrowd}} \begin{array}{|c|c|} \hline 3_{\color{red}1} & \\ \cline{1-1} 1_{\color{red}2} & 123_{\color{red}3} \\ \hline \end{array}$$

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**Example**

$$\begin{array}{|c|c} \hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{3} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array} \xleftarrow{\text{uncrowd}} \begin{array}{|c|c} \hline 3_{\color{red}1} & \\ \hline 1_{\color{red}2} & \multicolumn{1}{|c|}{123_{\color{red}3}} \\ \hline \end{array} \xrightarrow{\text{res}} (31)(3)(32) \xrightarrow{\star} \begin{array}{|c|c} \hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{3} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array}, \begin{array}{|c|c} \hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{3} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array}$$

# Hecke insertion (Buch 2008, Patrias, Pylyavskyy 2016)

**Insert $x$ to row $R$ of an increasing tableau**

- Try to append $x$ to the right of $R$ (record and terminate)
- Try to bump the smallest letter that is bigger (proceed to the next row)

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$$\boxed{2} \quad \rightarrow \quad \boxed{2}\boxed{3}$$

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**Theorem (Morse, Pan, Poh, S. 2019)**

*Let* $T \in \mathrm{SVT}(\lambda)$ *and* $[\mathbf{k}, \mathbf{h}]^t = \mathrm{res}(T)$*. Apply Hecke row insertion from the right on* $[\mathbf{k}, \mathbf{h}]^t$ *to obtain the pair of tableaux* $(P, Q)$*. Then* $Q = T$*.*

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# Future Work

- Crystal structure for the non-321 avoiding case (beyond skew shapes)

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- Demazure crystal structure to compute the intersection number?

Thank you !