The mysterious Kronecker coefficients

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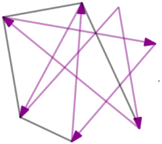
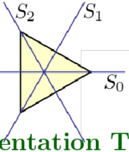
IMSc 2020

Complexity Theory

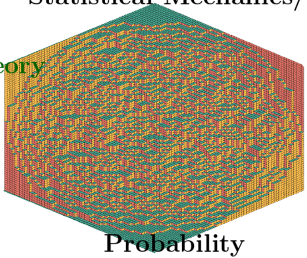
P vs NP





Representation Theory

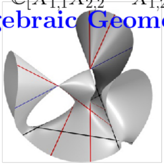
Statistical Mechanics/

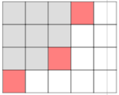


Probability

$\mathbb{C}[X_{1,1}X_{2,2} - X_{1,2}X_{2,1}]$

Algebraic Geometry





Algebraic Combinatorics

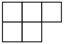
$$s_{(2,2)}(x_1,x_2,x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2$$

1 1	1 1	2 2	1 1	1 2	1 2
2 2	3 3	3 3	2 3	2 3	3 3

Intro

Permutations
$\pi = 42351$
$\pi : [1 \dots n] \to [1 \dots n]$

Partitions $\lambda = (3, 2)$



Symmetric Group S_n

Irreducible S_n modules
$\mathbb{S}_\lambda,\ \lambda \vdash n$

Standard **Y**oung **T**ableaux

1	3	4
2	5	

Irreducible GL_N modules V_λ

Semi-**S**tandard **Y**oung **T**ableaux

1	1	3
2	3	

Multiplicities

Decomposition of tensor product into GL_N–irreducibles:

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

Multiplicities $c^\nu_{\lambda\mu}$ – **Littlewood-Richardson coefficients**.

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Theorem (Littlewood-Richardson, 1934 [1])

The coefficient $c^{\nu}_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

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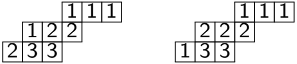

(LR tableaux of shape $(6,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

Corollary: $\textsc{ComputeLR}$ is in $\#\mathrm{P}$.

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S_n **tensor products decomposition** (diagonal action):

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus(\dots\dots\dots)}$$

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$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

E.g.: $\mathbb{S}_{(2,1)} \otimes \mathbb{S}_{(2,1)} = \mathbb{S}_{(3)} \oplus \mathbb{S}_{(2,1)} \oplus \mathbb{S}_{(1,1,1)}$ and so $g((2,1),(2,1),\nu) = 1$ for $\nu = (3),(2,1),(1,1,1)$.

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The problems

Problem (Murnaghan 1938, Stanley)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.

Conjecture

ComputeKRON *is in* #P .

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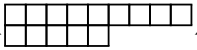
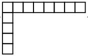
Results:
$\#\mathrm{P}$ formulas for $g(\lambda, \mu, \nu)$, when:

- $\nu = (n-k, k)$ () and $\lambda_1 \geq 2k-1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n-k, 1^k)$ (), [*Blasiak 2012, Blasiak-Liu 2014*]
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Ikenmeyer-Panova].

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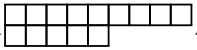
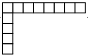
- $\nu = (n-k, k)$ () and $\lambda_1 \geq 2k-1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n-k, 1^k)$ (), [*Blasiak 2012, Blasiak-Liu 2014*]
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ComputeKron is #P–hard (and is in GapP).

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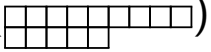
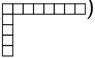
- $\nu = (n-k, k)$ () and $\lambda_1 \geq 2k-1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n-k, 1^k)$ (), [*Blasiak 2012, Blasiak-Liu 2014*]
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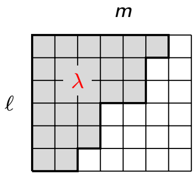
[Ikenmeyer-Panova, Adv.Math'16]:
$g((N-ab, a^b), (N-ab, a^b), (N-|\gamma|, \gamma)) > 0$ for almost all γ, a, b (with restrictions), related to [no occurrence-obstructions in] Geometric Complexity Theory.

Partitions inside a rectangle

$p_n := \#\{\lambda \vdash n\}$ $$\sum_{n=0}^{\infty} p_n q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}$$

$p_n(\ell, m) := \#\{\lambda \vdash n;\ \lambda \subset (m^\ell)\}$

$$\sum_{k\geq 0} p_n(\ell, m) q^n \ = \ \binom{m+\ell}{m}_q$$



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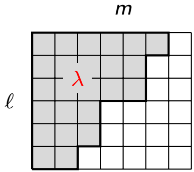

Hardy-Ramanujan:

$$p_n := \#\{\lambda \vdash n\} \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) ,$$

Trivial:

$$p_1 < p_2 < \cdots < p_n < p_{n+1} < \cdots$$

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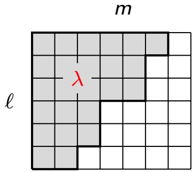

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Theorem (Sylvester 1878, Cayley's conjecture 1856)

The sequence $p_0(\ell, m), \ldots, p_{\ell m}(\ell, m)$ is unimodal, i.e.

$$p_0(\ell, m) \leq p_1(\ell, m) \leq \ \ldots \leq p_{\lfloor \ell m/2 \rfloor}(\ell, m) = p_{\lceil \ell m/2 \rceil}(\ell, m) \geq \ \cdots \geq \ p_{\ell m}(\ell, m)$$

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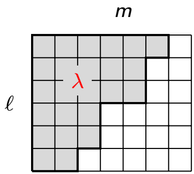

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"*I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. [...] I accomplished with scarcely an effort a task which I had believed lay outside the range of human power.*"

J.J. Sylvester, 1878.

Partitions inside a rectangle

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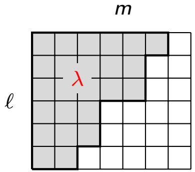

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Proofs:
Stanley [1. Hard Lefshetz, 2. linear algebra paradigm],
Kathy O'Hara [1991, combinatorial]
No known (symmetric) chain decomposition of the Partition Lattice

Partitions inside a rectangle

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Theorem (Pak-P (2014))

For all $m \geq \ell \geq 8$ *and* $2 \leq k \leq \ell m/2$, *we have:*

$$g(m^\ell, m^\ell, (m\ell - k, k)) = p_k(\ell, m) - p_{k-1}(\ell, m) > A \frac{2^{\sqrt{s}}}{s^{9/4}},$$

where $s = \min\{2k, \ell^2\}$, *and* $A > 0.004$ *is a universal constant.*

Lower bounds

Theorem[Pak-P (2014)] Let $\mu = \mu'$ be a self-conjugate partition and let $\widehat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, \ldots) \vdash n$. Let $\chi^\lambda[\pi]$ be the character of $\mathbb{S}^\lambda$ on the permutations of cycle type π. Then:

$$g(\lambda, \mu, \mu) \geq \left|\chi^\lambda[\widehat{\mu}]\right|, \quad \text{for every} \quad \lambda \vdash n,$$

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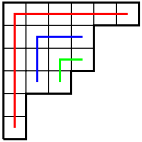
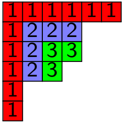
Example:

$\mu =$ 

$\lambda = \mu =$ 

$|\chi^\mu[(\widehat{\mu})]| = 1$

Corollary:[Bessenrodt–Behns] For every μ, s.t. $\mu = \mu'$ we have $\langle \chi^\mu \otimes \chi^\mu, \chi^\mu \rangle \neq 0$.

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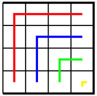
$\mu = (n^n)$:

$\mu =$ 

$$\lambda = (n^2 - k, k)$$

$$|\chi^\lambda[(1,3,5,\ldots)]| = |b_k(n) - b_{k-1}(n)|$$

$$\prod_{i=1}^{n} \left(1 + q^{2i-1}\right) = \sum_{k=0}^{n^2} b_k(n)\, q^k .$$

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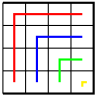
$\mu=(n^n)$:

$\mu=$ 

$$\lambda=(n^2-k,k)$$

$$|\chi^\lambda[(1,3,5,\ldots)]|=|b_k(n)-b_{k-1}(n)|$$

$$\prod_{i=1}^{n}\left(1+q^{2i-1}\right)\;=\;\sum_{k=0}^{n^2}\,b_k(n)\,q^k\,.$$

Lemma[Pak-P, Vallejo, etc]:

$$g((m^\ell),\mu,\mu)=p_k(m,\ell)-p_{k-1}(m,\ell)$$

Corollary[Stanley, 1982; Pak-P, 2016] The following polynomial in q is symmetric and unimodal

$$\binom{2n}{n}_q-\prod_{i=1}^{n}\left(1+q^{2i-1}\right).$$

Effective bounds

$$\prod_{i=1}^{n} \left(1+q^{2i-1}\right) \;=\; \sum_{k=0}^{n^2} b_k(n)\, q^k$$

1. **Theorem**[Pak-P, cor to Almkvist]

$$b_k(n) \,-\, b_{k-1}(n) \,\geq\, C\, 2^{\sqrt{2\,k}} \frac{1}{(2\,k)^{9/4}}$$

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3. **Semigroup property** for Kronecker coefficients: if $g(\lambda,\mu,\nu), g(\alpha,\beta,\gamma) > 0$, then: $g(\lambda+\alpha, \mu+\beta, \nu+\gamma) \;\geq\; \max\{g(\lambda,\mu,\nu),\, g(\alpha,\beta,\gamma)\}$.

$$\Longrightarrow g(m^\ell, m^\ell, (m\ell-k,k)) \geq g(\ell^\ell, \ell^\ell, (\ell^2-s,s)), \;\text{ where } s=\min\{k, \ell^2/2\}$$

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$\Longrightarrow$ **Theorem**[Pak-P (2016)]

$g(m^\ell, m^\ell, (m\ell - k, k)) = p_k(\ell, m) \;-\; p_{k-1}(\ell, m) \;>\; 0.004\, \frac{2^{\sqrt{s}}}{s^{9/4}}$, where $s = \min\{2k, \ell^2\}$,

The difference $p_{n+1}(m,\ell) - p_n(m,\ell)$

Theorem (Melczer-P-Pemantle)

Given m, ℓ and n, let $A := \ell/m$ and $B := n/m^2$ and define c, d as solutions of a given system of integral equations. Suppose m, ℓ and n go to infinity so that (A, B) remains in a compact subset of $\{(x, y) : x > 2y > 0\}$. Then

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Corollaries:
Sylvester's unimodality, [Pak-P] bound on the Kronecker to exact asymtptotics etc.

Corollary

The Kronecker coefficient of $S_{m\ell}$ for the (rectangle, rectangle, two-row) case is asymptotically given by

$$g((m^\ell),(m^\ell),(m\ell-n-1,n+1)) = p_{n+1}(\ell,m) - p_n(\ell,m) \sim \frac{d}{m} p_n(\ell,m)$$
$$\sim \frac{de^{m\left[cA+2dB-\log(1-e^{-c-d})\right]}}{2\pi m^3\sqrt{D}}.$$

Beyond explicit formulas: LR and Kronecker

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Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ of $GL(V)$:

$$V_\mu \otimes V_\nu = \bigoplus_\lambda V_\lambda^{c^{\lambda}_{\mu\nu}}$$

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Theorem [Stanley]

$$\max_{\lambda\vdash n}\max_{\mu\vdash n}\max_{\nu\vdash n} \; g(\lambda,\mu,\nu) \;=\; \sqrt{n!}\, e^{-O(\sqrt{n})}\,,$$

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Question: [Stanley] For which λ,μ,ν are these maxima achieved?

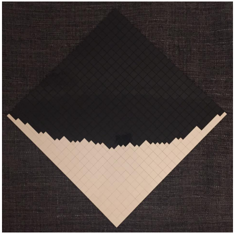
Maximal $\mathbf{D}(n) := \max_{\lambda \vdash n} f^\lambda$

Theorem [Vershik-Kerov, McKay]: The maximal dimension f^λ for $\lambda \vdash n$, denoted $\mathbf{D}(n)$ satisfies

$$\sqrt{n!}\, e^{-c_1 \sqrt{n}(1+o(1))} \le \mathbf{D}(n) \le \sqrt{n!}\, e^{-c_2 \sqrt{n}(1+o(1))}$$

for some $c_1 > c_2 > 0$. Moreover $\lambda^{(n)}$ satisfies $f^{\lambda^{(n)}} \ge \sqrt{n!} e^{-a\sqrt{n}}$ for some a iff it has the:

Vershik–Kerov–Logan–Shepp (VKLS) *shape* if .



$$\left| \frac{1}{\sqrt{n}} \left[\lambda^{(n)}\right] - \varphi \right| < C\, n^{1/6} \quad \text{for some} \quad C > 0,$$

where $\varphi : [0,2] \to [0,2]$ is the 135° rotation of $(x, y(x))$:

$$y(x) := \frac{2}{\pi} \left(x \arcsin \frac{x}{\sqrt{2}} + \sqrt{2 - x^2} \right), \ -\sqrt{2} \le x \le \sqrt{2}$$

$$\mathbf{D}(n) \sim e^{\frac{1}{2} n \log(n) - \frac{1}{2} n + O(\sqrt{n})}$$

Largest Kroneckers

Identities:

$$\sum_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu)^2 = \sum_{\alpha\vdash n} z_\alpha \geq z_{1^n} = n!,$$

where $z_\alpha = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ when $\alpha = (1^{m_1} 2^{m_2} \ldots)$,
so $\max g(\lambda,\mu,\nu) \geq \sqrt{n!/p(n)^3} = e^{\frac{1}{2} n \log(n) - \frac{1}{2} n - O(\sqrt{n})}$.

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Theorem (Pak-Panova-Yeliussizov)

Let $\{\lambda^{(n)} \vdash n\}$, $\{\mu^{(n)} \vdash n\}$, $\{\nu^{(n)} \vdash n\}$ be three partitions sequences, such that

$$(*) \qquad g\big(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}\big) = \sqrt{n!}\, e^{-O(\sqrt{n})}.$$

Then all three partition sequences are Plancherel (i.e. VKLS shape). Conversely, for every two Plancherel partition sequences $\{\lambda^{(n)} \vdash n\}$ and $\{\mu^{(n)} \vdash n\}$, there exists a Plancherel partition sequence $\{\nu^{(n)} \vdash n\}$, s.t. (*) holds.

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Theorem[PPY]: Let $\mu, \nu \vdash n$. Suppose $f^\mu, f^\nu \geq \mathbf{D}(n)/a$ for some $a \geq 1$. Then there exist $\lambda \vdash n$, s.t.

$$f^\lambda \,\geq\, \frac{\mathbf{D}(n)}{a\sqrt{p(n)}} \quad \text{and} \quad g(\lambda,\mu,\nu) \,\geq\, \frac{\mathbf{D}(n)}{a^2\, p(n)}\,.$$

Small number of rows

Theorem (Pak-P'20)

Let $\lambda, \mu, \nu \vdash n$ such that $\ell(\lambda) = \ell$, $\ell(\mu) = m$, and $\ell(\nu) = r$. Then:

$$g(\lambda, \mu, \nu) \le \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r}.$$

Corollary

Let $\lambda = (\ell^2)^\ell$, where $\ell = \sqrt[3]{n}$, then

$$g(\lambda, \lambda, \lambda) \le 4^n.$$

(Compare to $f^\lambda = \exp[\frac{1}{3} n \log(n) + O(n^{2/3})]$ and $\mathbf{D}(n) = \exp[\frac{1}{2} n \log n + O(\sqrt{n})$)

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Proof via contingency arrays:

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[Barvinok]: The number of 3d contingency tables with marginals (α, β, γ) is

$$\leq \exp\left(\max_{Z \in P(\alpha,\beta,\gamma)} \sum_{i,j,k} (Z_{ijk} + 1) \log(Z_{ijk} + 1) - Z_{ijk} \log(Z_{ijk})\right)$$

The elusive lower bound

$$g(\lambda,\lambda,\lambda) \geq 1 \quad \text{for } \lambda = \lambda'$$

(from $g(\lambda,\lambda,\lambda) \geq |\chi^\lambda(2\lambda_1 - 1, 2\lambda_2 - 3, \ldots)| = 1$)

The elusive lower bound

$Pyr(\alpha, \beta, \gamma) := \#$ of pyramids (solid partitions, plane partitions) with marginals α, β, γ.

Theorem [Manivel, Vallejo]

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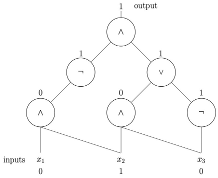
Conjecture [Pak-P]:

$$\sum_{\lambda\vdash n, \lambda=\lambda'} g(\lambda,\lambda,\lambda) = \exp\left(\frac{1}{2} n\log n + O(n)\right).$$

Computational Complexity

Input: I, size$(I) = n$ (bits)



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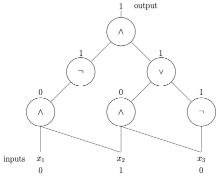

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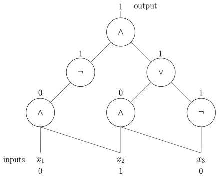

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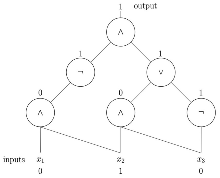

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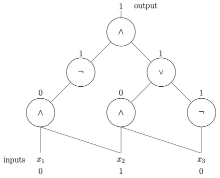

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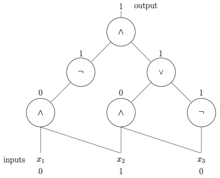

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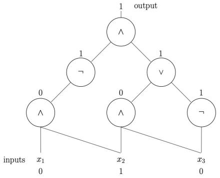

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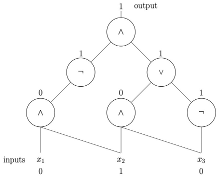

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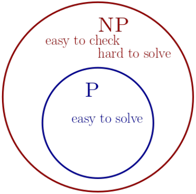

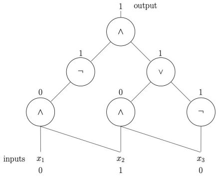

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The P vs NP Millennium Problem:
Is P = NP ?

Complexity of Computing Multiplicities

Dimension of irreducible representations:

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$f^{(3,2)} = \frac{5!}{4.3.2.1.1} = 5$

$$\begin{array}{ccc}1&2&3\\4&5&\end{array} \quad \begin{array}{ccc}1&2&4\\3&5&\end{array} \quad \begin{array}{ccc}1&2&5\\3&4&\end{array} \quad \begin{array}{ccc}1&3&4\\2&5&\end{array} \quad \begin{array}{ccc}1&3&5\\2&4&\end{array}$$

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Dimension of irreducible representations: $\in$ P

$$\dim \mathbb{S}_\lambda = f^\lambda = \#SYT(\lambda) = \frac{n!}{\prod_{(i,j)\in\lambda}(\lambda_i - i + \lambda'_j - j + 1)}.$$

$f^{(3,2)} = \frac{5!}{4.3.2.1.1} = 5$

1	2	3
4	5	

1	2	4
3	5	

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Theorem (Narayanan'06)

ComputeKostka *is #P-complete. (input – binary)*

Theorem (Pak-Panova'20+)

ComputeKostka *is strongly #P-complete. (input – unary)*

Complexity of Computing LR

Littlewood-Richardson coefficients: $c^{\lambda}_{\mu\nu} = \mathrm{mult}_{\lambda} V_{\mu} \otimes V_{\nu} = \#LR - tableaux$

$c^{(6,4,3)}_{(4,3,2)(3,1)} = 2$:

```
        1 1 1          1 1 1
    1 2 2          2 2 2
  2 3 3          1 3 3
```

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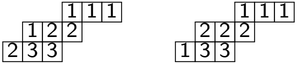

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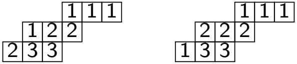

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Theorem (Pak-Panova'20+)

ComputeLR *is strongly #P-hard.*

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$\chi^\lambda(\alpha) =$ **character** of $\mathbb{S}_\lambda$ at permutation of cycle type α.
CharPos:
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ComputeKron *is in* GapP *(= F − G, where F, G ∈ #P).*

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Conjecture

ComputeKron *is in* #P .

The reduced (but not simpler!) Kronecker coefficients

$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g\big(\alpha[n],\beta[n],\gamma[n]\big), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n \geq |\alpha|+\alpha_1,$$

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Conjecture (Kirillov, Klyachko)

The reduced Kronecker coefficients satisfy the saturation property:

$$\overline{g}(N\alpha,N\beta,N\gamma) > 0 \quad \textit{for some } N \geq 1 \quad \Longrightarrow \quad \overline{g}(\alpha,\beta,\gamma) > 0\,.$$

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Theorem (Pak-P, '20)

For all $k\geq 3$*, the triple of partitions* $\big(1^{k^2-1},1^{k^2-1},k^{k-1}\big)$ *is a counterexample to the Conjecture. For every partition* γ *s.t.* $\gamma_2\geq 3$*, there are infinitely many pairs* $(a,b)\in\mathbb{N}^2$ *s.t.* (a^b,a^b,γ) *is a counterexample.*

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Let a, b be s.t. $b \geq \max\{3\ell(\gamma)^{3/2}, |\gamma|/(3\sqrt{2d(\gamma[n])}-6)\}$ and $|\gamma|/(6b) \leq a \leq \sqrt{d(\gamma[n])/2}$. By [Ikenmeyer-P, Adv.Math'16] for $N \geq 3\ell^2/a$:

$$\overline{g}(N\alpha,N\alpha,\gamma) \geq g((Na)^{b+1},(Na)^{b+1},N\gamma[n]) > 0.$$

Bounds

Theorem

We have:

$$\max_{a+b+c\leq 3n} \max_{\alpha\vdash a} \max_{\beta\vdash b} \max_{\gamma\vdash c} \overline{g}(\alpha,\beta,\gamma) = \sqrt{n!}\ e^{O(n)}.$$

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[Bowman-DeVisscher-Orellana]:

$$\overline{g}(\alpha,\beta,\gamma) = \sum_{m=0}^{\lfloor k/2\rfloor} \sum_{\pi\vdash q+m-b} \sum_{\rho\vdash q+m-a} \sum_{\sigma\vdash m} \sum_{\lambda,\mu,\nu\vdash k-2m} c^{\alpha}_{\nu\pi\rho}\, c^{\beta}_{\mu\pi\sigma}\, c^{\gamma}_{\lambda\rho\sigma}\, g(\lambda,\mu,\nu),$$

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Lower bound: $\overline{g}(\alpha,\beta,\gamma) \geq g(\alpha,\beta,\gamma)$
$+$
[Stanley:] $\max g(\alpha,\beta,\gamma) \geq \sqrt{n!}e^{O(n)}$

Complexity

Theorem

Computing the reduced Kronecker coefficients $\overline{g}(\alpha, \beta, \gamma)$ *is strongly #P-hard.*

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$$g(\lambda, \mu, \nu) = \sum_{i=1}^{\ell(\mu)\,\ell(\nu)} (-1)^i \, \overline{g}\left(\lambda^i, \mu^1, \nu^1\right),$$

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[Ikenmeyer-Mulmuley-Walter]: $\textsc{ComputeKron}$ is strongly #P -hard.

Thank you!