Intermediate Algebraic Structure in the Restriction Problem

Nate Harman

University of Chicago

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Polynomial Representations of $GL_n(\mathbb{C})$

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Such representations are called *polynomial representations*.

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$$\mathrm{tr}(\phi_\lambda(\mathrm{diag}(x_1, x_2, \ldots, x_n)) = s_\lambda(x_1, x_2, \ldots, x_n)$$

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- $d := |\lambda|$ is called the *degree*. It is also the degree of the matrix entries $\phi_\lambda(g)_{i,j}$ as polynomials in the matrix entries $g_{i,j}$.
- The degree also controls the dimension growth as we vary n:

$$\dim(W_\lambda(n)) = \frac{f_\lambda}{d!} n^d + O(n^{d-1})$$

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- As n varies the representations $S^{\lambda[n]}$ are functorial in a certain sense made precise by the theory of *FI*-modules.
- For n sufficiently large the dimension of $S^{\lambda[n]}$ is a polynomial in n. Explicitly if $d = |\lambda|$ then

$$\dim(S^{\lambda[n]}) = \frac{f_\lambda}{d!} n^d + O(n^{d-1})$$

The Restriction Problem

S_n sits naturally inside $GL_n(\mathbb{C})$ as the group of permutation matrices.

Goal: Understand the restriction of polynomial representations from $GL_n(\mathbb{C})$ to S_n.

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The Restriction Problem: Find a positive combinatorial interpretation for the multiplicities $m_{\lambda,\mu}$.

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Alternatively we could define R_n as the monoid of partially defined injective functions from $\{1, 2, \ldots, n\}$ to itself.

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Easy observation: Polynomial $GL_n(\mathbb{C})$ representations can be "restricted" to the R_n.

Representations of the Rook Monoid

Theorem (Munn, Solomon.)

1. *If* λ *is a partition of* $k \leq n$ *the* S_n*-representation* $\mathrm{Ind}_{S_k \times S_{n-k}}^{S_n}(S^\lambda \boxtimes S^{(n-k)})$ *can be upgraded to an irreducible representation* $V_\lambda(n)$ *of* R_n*.*

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2. *As λ runs over all partitions of size at most n these $V_{\lambda}(n)$ form a complete list of the irreducible representations of R_n.*

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In particular note that restriction from R_n to S_n is just given by the Pieri rule.

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Any representation M of R_n has a canonical filtration given by:

$$M_{\geq r} := \{m \in M \mid I_{<r}(n) \cdot m = 0\}$$

In terms of the classification off irreducible representations $M_{\geq r}$ is the sum of all the irreducible subspaces of M of the form V_λ with $|\lambda| \geq r$.

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In the context of the restriction problem, if λ is a partition of degree d and ℓ parts then:

$$W_\lambda(n)_{\geq d} \cong V_\lambda(n) \text{ and } W_\lambda(n)_{\geq \ell} = W_\lambda(n)$$

Functoriality and Representation Stability

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Theorem (Church-Ellenberg-Farb)

If V is a finitely generated FI-module (over $\mathbb{C}$) then for $n \gg 0$

$$V([n]) \cong \bigoplus_{\lambda} c_\lambda S^{\lambda[n]}$$

where c_λ is independent of n, and nonzero for only finitely many λ.

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$FI\sharp$-modules are then functors from $FI\sharp$ to vector spaces.

$FI\sharp$-modules cont.

$FI\sharp$-modules are **much** more rigid than FI-modules in general.

Theorem (Church-Ellenberg-Farb)

1. *For λ fixed and n varying, the irreducible rook monoid representations $V_\lambda(n)$ form an irreducible $FI\sharp$-module V_λ.*

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Fact (Church-Ellenberg-Farb): This composition is finitely generated.

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4. If we look at all the sum of all the weight spaces of W_λ corresponding to a single orbit, this forms an S_n-subrepresentation.

An example, or why I am skeptical of symmetric functions

$$S_{3,1}(x_1, x_2, x_3, x_4) = 3m_{(1,1,1,1)} + 2m_{(2,1,1)} + m_{(2,2)} + m_{(3,1)}$$

$3m_{(1,1,1,1)} = 3x_1x_2x_3x_4$ corresponds to the 3 dimensional $(1,1,1,1)$-weight space being isomorphic to $S^{(3,1)}$ as a representation of S_4.

$2m_{(2,1,1)} = 2x_1^2x_2x_3 + 2x_1^2x_2x_4 + \cdots + 2x_4^2x_2x_3$ is the T_n character of a 24 dimensional space with an S_4 action isomorphic to

$$S^{1,1,1,1} \oplus 3S^{(2,1,1)} \oplus 3S^{(3,1)} \oplus 2S^{(2,2)} \oplus S^{(4)}$$

$m_{(2,2)} = x_1^2x_2^2 + \cdots + x_3^2x_4^2$ is the T_n character of a 6 dimensional space with an S_4 action isomorphic to $S^{(2,1,1)} \oplus S^{(3,1)}$

$m_{(3,1)} = x_1^3x_2 + \cdots + x_4^3x_3$ is the T_n character of a 12 dimensional space with an S_4 action isomorphic to $S^{(3,1)} \oplus S^{(2,2)} \oplus 2S^{(3,1)} \oplus S^{(4)}$

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As a consequence of this semidirect product structure, each irreducible M_n representation has a single S_n-orbit of weights for T_n that all occur with equal multiplicity.

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- In particular the sum $\sum_{i=0}^{\infty} i|\lambda_i|$ is the degree of the representation.
- As a representation of S_n, the irreducible representation indexed by $(\lambda_0, \lambda_1, \lambda_2, \dots)$ is isomorphic to

$$\mathrm{Ind}_{S_{|\lambda_0|} \times S_{|\lambda_1|} \times \dots}^{S_n} (S^{\lambda_0} \boxtimes S^{\lambda_1} \boxtimes \dots)$$

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Restriction to Monomial Matrices

We say that an irreducible representation of M_n is *strongly polynomial* if λ_0 is a (possibly empty) horizontal strip. In this case we'll denote it as $V(\lambda_1, \lambda_2, \lambda_3, \dots)_n$, omitting all trailing empty partitions.

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2. *$W_\lambda(n)$ contains $V(\lambda)_n$ as a summand provided $n \geq |\lambda|$.*
3. *As n varies the M_n representations $V(\lambda_1, \lambda_2, \lambda_3, \ldots)_n$ form an irreducible FI_{mon}-module, where FI_{mon} is a monomial matrix analog of FI and $FI\sharp$.*

Restriction to Monomial Matrices

We say that an irreducible representation of M_n is *strongly polynomial* if λ_0 is a (possibly empty) horizontal strip. In this case we'll denote it as $V(\lambda_1, \lambda_2, \lambda_3, \ldots)_n$, omitting all trailing empty partitions.

Theorem (H?)

1. *$W_\lambda(n)$ restricted to M_n is decomposes as a direct sum of degree $|\lambda|$ strictly polynomial irreducible representations.*
2. *$W_\lambda(n)$ contains $V(\lambda)_n$ as a summand provided $n \geq |\lambda|$.*
3. *As n varies the M_n representations $V(\lambda_1, \lambda_2, \lambda_3, \ldots)_n$ form an irreducible FI_{mon}-module, where FI_{mon} is a monomial matrix analog of FI and $FI\sharp$.*

Definition/Problem: Call these representations of M_n on an S_n orbit of weight spaces in W_λ *Kostka modules*. How do they decompose into irreducible M_n representations?

An example revisited

$$S_{3,1} = 3m_{(1,1,1,1)} + 2m_{(2,1,1)} + m_{(2,2)} + m_{(3,1)}$$

$3m_{(1,1,1,1)}$ is the T_n character the Kostka module $K((3,1),(1,1,1,1))$, which is isomorphic to $V((3,1))_n$ as a representation of M_n.

$2m_{(2,1,1)}$ is the T_n character the Kostka module $K((3,1),(2,1,1))$, which is isomorphic to $V((1,1),(1))_n \oplus V((2),(1))_n$ as a representation of M_n.

$m_{(2,2)}$ is the T_n character the Kostka module $K((3,1),(2,2))$, which is isomorphic to $V(\emptyset,(1,1))_n$ as a representation of M_n.

$m_{(3,1)}$ is the T_n character the Kostka module $K((3,1),(3,1))$, which is isomorphic to $V((1),\emptyset,(1))_n$ as a representation of M_n.

Thanks!