SYMMETRIC GROUP CHARACTERS AS SYMMETRIC FUNCTIONS

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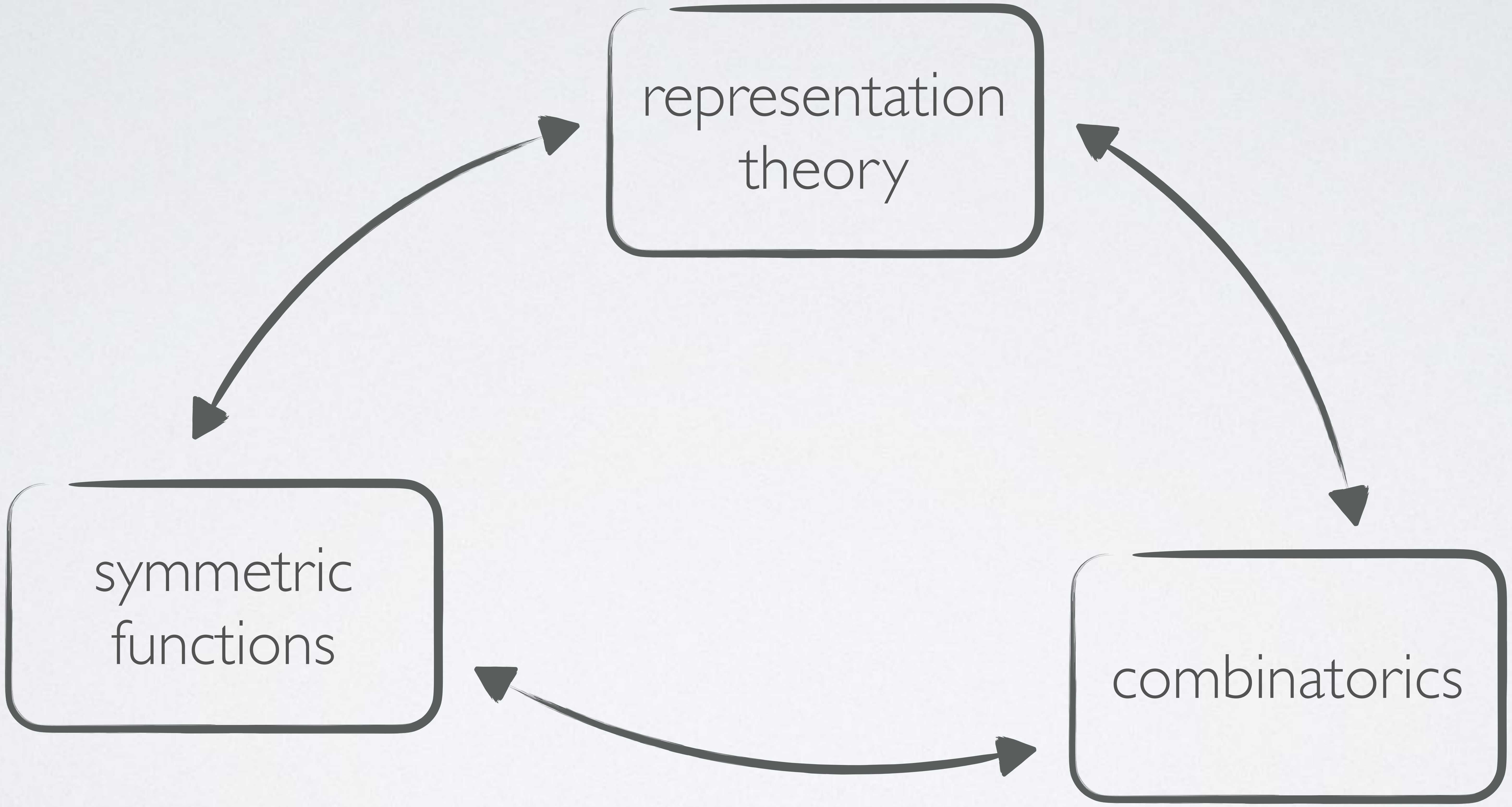
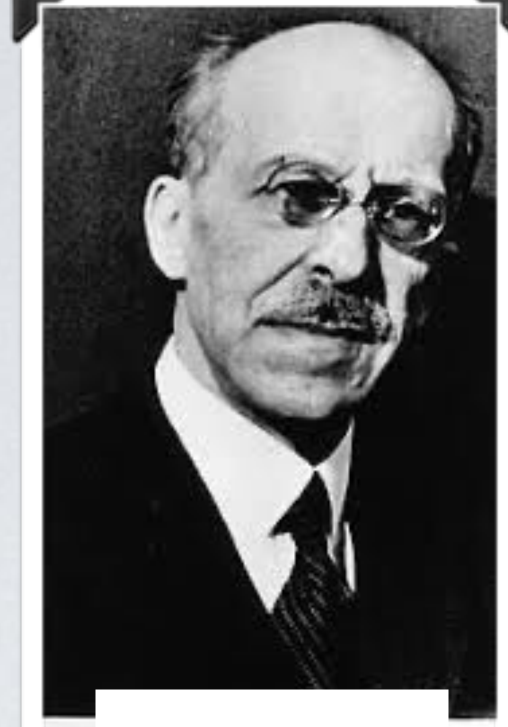
joint work with Rosa Orellana

Combinatorial representation theory



Basic idea of representation theory:

Describe the maps from a group to the general linear group (the possible linear actions of the group on a vector space).

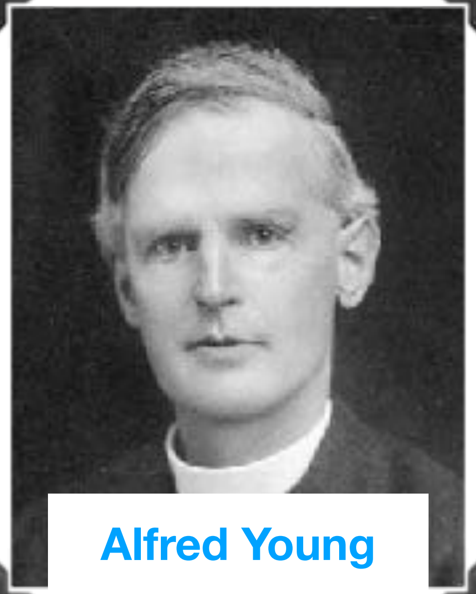


Issai Schur

Gilbert de Beauregard Robinson

Archibald Read Richardson

Hermann Weyl



Alfred Young

Heinrich Maschke

Georg Frobenius

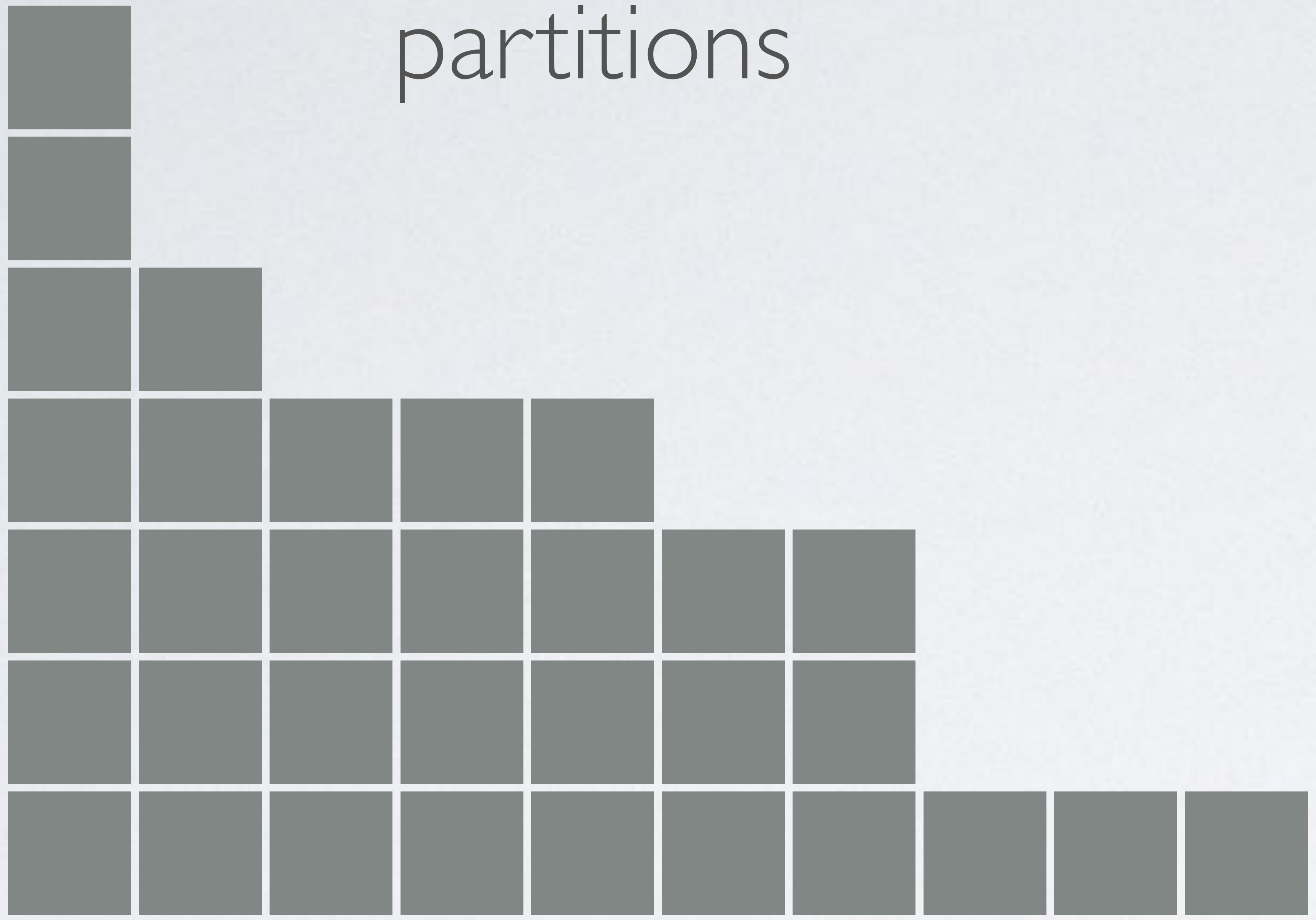
Dudley Littlewood

Francis Dominique Murnaghan

- all Gl_n and S_k representations can be broken down into irreducible representations
- the irreducible representations of general linear group Gl_n and symmetric group S_k are closely related
- irreducible representations of Gl_n and S_k are indexed by partitions
- the maps are characterized by a small number of values called ‘characters’

Combinatorics

partitions



$\lambda = (10, 7, 7, 5, 2, 1, 1)$

$\lambda \vdash 33$

is a partition of

column strict tableaux

5					
3	3	6			
2	2	3	5	5	
1	1	1	1	2	3

shape $\lambda = (6, 5, 3, 1)$

content $(4, 3, 4, 0, 3, 1)$

standard tableaux $\Longrightarrow$ content $= (1^n)$

6					
5	8	12			
3	4	11	14	15	
1	2	7	9	10	13

The Hopf algebra of symmetric functions

symmetric polynomials ⟷ get rid of variables

power sum

$$p_r(x_1, x_2, \ldots, x_n) = x_1^r + x_2^r + \cdots + x_n^r$$
$$p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{\ell(\mu)}}$$

every symmetric polynomial can be written as a polynomial in:

$$p_1(x_1, x_2, \ldots, x_n), p_2(x_1, x_2, \ldots, x_n), \ldots, p_n(x_1, x_2, \ldots, x_n)$$

$p_{a_1} p_{a_2} \cdots p_{a_r}$ span ⟷ $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{\ell(\mu)}}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{\ell(\mu)}$ basis

$$Sym = \mathbb{Q}[p_1, p_2, p_3, \ldots]$$

$$p_\lambda p_\mu = p_{\lambda \uplus \mu}$$

$$\Delta(p_r) = p_r \otimes 1 + 1 \otimes p_r$$

$$\frac{p_\mu}{z_\mu} * \frac{p_\gamma}{z_\gamma} = \begin{cases} \frac{p_\mu}{z_\mu} & \text{if } \gamma = \mu \\ 0 & \text{else} \end{cases}$$

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$$p_r(x_1, x_2, \ldots, x_n) = x_1^r + x_2^r + \cdots + x_n^r$$
$$p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{\ell(\mu)}}$$

homogeneous

$$h_r(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}$$
$$h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_{\ell(\mu)}}$$

$$h_r = \sum_{\lambda \vdash r} \frac{p_\lambda}{z_\lambda}$$

Schur function

$$s_\lambda(x_1, x_2, \ldots, x_n) = \sum_{T \in CST_n^\lambda} x_1^{m_1(T)} x_2^{m_2(T)} \cdots x_n^{m_n(T)}$$

CST_n^λ = column strict tableaux, shape λ , entries in $\{1, 2, \ldots, n\}$

$m_i(T)$ = number of cells with label i in T

$$s_\lambda = \sum_{\mu \vdash |\lambda|} \chi^\lambda(\mu) \frac{p_\mu}{z_\mu}$$

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Schur functions encode characters

$$s_\lambda(x_1, x_2, \ldots, x_n) = \sum_{T \in CST_n^\lambda} x_1^{m_1(T)} x_2^{m_2(T)} \cdots x_n^{m_n(T)}$$

CST_n^λ = column strict tableaux, shape λ, entries in $\{1, 2, \ldots, n\}$

$m_i(T)$ = number of cells with label i in T

general linear group characters

$A \in Gl_n(\mathbb{C})$

$s_\lambda[\Xi_A]$ are Gl_n characters

$\Xi_A = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ eigenvalues(A)

symmetric group characters

$\sigma \in S_k$

$p_\mu(x_1, x_2, \ldots, x_n) = \sum_{\lambda \vdash k} \chi_{S_k}^\lambda(\mu) s_\lambda(x_1, x_2, \ldots, x_n)$

$\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}) \vdash k$ cycle type(σ)

$\chi_{S_k}^\lambda(\mu)$ are S_k characters

Frobenius character map

$\chi : S_k \to \mathbb{Z}$ character of a symmetric group representation

$$\mathcal{F}_k(\chi) = \frac{1}{k!} \sum_{\sigma \in S_k} \chi(\sigma) p_{cyc(\sigma)}$$

irreducibles are encoded by Schur functions

$$\mathcal{F}_k(\chi^\lambda) = s_\lambda$$

induction, restriction, tensor = product, coproduct, inner product

$$\mathcal{F}_{k+\ell}(Ind_{S_k \times S_\ell}^{S_{k+\ell}} \chi_1 \otimes \chi_2) = \mathcal{F}_k(\chi_1)\mathcal{F}_\ell(\chi_2)$$

$$\mathcal{F}(\oplus_{r=0}^{k} Res_{S_{k-r} \times S_r}^{S_k} \chi) = \Delta(\mathcal{F}_k(\chi))$$

$$\mathcal{F}_k(\chi_1 \otimes \chi_2) = \mathcal{F}_k(\chi_1) * \mathcal{F}_k(\chi_2)$$

Schur-Weyl duality

general linear group ⟷ symmetric group

$A \in Gl_n(\mathbb{C})$

$\sigma \in S_k$

$V = \text{span}_{\mathbb{C}}\{v_1, v_2, \ldots, v_n\}$

$$A(v_i) = \sum_{j=1}^{n} a_{ij} v_j$$

$$\sigma(w_1 \otimes w_2 \otimes \cdots \otimes w_k) = w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \cdots \otimes w_{\sigma(k)}$$

$$V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}}$$

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} W^{\lambda}_{Gl_n(\mathbb{C})} \otimes W^{\lambda}_{S_k}$$

multiplicity of tensor of irreducibles of one

=

restriction of irreducibles on the other

Unsolved problems in combinatorial representation theory

TENSOR

$W^{\lambda}_{S_n} \otimes W^{\mu}_{S_n}$

RESTRICTION

$Res \downarrow^{Gl_n}_{S_n} W^{\lambda}_{Gl_n}$

PLETHYSM

composition of characters

$s_{\lambda}[s_{\mu}]$

partition algebra SW-duality (1990's +)

symmetric group $\longleftrightarrow$ partition algebra

$$\sigma \in S_n$$

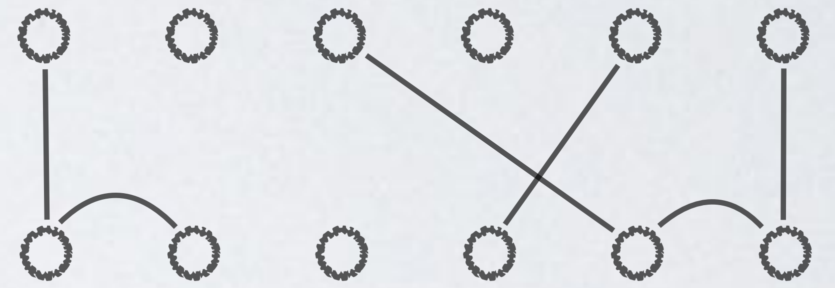
$$V = \text{span}_{\mathbb{C}}\{v_1, v_2, \ldots, v_n\}$$

$$\sigma(v_i) = v_{\sigma(i)}$$

$$d(v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4} \otimes v_{i_5} \otimes v_{i_6}) = \sum_{\vec{i}'} c^{\vec{i}}_{\vec{i}'} v_{i'_1} \otimes v_{i'_2} \otimes v_{i'_3} \otimes v_{i'_4} \otimes v_{i'_5} \otimes v_{i'_6}$$

$$c^{\vec{i}}_{\vec{i}'} = \begin{cases} 1 & \text{if } i_r = i_s \text{ when } r, s \text{ in same block of } d \\ 0 & \text{else} \end{cases}$$

$$v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4} \otimes v_{i_5} \otimes v_{i_6}$$

$d =$ 

$$v_{i'_1} \otimes v_{i'_2} \otimes v_{i'_3} \otimes v_{i'_4} \otimes v_{i'_5} \otimes v_{i'_6}$$

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash n} W^{\lambda}_{S_n} \otimes W^{\lambda}_{P_k(n)}$$

multiplicity of tensor of irreducibles of one
=
restriction of irreducibles on the other

$$V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}}$$

Symmetric group characters as symmetric functions

Question: Is there a basis of the symmetric functions takes the place of the Schur functions in this setup?

Original answer: $\tilde{s}_\lambda := \lim_{n\to\infty} \mathcal{F}_n^{-1}(s_{(n-|\lambda|,\lambda)})$

symmetric group characters

$\sigma \in S_n \subset Gl_n(\mathbb{C})$

$\Xi_\mu = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ eigenvalues(σ)

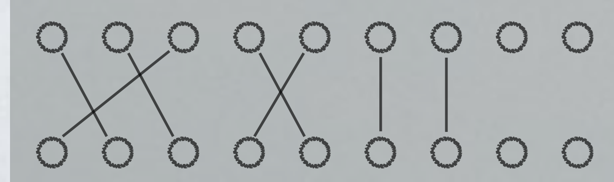
eigenvalues depend only on cycle type

$\tilde{s}_\lambda[\Xi_\mu]$ are S_n characters

$\chi_{S_n}^{(n-|\lambda|,\lambda)}(\mu)$

partition algebra characters

$d_{(3,2,1,1)} =$



$d \in P_k(n)$

type(d)

$$p_\mu(x_1, x_2, \ldots, x_n) = \sum_{\lambda \vdash n} \chi^\lambda_{P_k(n)}(d_\mu)\tilde{s}_{\overline{\lambda}}(x_1, x_2, \ldots, x_n)$$

$\chi^\lambda_{P_k(n)}(d_\mu)$ are $P_k(n)$ characters

Practical definitions of the irreducible character basis

orthonormal wrt scalar product $\langle f, g \rangle_{@} := \frac{1}{k!} \sum_{\sigma \in S_k} f[\Xi_\sigma] g[\Xi_\sigma]$ + Gram–Schmidt orthonormalization

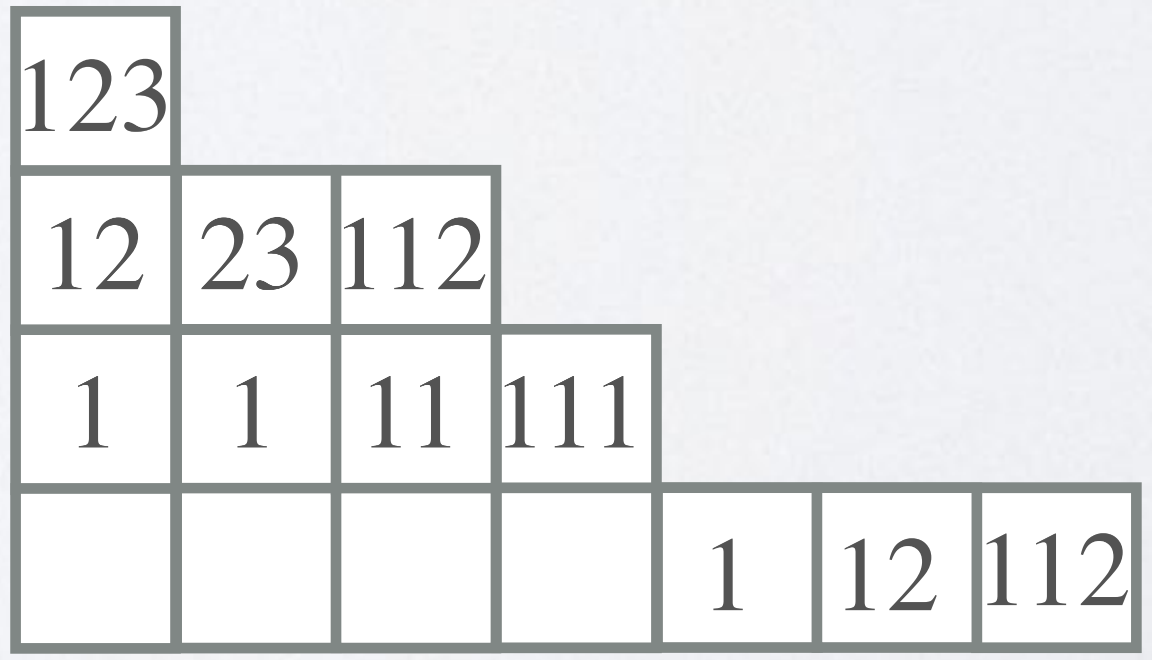
various transition coefficients

$$\tilde{s}_\lambda = \sum_{\gamma \vdash n} \chi^{\lambda}_{S_n}(\gamma) \frac{\mathbf{p}_\gamma}{z_\gamma} \qquad \mathbf{p}_\gamma = \prod_{i \geq 1} \mathbf{p}_{i^{n_i(\gamma)}} \qquad \mathbf{p}_{i^r} = \sum_{k=0}^{r} (-1)^{r-k} i^k \binom{r}{k} \left(\frac{1}{i} \sum_{d|i} \mu(i/d) p_d \right)_k$$

$$p_\mu = \sum_{|\lambda| \leq |\mu|} \chi^{(n-|\lambda|,\lambda)}_{P_{|\mu|}(n)}(d_\mu) \tilde{s}_\lambda$$

$$h_\mu = \sum_{T \in MST_\mu} \tilde{s}_{\overline{sh(T)}}$$

MST_μ =multiset tableaux of content μ



content $(15, 6, 2)$

Expansion formula into the irreducible character basis

$$f = \sum_{\lambda} \left\langle f, s_{(n-|\lambda|,\lambda)}[1 + h_1 + h_2 + \cdots] \right\rangle \tilde{s}_\lambda = \sum_{\lambda} \langle f, \tilde{s}_\lambda \rangle_{@} \, \tilde{s}_\lambda$$

$$s_\mu = \sum_{|\lambda| \leq |\mu|} r_{\lambda\mu} \tilde{s}_\lambda$$ "restriction problem" $$Res\Big\downarrow_{S_n}^{Gl_n} W^\mu_{Gl_n} \cong \bigoplus_\lambda (W^\lambda_{S_n})^{r_{\lambda\mu}}$$

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_\gamma \overline{g}_{\lambda\mu\gamma} \tilde{s}_\gamma$$ "tensor problem" $$W_{S_n}^{(n-|\lambda|,\lambda)} \otimes W_{S_n}^{(n-|\mu|,\mu)} \cong \bigoplus_\gamma (W_{S_n}^{(n-|\gamma|,\gamma)})^{\overline{g}_{\lambda\mu\gamma}}$$

Combinatorial interpretation of $\tilde{s}_\lambda \tilde{s}_{a_1} \tilde{s}_{a_2} \cdots \tilde{s}_{a_\ell}$

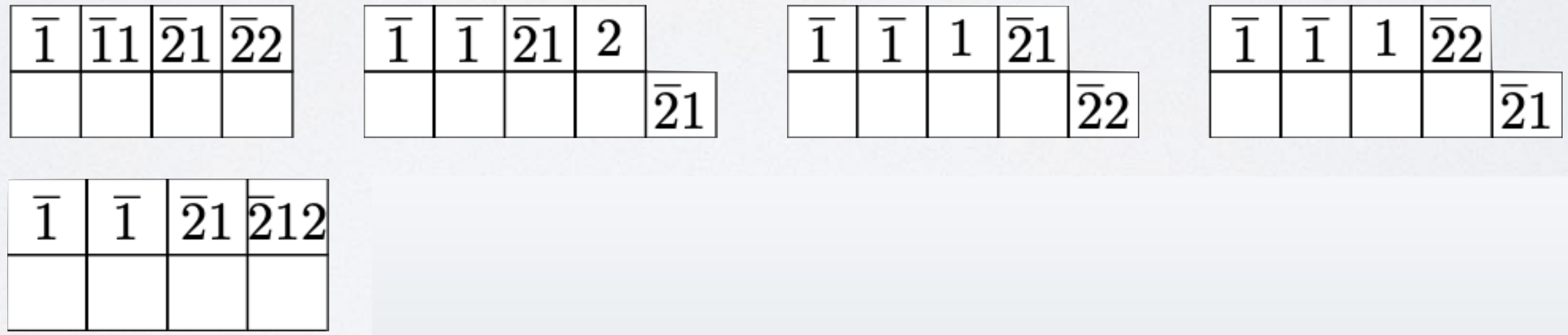
coefficients are $\overline{g}_{\nu\lambda(a_1)(a_2)\cdots(a_\ell)}$ = number of multiset tableaux satisfying

- shape $(n-|\nu|,\nu)$
- content λ barred $(a_1, a_2, \ldots, a_\ell)$ unbarred
- lattice condition
- no singletons first row
- no repeated entries in any cell

Example: $\lambda = (2,2)$ $a_1 = 2$ $a_2 = 1$

coefficient of $\tilde{s}_4$ in $\tilde{s}_{(2,2)}\tilde{s}_2\tilde{s}_1 = 5$

$\nu = (4)$



the Littlewood-Richardson rule

$$s_\lambda s_{\mu_1} s_{\mu_2} \cdots s_{\mu_r} = s_\lambda s_\mu + \text{other stuff}$$

combinatorial interpretation of LHS = skew column strict tableaux of shape γ/λ content μ

"lattice"

not "lattice"

$$s_\lambda s_\mu = \sum_T s_{sh(T)}$$

where the sum is over all "lattice" column strict tableaux of shape γ/λ and content μ

2	3	3			
	1	2	2		
			1	1	1

"lattice"

2	3	3			
	1	1	2		
			1	1	2

not "lattice"

a column strict tableau is "lattice" if the last r letters of the reading word contains at least as many i's as i+1's

Combinatorial interpretation of $\tilde{s}_\lambda \tilde{s}_{a_1} \tilde{s}_{a_2} \cdots \tilde{s}_{a_\ell}$

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coefficient of $\tilde{s}_4$ in $\tilde{s}_{(2,2)} \tilde{s}_2 \tilde{s}_1 = 5$

$\nu = (4)$

$\bar{1}$	$\bar{1}1$	$\bar{2}1$	$\bar{2}2$	

$\bar{1}$	$\bar{1}$	$\bar{2}1$	2	
				$\bar{2}1$

$\bar{1}$	$\bar{1}$	1	$\bar{2}1$	
				$\bar{2}2$

$\bar{1}$	$\bar{1}$	1	$\bar{2}2$	
				$\bar{2}1$

$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}12$

(vague) Conjecture: There is an additional condition such that this set of tableau + this condition is a combinatorial interpretation for the coefficient $\overline{g}_{\nu\lambda\mu}$

Takeaway message

The following concepts have developed about representation/combinatorics of the Littlewood-Richardson rule over nearly 80+ years of mathematics:

operations of adding/removing a corner/row (Pieri rule)
that are compatible with RSK insertion and evacuation on tableaux
reading words/lattice, plactic monoid and Jeu de Taquin

Similar operations may need to be developed for set-valued and multiset valued tableaux to arrive at a combinatorial Kronecker product rule, restriction problem and plethysm

We have guides for developing this combinatorics: representation theory of partition algebra, multiset partition algebra, symmetric group character basis and we need to develop others

Thank you!