Category $\mathcal{O}$

Volodymyr Mazorchuk

(Uppsala University)

Bigrassmannian permutations and Verma modules

Coauthors

Joint work with:

Hankyung Ko and Rafael Mrđen

Both: postdocs at Uppsala University

Category $\mathcal{O}$

$\mathfrak{g} = \mathfrak{sl}_n$ – special linear Lie algebra over $\mathbb{C}$

$\mathfrak{sl}_n = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ — standard triangular decomposition.

$W \cong S_n$ — the Weyl group.

$\mathcal{O}$ — the associated BGG category $\mathcal{O}$ consisting of all

- finitely generated;
- $\mathfrak{h}$-diagonalizable;
- $U(\mathfrak{n}_+)$-locally finite modules.

$\mathcal{O}_0$ — the principal blocks of $\mathcal{O}$, i.e. the indecomposable direct summand containing the trivial (one-dimensional) $\mathfrak{g}$-module.

Fact. The category $\mathcal{O}_0$ is equivalent to A-mod, for a unique, up to isomorphism, basic, finite dimensional, associative algebra A.

Elementary combinatorics via Bruhat order

Fact. Simple objects in $\mathcal{O}_0$ are indexed by elements in W via the convention $W \ni w \mapsto L_w$.

Note. L_w is the simple highest weight module with highest weight $w \cdot 0$.

Δ_w — the Verma module covering L_w.

Note. Δ_w is the universal highest weight module with highest weight $w \cdot 0$.

$\prec$ — Bruhat order on W.

BGG structure Theorem. For $x, y \in W$, the following are equivalent:

- $[\Delta_x : L_y] > 0$.
- $x \prec y$.
- $\Delta_y \subset \Delta_x$.

Additionally. $\dim \mathrm{Hom}_{\mathfrak{g}}(\Delta_y, \Delta_x) \leq 1$ and any non-zero hom is injective.

Original motivation

w_0 — the longest element in W.

Question from S. Orlik and M. Strauch: Do there exist $x, y \in W$ such that $x \prec y$ and $\mathrm{Ext}^1_{\mathcal{O}}(\Delta_y/\Delta_{w_0}, \Delta_x) \neq 0$?

Observation: "No" in ranks up to 2.

Example: In general, "yes", e.g., for S_4 corresponding to $r — s — t$, we have $\mathrm{Ext}^1_{\mathcal{O}}(\Delta_{sw_0}/\Delta_{w_0}, \Delta_s) \neq 0$ and $s \prec sw_0$.

Remark. In this example $\Delta_{sw_0}/\Delta_{w_0} = L_{sw_0}$.

New question: Describe $\mathrm{Ext}^1_{\mathcal{O}}(L_y, \Delta_x)$?

$\overline{\ell}(w)$ — the number of different simple refl. in a reduced word for w.

Known case. [M., 2007] $\dim \mathrm{Ext}^1_{\mathcal{O}}(L_{w_0}, \Delta_x) = \overline{\ell}(w_0 x)$.

Fact: If $y \neq w_0$, then any non-split M such that $0 \to \Delta_x \to M \to L_y \to 0$ is a submodule of Δ_e.

Real question: Describe the socle of Δ_e/Δ_x.

Hecke algebra combinatorics

Recall: the Hecke algebra $\mathbf{H} = \mathbf{H}(W, S)$ is the associative algebra over $\mathbb{A} := \mathbb{Z}[v, v^{-1}]$ generated by H_s, where $s \in S$, subject to the relations:

- $(H_s + v)(H_s - v^{-1}) = 0$;
- the braid relations.

Std basis: $H_w = H_{s_1} H_{s_2} \cdots H_{s_k}$, for $w \in W$, $w = s_1 s_2 \cdots s_k$ reduced.

Bar involution: $\overline{\cdot} : \mathbf{H} \to \mathbf{H}$ given by $\overline{v} = v^{-1}$ and $\overline{H_s} = H_s^{-1}$, for $s \in S$.

Kazhdan-Lusztig basis: for $w \in W$, there is a unique $\underline{H}_w \in \mathbf{H}$ such that

- $\overline{\underline{H}_w} = \underline{H}_w$.
- $\underline{H}_w \in H_w + \sum_{x \in W} v\mathbb{Z}[v] H_x$.

Kazhdan-Lusztig polynomials: $h_{x,w} \in \mathbb{Z}[v]$ s.t. $\underline{H}_w = \sum_{x \in W} h_{x,w} H_x$.

Kazhdan-Lusztig conjecture. [Beilinson-Bernstein, Brylinski-Kashiwara, 1981] The iso $[{}^{\mathbb{Z}}\mathcal{O}_0] \cong \mathbf{H}$ sending $[\Delta_w]$ to H_w, sends $[P_w]$ to $\underline{H}_w$.

Kazhdan-Lusztig cells

Consequence 1: All coefficients of KL-polynomials are non-negative integers.

Consequence 2: All structure constants $\gamma_{x,y}^{w} \in \mathbb{Z}[v, v^{-1}]$ with respect to the KL-basis have non-negative coefficients, i.e.

$$\underline{H}_x \underline{H}_y = \sum_{w} \gamma_{x,y}^{w} \underline{H}_w, \qquad \text{with} \qquad \gamma_{x,y}^{w} \in \mathbb{Z}_{\geq 0}[v, v^{-1}].$$

Kazhdan-Lusztig preorders on W:

- $y \leq_L w$ provided that there is x such that $\gamma_{x,y}^{w} \neq 0$;
- $x \leq_R w$ provided that there is y such that $\gamma_{x,y}^{w} \neq 0$;
- $\leq_J$ — the minimal preorder containing both $\leq_L$ and $\leq_R$.

Kazhdan-Lusztig cells:

- left cells: the equivalence classes for $\leq_L$;
- right cells: the equivalence classes for $\leq_R$;
- two-sided cells: the equivalence classes for $\leq_J$.

Cells in type A via Robinson-Schensted

Robinson-Schensted map: the bijection $\mathbf{RS} : S_n \to \coprod_{\lambda \vdash n} \mathrm{SYT}_\lambda \times \mathrm{SYT}_\lambda$.

Notation: $\mathbf{RS}(w) =: (\mathbf{p}_w, \mathbf{q}_w)$.

Theorem. [KL, 1979] For $x, y \in S_n$, we have:

- $x \sim_L y$ if and only if $\mathbf{q}_x = \mathbf{q}_y$;
- $x \sim_R y$: if and only if $\mathbf{p}_x = \mathbf{p}_y$;
- $x \sim_J y$: if and only if $\mathrm{shape}(\mathbf{RS}(x)) = \mathrm{shape}(\mathbf{RS}(y))$.

The last cell: w_0 is the maximum element for $\leq_L$, $\leq_R$, $\leq_J$.

The penultimate cell J: all $w \in S_n$ with $\mathrm{shape}(\mathbf{RS}(w)) = (2, 1^{n-2})$.

Easy: **J** has $(n-1)^2$ elements and both the left and the right cells in **J** are naturally indexed by simple reflections.

Small rank examples:

Rank 1: Dynkin diagram s, we have $\mathbf{J} = \{e\}$.

Rank 2: Dynkin diagram s — t , we have

$\mathbf{J}$:

s	ts
ts	t

Rank 3: Dynkin diagram r — s — t , we have

$\mathbf{J}$:

sts	$stsr$	$strsr$
$rsts$	$rstsr$	$trsr$
$rstrs$	$rstr$	rsr

The socle of the cokernel of an inclusion of Verma modules

Principal observation 1: If L_x appears in the socle of Δ_e/Δ_w, then $x \in \mathbf{J}$.

About proof: Uses derived category and (derived) twisting functors.

Recall:

- left ascent set: $\mathbf{LA}(w) := \{s : \ell(sw) > \ell(w)\}$.
- right ascent set: $\mathbf{RA}(w) := \{s : \ell(ws) > \ell(w)\}$.

Note: left/right cells in $\mathbf{J}$ are indexed by the right/left (singleton!) ascent sets.

Next observation:

- If $sx < x$, then the socle of Δ_e/Δ_x contains some L_y such that $sx > y$.
- If $xs < x$, then the socle of Δ_e/Δ_x contains some L_y such that $xs > y$.

Bigrassmannian permutations

Recall:

- left descent set: $\mathbf{LD}(w) := \{s : \ell(sw) < \ell(w)\}$.
- right descent set: $\mathbf{RD}(w) := \{s : \ell(ws) < \ell(w)\}$.

Definition. $w \in W$ is called bigrassmannian provided that both $\mathbf{LD}(w)$ and $\mathbf{RD}(w)$ are singletons.

Corollary. If $w \in W$ is such that Δ_e/Δ_w has simple socle, then w is bigrassmannian.

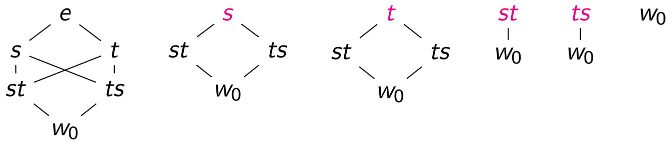
Note. Bigrassmannian permutations in S_n are exactly the join-irreducible elements w.r.t. the Bruhat order.

$\mathfrak{sl}_3$-example

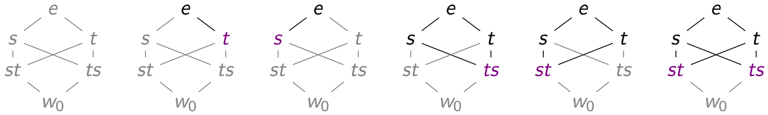
$\mathfrak{g} = \mathfrak{sl}_3$

$W = S_3 = \{e, s, t, st, ts, w_0 = sts = tst\}$, with colored bigrassmannians.

Loewy structure of the Verma modules:



Socle of the cokernel of an inclusion into Δ_e:



Kazhdan-Lusztig polynomials for the penultimate cell

Computations for $h_{e,w}$, where $w \in \mathbf{J}$, in ranks 1, 2, 3 and 4:

v^0

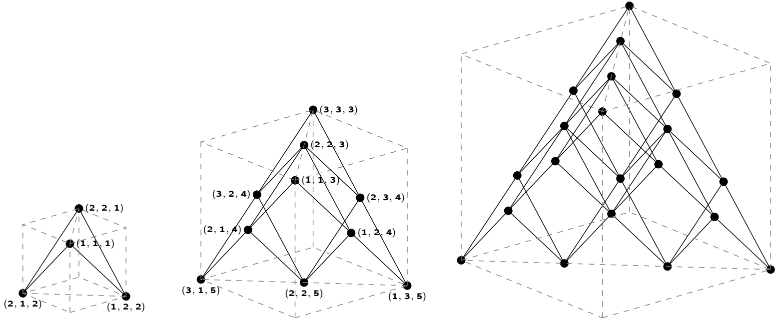
v	v^2
v^2	v

v^3	v^4	v^5
v^4	$v^5 + v^3$	v^4
v^5	v^4	v^3

v^6	v^7	v^8	v^9
v^7	$v^8 + v^6$	$v^9 + v^7$	v^8
v^8	$v^9 + v^7$	$v^8 + v^6$	v^7
v^9	v^8	v^7	v^6

Length of w_0: 1, 3, 6 and 10.

Graded picture: (solid lines represent "inclusions" inside Δ_e)

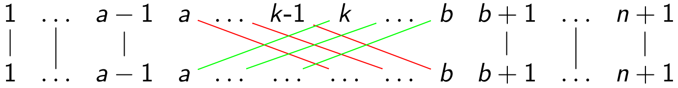


Enumerations

Principal observation 2: The patter of the previous slide extends to all ranks n, in particular, $\sum_{w \in \mathbf{J}} h_{e,w}(1)$ is the tetrahedral number $\frac{n(n+1)(n+2)}{6}$.

Principal observation 3: It is known that the number of bigrassmannian permutations in rank n is the tetrahedral number $\frac{n(n+1)(n+2)}{6}$.

Construction of bigrassmannian permutations: $\beta_{a,k,b}$, where $1 \leq a < k \leq b \leq n+1$:



The main result, part I: simple socle

Terminology: Simples of the form L_w, where $w \in \mathbf{J}$, are called penultimate.

Main Theorem, Part I. Let $w \in S_n$.

- Δ_e/Δ_w has simple socle if and only if w is bigrassmannian.
- The correspondence $w \mapsto \mathrm{soc}(\Delta_e/\Delta_w)$ is a bijection from the set of all bigrassmannian elements in S_n to the set of all penultimate subquotients of Δ_e.

Remark. Bigrassmannian permutations with fixed left and right descents form a chain w.r.t. the Bruhat order. They correspond to the same simple module L_w with $w \in \mathbf{J}$ (the descent of a bigrassmannian is the ascent of w) which appears, as a subquotient of Δ_e in different degrees.

The main result, part II: general case

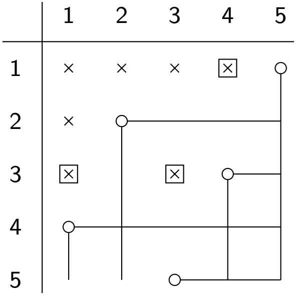
Main Theorem, Part II. Let $w \in S_n$. The socle of Δ_e/Δ_w corresponds, under the bijection from Part I, to the Bruhat maximal bigrassmannian elements in the set of all elements that are Bruhat smaller than or equal to w.

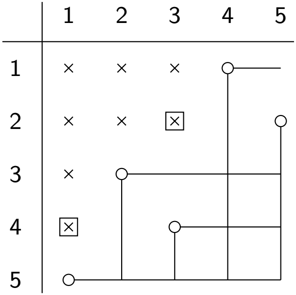
Application. Let $x, w \in S_n$ and $x \neq w_0$. Then $\dim \mathrm{Ext}^1_{\mathcal{O}}(L_x, \Delta_w) \leq 1$, moreover, $\dim \mathrm{Ext}^1_{\mathcal{O}}(L_x, \Delta_w) = 1$ if and only if x corresponds, under the bijection from Part I, to a Bruhat maximal bigrassmannian element in the set of all elements that are Bruhat smaller than or equal to w.

Essential set of a permutation

Definition via example:

Essential sets for $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}$:





Socles via essential sets

Notation: For $w \in S_n$:

- **BM**(w) — the set of all Bruhat maximal bigrassmannian elements in the set of all elements that are Bruhat smaller than or equal to w.

Theorem. [Kobayashi, 2010] For $w \in S_n$, the map $x \mapsto (\mathbf{RD}(x), \mathbf{LD}(x))$ is a bijection between $\mathbf{BM}(w)$ and the essential set of w.

Ungraded socle. For $w \in S_n$, the simple constituents of the (ungraded) socle of Δ_e/Δ_w are in bijection with the essential set of w.

Rank function

Fulton's rank function: for $w \in S_n$, is defined via

$$r_w(i,j) := |\{k \leq i \colon w(k) \leq j\}|, \qquad 1 \leq i,j \leq n.$$

The corank function: for $w \in S_n$, is defined via

$$t_w(i,j) := \min\{i,j\} - r_w(i,j).$$

Combinatorial description: $r_w(i,j)$ is the number of $\circ$ to the north-west of (i,j). $t_w(i,j)$ is the number of $\circ$ to the north-east of (i,j), if $i \leq j$, otherwise, to the south-west of (i,j).

Example: $t_w(i,j)$ for $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}$:

	1	2	3	4	5
1	1	1	1	$\boxed{1}$	0
2	1	1	1	1	0
3	$\boxed{1}$	1	$\boxed{2}$	1	0
4	0	0	1	1	0
5	0	0	0	0	0

	1	2	3	4	5
1	1	1	1	0	0
2	1	2	$\boxed{2}$	1	0
3	1	1	2	1	0
4	$\boxed{1}$	1	1	1	0
5	0	0	0	0	0

Graded shifts via rank function

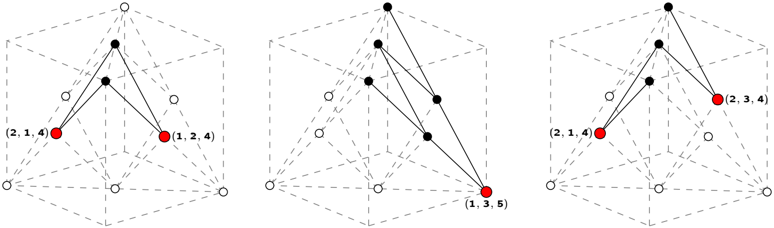
Notation: $w_{i,j}$ — the unique element in $\mathbf{J}$ with left ascent $(i, i+1)$ and right ascent $(j, j+1)$.

Graded Socle. For $w \in S_n$, the socle of $\Delta_e/(\Delta_w\langle -\ell(w)\rangle)$ is

$$\bigoplus_{(i,j)\in \mathrm{Ess}(w)} L_{w_{j,i}} \left\langle -\frac{(n-1)(n-2)}{2} - |i-j| - 2(t_w(i,j)-1) \right\rangle .$$

Examples

Socles of the quotients Δ_e/Δ_w for $n = 4$, for $w = s_1s_2s_1, s_1s_2s_3, s_2s_3s_1$:



THANK YOU!!!