SELBERG INTEGRALS

SCHUR POLYNOMIALS

COMBINATORIAL CONNECTIONS

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IMSc Algebraic Combinatorics Seminar

JACK POLYNOMIALS

$\alpha \in \mathbb{R}$

(a) $P_\lambda^{(\alpha)} = m_\lambda + \sum_{\mu < \lambda} k_{\lambda\mu}(\alpha)\, m_\mu$

(b) If $\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu}\, \alpha^{\ell(\lambda)} z_\lambda$

then $P_\lambda^{(\alpha)}$ are orthogonal w.r.t $\langle , \rangle_\alpha$.

$P_\lambda^{(0)} = e_{\lambda'}$ $\qquad P_\lambda^{(1)} = s_\lambda$ $\qquad P_\lambda^{(\infty)} = m_\lambda$ $\qquad P_{(1^r)}^{(\alpha)} = e_r$ $\qquad P_{(r)}^{(\alpha)} \sim h_r^{(\cdot\cdot)}$

Pieri rule (Stanley)

$$P_\lambda^{(\alpha)} e_r = \sum_{\substack{\nu \\ \nu/\lambda \text{ vertical} \\ r \text{ strip}}} f_{\nu\lambda}(\alpha)\, P_\nu^{(\alpha)}$$

product of upper/lower hook lengths

KADELL INTEGRAL

$\underline{x} = (x_1, \dots, x_n)$

$\delta = (n-1, n-2, \dots, 0)$

Let $W(\alpha, \beta, \gamma; \underline{x}) = \prod_i x_i^{\alpha-1} (1-x_i)^{\beta-1} \left|\Delta(\underline{x})\right|^{2\gamma}$

$\Delta(\underline{x}) = \prod_{1 \le i < j \le n} (x_i - x_j)$

$a_\delta = \det\left(x_i^{n-1-j}\right)_{i,j=0}^{n-1}$

Then

$$\int_{[0,1]^n} P_\lambda^{(1/\gamma)}(\underline{x})\, W(\alpha, \beta, \gamma; \underline{x})\, d\underline{x}$$

$$= n! \prod_{i<j} \frac{\Gamma(\lambda_i - \lambda_j + (j-i+1)\gamma)}{\Gamma(\lambda_i - \lambda_j + (j-i)\gamma)} \prod_{i=1}^{n} \frac{\Gamma(\lambda_i + \alpha + (n-i)\gamma)\,\Gamma(\beta + (n-i)\gamma)}{\Gamma(\lambda_i + \alpha + \beta + (2n-i-1)\gamma)}$$

$(\lambda_i - \lambda_j + (j-i)\gamma)_\gamma$

Generalizes Selberg 1944, Aomoto 1987.

DETERMINANTAL APPROACH

$\underline{y} = (y_1, \dots, y_n)$

$\underline{x} = (x_1, \dots, x_n)$

$$S_\lambda = \frac{\det\left(x_j^{\lambda_i + \delta_i}\right)}{\Delta(\underline{x})} = \frac{a_{\lambda+\delta}}{a_\delta}$$

$$f(\underline{y}) = \det\left(\int_0^1 \frac{x_i^i \, y_i^j}{1 - x_i y_i} \, dx_i \right)_{i,j=0\dots n-1}$$

$$= \int_0^1 \cdots \int_0^1 \prod_i \frac{x_i^i}{1 - x_i y_i} \underbrace{\det(y_i^j)}_{= \pm\Delta(\underline{y})} \, d\underline{x}$$

$$= \sum_{k_1, \dots, k_n = 1}^{\infty} \prod_i \frac{1}{k_i + i} \det\left(y_i^{k_i - 1 + j}\right)$$

$$\prod_i \frac{y_i^{k_i}}{k_i + i} \pm \Delta(\underline{y})$$

$$\sum_{\sigma \in S_n} f(\sigma . y)$$

$$= \int_{[0,1]^n} \prod_i x_i^i \, \Delta(\underline{y}) \det\left(\frac{1}{1 - x_i y_j}\right) d\underline{x}$$

$$= \sum_{k_1 \dots k_n = 1}^{\infty} \prod_i \frac{1}{k_i + i} \Delta(\underline{y}) \det\left(y_j^{k_i - 1}\right)$$

$$= \int_{[0,1]^n} \prod_i x_i^i \, \Delta(\underline{y}) \frac{\Delta(\underline{x}) \Delta(\underline{y})}{\prod_{i,j} (1 - x_i y_j)} \, d\underline{x}$$

$$= \sum_{k_1 \dots k_n = 1}^{\infty} \prod_i \frac{1}{k_i + i} \Delta(\underline{y})^2 \, S_{k-1-\delta}(\underline{y}) \, (-1)^{|k|}$$

$$k - 1 - \delta = (k_1 - 1 - \delta_1, k_2 - 1 - \delta_2, \dots)$$

$$= \frac{1}{n!}\int_{[0,1]^n} \frac{\Delta(\underline{x})^2\Delta(\underline{y})^2}{\prod_{i,j}(1-x_iy_j)}\, d\underline{x} \qquad = \sum_{k_1>k_2>\cdots>k_n\geq 1} \det\left(\frac{1}{k_j+i}\right)\Delta(\underline{y})^2 s_{k-1-\delta}(\underline{y})$$

$$= \sum_{\lambda} \left(\frac{1}{n!}\int s_\lambda(\underline{x})\Delta(x)^2\, d\underline{x}\right)\Delta(\underline{y})^2 s_\lambda(\underline{y}) \qquad = \sum_{k_1>\cdots>k_n} \frac{\Delta(k)\Delta(\delta)}{\prod_{i,j}(k_i+j)}\Delta(\underline{y})^2 s_{k-1-\delta}(\underline{y})$$

$$\frac{1}{\prod_{i,j}(1-x_iy_j)^\gamma} = \sum_\lambda \bigcirc\, P_\lambda^{(1/\gamma)}(x)\, P_\lambda^{(1/\gamma)}(y)$$

$k = \lambda + 1 + \delta \qquad \delta = (n-1, n-2, \cdots, 0)$

$\therefore k_i = \lambda_i + 1 + n - 1 - i = \lambda_i + n - i$

$$\int_{[0,1]^n} s_\lambda(\underline{x})\Delta(x)^2\, d\underline{x} = \frac{n!\,\Delta(k)\Delta(\delta)}{\prod_{i,j}(k_i+j)} = \Delta(k)\prod_{i=0}^{n-1}\frac{\Gamma(k_i)\Gamma(i+2)}{\Gamma(k_i+n)}$$

$$\Delta(k) = \prod_{i<j}(\lambda_i \cancel{+n} - i - \lambda_j \cancel{-n} + j)$$

Question 1: Does this generalize to prove the Kadell integral?

VARIATIONS ON THE THEME

$$\det\left(\int_0^1 \frac{x_i^{\mu_i} y_i^{\delta_j}}{1 - x_i y_i} dx_i \right) \to$$

where $\mu = \nu + \delta$

$$\int_{[0,1]^n} S_\lambda(\underline{x})\, S_\nu(\underline{x})\, \Delta(\underline{x})^2 d\underline{x} = \frac{\Delta(\lambda+1+\delta)\,\Delta(\nu+\delta)}{\prod_{i,j} (\lambda_i+\nu_j+2n-i-j+1)}$$

Agrees with $\beta=\gamma=1$ case of

Hua-Kadell

2011 Alba-Fateev-Litvinov-Tarnopolsky

$$\int_{[0,1]^n} P_\lambda^{(1/\gamma)}(\underline{x})\, P_\nu^{(1/\gamma)}\left(\underline{x}+\frac{\beta}{\gamma}-1\right) W(\alpha,\beta,\gamma;\underline{x})\, d\underline{x} = \text{product of } \Gamma\text{-factors}$$

$$\det\left(\int_0^1 \frac{(1-x_i)^{\mu_i} y_i^{\delta_j}}{1-x_i y_i} dx_i \right) \longrightarrow \qquad \text{where } \mu = \nu + \delta$$

$$\int_{[0,1]^n} S_\lambda(\underline{x})\, S_\nu(1-\underline{x})\, \Delta(\underline{x})^2 d\underline{x} = \prod_i (\lambda_i+n-i)!\,(\nu_i+n-i)! \det\left(\frac{1}{(\lambda_i+\nu_j+2n-i-j+1)!} \right)$$

has a combinatorial interpretation.

$$\int_{[0,1]^n} S_\lambda(1-\underline{x})\, \Delta(x)^2 d\underline{x} = \int S_\lambda(\underline{x})\, \Delta(\underline{x})^2 dx$$

$$\frac{a_{\lambda+\delta}(1-x)}{a_\delta}$$

$$S_\lambda(1-\underline{x}) = \sum C_{\lambda\nu} S_\nu(x)$$

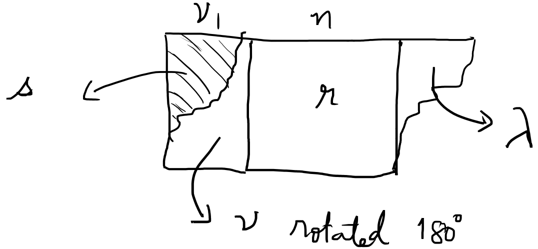
$$\sum_\nu \boxed{C_{\lambda\nu}} I_\nu = I_\lambda$$

Reordering the columns $j \leftrightarrow n+1-j$ we get

$$\det\left(\frac{1}{(r_i - s_j + j - i)!}\right) = \frac{|SYT(r/s)|}{|r/s|!}$$

where $r_i = \lambda_i + \nu_1 + n$

$s_j = \nu_1 - \nu_{n+1-j}$



$$|SYT(r/s)| = \int s_\lambda(x)\, \underbrace{\boxed{s_\nu(1-x)}}_{\sum c_{\nu\mu} s_\mu(x)}\, \Delta(x)^2 = \sum c_{\nu\mu} \int s_\lambda s_\mu \Delta(x)^2$$

HOOK LENGTH FORMULAS

Naruse

Morales-Pak-Panova

$$|SYT(\lambda/\mu)| = \sum_{D \in \mathcal{E}(\lambda/\mu)} \frac{1}{\prod_{s \in D} h(s)}$$

NUMBER THEORETIC CONNECTION

Apéry 78
Beukers 79

$$\iint \frac{x^i y^j}{1-xy}\,dx\,dy = \underbrace{A}_{\text{integer}}\zeta(2) - \underbrace{B}_{\text{rational no}}$$

$$\det\left(\int_0^1\!\!\int_0^1 \frac{x^i y^j}{1-xy}\,dx\,dy \right) = f(\zeta(2)) \in \mathbb{Q}(\zeta(2))$$

(with known denominator)

$$= \frac{1}{n!^2} \iint_{[0,1]^n\,[0,1]^n} \frac{\Delta(\underline{x})^2\,\Delta(\underline{y})^2}{\prod_{i,j}(1-x_i y_j)}\,d\underline{x}\,d\underline{y} = \text{"nice" product!!}$$

because (a) can be thought of as special value of a hypergeometric fn with Jack Poly argument

Datsenko-Fateev (b)

$$\int_{[0,1]^n} \int_{[1,\infty]^n} \prod x_i^{\alpha}(1-x_i)^{\beta} \prod y_i^{\alpha'}(1-y_i)^{\beta'} |\Delta(\underline{x})|^{2\gamma} |\Delta(\underline{y})|^{2\gamma'} \prod_{i,j} |x_i - y_j|^{-1} \, d\underline{x} \, d\underline{y}$$

= "nice" product

when $\left\{ \begin{array}{l} \alpha, \alpha' \\ \beta, \beta' \\ \gamma, \gamma' \end{array} \right\}$ are related

(not true for us)

Just the expansion as a sum over k gives the

Estimate $\quad 0 \neq f(\zeta(2)) \lesssim e^{-2n^2 + O(n \log n)}$

and known denominator

$\Rightarrow \quad \underline{\zeta(2) \notin \mathbb{Q}}$