The mysterious Kronecker coefficients

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IMSc 2020
Complexity Theory
\[ P \text{ vs } NP \]
Representation Theory
\[ \mathbb{C}[X_{1,1}X_{2,2} - X_{1,2}X_{2,1}] \]
Algebraic Geometry
Algebraic Combinatorics
\[ s_{(2,2)}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + x_1^3x_2^3 + x_1x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 \]
\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]
Intro

Permutations
\( \pi = 42351 \)
\( \pi : [1 \ldots n] \rightarrow [1 \ldots n] \)

Partitions \( \lambda = (3, 2) \)

Symmetric Group \( S_n \)
Irreducible \( S_n \) modules \( S_\lambda, \lambda \vdash n \)

Irreducible \( GL_N \) modules \( V_\lambda \)

Standard Young Tableaux
\[
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 \\
\end{array}
\]

Semi-Standard Young Tableaux
\[
\begin{array}{ccc}
1 & 1 & 3 \\
2 & 3 \\
\end{array}
\]
Multiplicities

Decomposition of tensor product into $GL_N$-irreducibles:

$$V_\lambda \otimes V_\mu = \bigoplus \nu V_\nu^{\oplus c_{\lambda \mu}^{\nu}}$$

Multiplicities $c_{\lambda \mu}^{\nu}$ – **Littlewood-Richardson coefficients**.
Multiplicities

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$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} V_\nu \oplus c_{\lambda\mu}^\nu$$

Multiplicities $c_{\lambda\mu}^\nu$ – **Littlewood-Richardson coefficients**.

**Theorem (Littlewood-Richardson, 1934)**

The coefficient $c_{\lambda\mu}^\nu$ is equal to the number of LR tableaux of shape $\nu/\mu$ and type $\lambda$.

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1 First proof (with gaps) by Robinson’38, first rigorous proof by Schützenberger’77 and Thomas’74.
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Multiplicities $c_{\lambda\mu}^\nu$ – Littlewood-Richardson coefficients.

Theorem (Littlewood-Richardson, 1934 $^1$)

The coefficient $c_{\lambda\mu}^\nu$ is equal to the number of LR tableaux of shape $\nu/\mu$ and type $\lambda$.

$$\begin{array}{ll}
1 & 1 \\
1 & 1 \\
2 & 3 & 3
\end{array}$$

(LR tableaux of shape $(6, 4, 3)/(3, 1)$ and type $(4, 3, 2)$. $c_{(3,1)(4,3,2)}^{(6,4,3)} = 2$)

Corollary: $\text{COMPUTE LR}$ is in $\#P$.

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Multiplicities

Decomposition of tensor product into $GL_N$–irreducibles:

$$V_\lambda \otimes V_\mu = \bigoplus\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

Multiplicities $c^\nu_{\lambda\mu}$ – Littlewood-Richardson coefficients.

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The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape $\nu/\mu$ and type $\lambda$.

$S_n$ tensor products decomposition (diagonal action):

$$S_\lambda \otimes S_\mu = \bigoplus_{\nu \vdash n} S_\nu^{\oplus (\ldots . .\ldots.)}$$
Multiplicities

Decomposition of tensor product into $GL_N$–irreducibles:

$$V_\lambda \otimes V_\mu = \bigoplus \nu V_\nu^{c_{\lambda\mu}^\nu}$$

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$S_n$ tensor products decomposition (diagonal action):

$$S_\lambda \otimes S_\mu = \bigoplus \nu S_\nu^{g(\lambda, \mu, \nu)}$$

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $S_\nu$ in $S_\lambda \otimes S_\mu$

E.g.: $S_{(2,1)} \otimes S_{(2,1)} = S_{(3)} \oplus S_{(2,1)} \oplus S_{(1,1,1)}$ and so $g((2,1), (2,1), \nu) = 1$ for $\nu = (3), (2,1), (1,1,1)$.

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The problems

Problem (Murnaghan 1938, Stanley)

*Find a positive combinatorial interpretation for* \( g(\lambda, \mu, \nu) \), *i.e. a family of combinatorial objects* \( O_{\lambda, \mu, \nu} \) *s.t.* \( g(\lambda, \mu, \nu) = \#O_{\lambda, \mu, \nu} \).

**Conjecture**

\( \text{ComputeKRON} \) *is in* \( \#P \).
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**Conjecture**

$\text{COMPUTEKRON}$ \textit{is in} $\#P$.

**Results:**

$\#P$ formulas for $g(\lambda, \mu, \nu)$, when:

- $\nu = (n - k, k)$ (where something is written in red), and $\lambda_1 \geq 2k - 1$, \cite{Ballantine-Orellana, 2006}
- $\nu = (n - k, 1^k)$ (where something is written in red), \cite{Blasiak 2012, Blasiak-Liu 2014}
- Other special cases \cite{Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Ikenmeyer-Panova}. 
The problems

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*Find a positive combinatorial interpretation for* $g(\lambda, \mu, \nu)$, *i.e. a family of combinatorial objects* $\mathcal{O}_{\lambda, \mu, \nu}$, *s.t.* $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda, \mu, \nu}$.

Conjecture

**ComputeKRON** *is in* $\mathcal{NP}$.

Results:

$\mathcal{NP}$ formulas for $g(\lambda, \mu, \nu)$, when:

- $\nu = (n - k, k)$ (rectangle) and $\lambda_1 \geq 2k - 1$, [Ballantine–Orellana, 2006]
- $\nu = (n - k, 1^k)$ (rectangle), [Blasiak 2012, Blasiak-Liu 2014]
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Ikenmeyer-Panova].

[Ikenmeyer-Mulmuley-Walter] **KRONPOS** is **NP**–hard,

**ComputeKRON** *is* $\mathcal{NP}$–hard (and is in Gap$\mathcal{P}$).
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*Find a positive combinatorial interpretation for* $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $O_{\lambda, \mu, \nu}$, s.t. $g(\lambda, \mu, \nu) = \#O_{\lambda, \mu, \nu}$.

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[Ikenmeyer-Panova, Adv.Math’16]:

$g((N - ab, a^b), (N - ab, a^b), (N - |\gamma|, \gamma)) > 0$ for almost all $\gamma, a, b$ (with restrictions), related to [ no occurrence-obstructions in] Geometric Complexity Theory.
Partitions inside a rectangle

\[ p_n := \# \{ \lambda \vdash n \} \]
\[ \sum_{n=0}^{\infty} p_n q^n = \prod_{i=1}^{\infty} \frac{1}{1 - q^i} \]

\[ p_n(\ell, m) := \# \{ \lambda \vdash n; \lambda \subset (m^\ell) \} \]
\[ \sum_{k \geq 0} p_n(\ell, m) q^n = \binom{m + \ell}{m}_q \]
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Hardy-Ramanujan:

\[ p_n := \#\{\lambda \vdash n\} \sim \frac{1}{4n\sqrt{3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right) , \]

Trivial:

\[ p_1 < p_2 < \cdots < p_n < p_{n+1} < \cdots \]
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**Theorem (Sylvester 1878, Cayley's conjecture 1856)**

The sequence \( p_0(\ell, m), \ldots, p_{\ell m}(\ell, m) \) is unimodal, i.e.

\[ p_0(\ell, m) \leq p_1(\ell, m) \leq \cdots \leq p_{\lfloor \ell m/2 \rfloor}(\ell, m) = p_{\lceil \ell m/2 \rceil}(\ell, m) \geq \cdots \geq p_{\ell m}(\ell, m) \]
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“\textit{I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. [...] I accomplished with scarcely an effort a task which I had believed lay outside the range of human power.}”

J.J. Sylvester, 1878.

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Partitions inside a rectangle

$$p_n := \#\{\lambda \vdash n\}$$
$$\sum_{n=0}^{\infty} p_n q^n = \prod_{i=1}^{\infty} \frac{1}{1 - q^i}$$

$$p_n(\ell, m) := \#\{\lambda \vdash n; \lambda \subset (m^\ell)\}$$
$$\sum_{k \geq 0} p_n(\ell, m) q^n = \left(\begin{array}{c} m + \ell \\ m \end{array}\right)_q$$

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Proofs:
Stanley [1. Hard Lefshetz, 2. linear algebra paradigm],
Kathy O’Hara [1991, combinatorial]
No known (symmetric) chain decomposition of the Partition Lattice
The mystery
Partitions
Inequalities and Asymptotics
Hardness
Reduced Kroneckers

Partitions inside a rectangle

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**Theorem (Pak-P (2014))**

For all $m \geq \ell \geq 8$ and $2 \leq k \leq \ell m/2$, we have:

$$g(m^\ell, m^\ell, (m\ell - k, k)) = p_k(\ell, m) - p_{k-1}(\ell, m) > A \frac{2\sqrt{s}}{s^{9/4}},$$

where $s = \min\{2k, \ell^2\}$, and $A > 0.004$ is a universal constant.
Lower bounds

**Theorem** [Pak-P (2014)] Let $\mu = \mu'$ be a self-conjugate partition and let $\hat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, \ldots) \vdash n$. Let $\chi^\lambda[\pi]$ be the character of $\mathbb{S}^\lambda$ on the permutations of cycle type $\pi$. Then:

$$g(\lambda, \mu, \mu) \geq |\chi^\lambda[\hat{\mu}]|, \quad \text{for every } \lambda \vdash n,$$

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**Theorem** [Pak-P (2014)] Let \( \mu = \mu' \) be a self-conjugate partition and let \( \hat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, \ldots) \vdash n \). Let \( \chi^\lambda[\pi] \) be the character of \( S^\lambda \) on the permutations of cycle type \( \pi \). Then:

\[
g(\lambda, \mu, \mu) \geq |\chi^\lambda[\hat{\mu}]|, \quad \text{for every } \lambda \vdash n,
\]

**Example:**

\[
\lambda = \mu = \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & & \\
1 & 2 & 3 & 3 & & \\
1 & 2 & 3 & & & \\
1 & & & & & \\
1 & & & & & \\
\end{array}
\]

\[
|\chi^\mu[\hat{\mu}]| = 1
\]

**Corollary:** [Bessenrodt–Behns] For every \( \mu \), s.t. \( \mu = \mu' \) we have \( \langle \chi^\mu \otimes \chi^\mu, \chi^\mu \rangle \neq 0 \).
**Lower bounds**

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$$\mu = (n^n): \quad \lambda = (n^2 - k, k)$$

$$|\chi^\lambda[(1, 3, 5, \ldots)]| = |b_k(n) - b_{k-1}(n)|$$

$$\prod_{i=1}^{n} (1 + q^{2i-1}) = \sum_{k=0}^{n^2} b_k(n) q^k.$$
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**Lemma** [Pak-P, Vallejo, etc]:

$$g((m^\ell), \mu, \mu) = p_k(m, \ell) - p_{k-1}(m, \ell)$$

**Corollary** [Stanley, 1982; Pak-P, 2016] The following polynomial in $q$ is symmetric and unimodal

$$\binom{2n}{n}_q - \prod_{i=1}^{n} (1 + q^{2i-1}).$$
Effective bounds

\[ \prod_{i=1}^{n} (1 + q^{2i-1}) = \sum_{k=0}^{n^2} b_k(n) q^k \]

1. **Theorem [Pak-P, cor to Almkvist]**

\[ b_k(n) - b_{k-1}(n) \geq C2^{\sqrt{2k}} \frac{1}{(2k)^{9/4}} \]
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2. **Theorem** for \( \ell = m = n \):

   \[ g(n^n, n^n, (n^2 - k, k)) = p_k(n, n) - p_{k-1}(n, n) > C \frac{2^{\sqrt{2k}}}{(2k)^{9/4}} . \]
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3. **Semigroup property** for Kronecker coefficients: if \( g(\lambda, \mu, \nu), g(\alpha, \beta, \gamma) > 0 \), then:

\[ g(\lambda + \alpha, \mu + \beta, \nu + \gamma) \geq \max \{ g(\lambda, \mu, \nu), g(\alpha, \beta, \gamma) \}. \]

\[ \implies g(m^\ell, m^\ell, (m\ell - k, k)) \geq g(\ell^\ell, \ell^\ell, (\ell^2 - s, s)), \text{ where } s = \min\{k, \ell^2/2\}. \]
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\[ \implies \text{ Theorem} \ [\text{Pak-P (2016)}] \]

\[ g(m^\ell, m^\ell, (m^\ell - k, k)) = p_k(\ell, m) - p_{k-1}(\ell, m) > 0.004 \frac{2^{\sqrt{s}}}{s^{9/4}}, \text{ where } s = \min\{2k, \ell^2\}, \]
The difference $p_{n+1}(m, \ell) - p_n(m, \ell)$

Theorem (Melczer-P-Pemantle)

Given $m, \ell$ and $n$, let $A := \ell/m$ and $B := n/m^2$ and define $c, d$ as solutions of a given system of integral equations. Suppose $m, \ell$ and $n$ go to infinity so that $(A, B)$ remains in a compact subset of $\{(x, y) : x > 2y > 0\}$. Then

$$p_{n+1}(\ell, m) - p_n(\ell, m) \sim \frac{d}{m} p_n(\ell, m) \sim \frac{d e^{m\left[ cA + 2dB - \log(1 - e^{-c-d}) \right]}}{2\pi m^3 \sqrt{D}}.$$
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**Corollaries:**
Sylvester’s unimodality, [Pak-P] bound on the Kronecker to exact asymptotics etc.

**Corollary**

The Kronecker coefficient of $S_{m\ell}$ for the (rectangle, rectangle, two-row) case is asymptotically given by

$$g((m\ell), (m\ell), (m\ell - n - 1, n + 1)) = p_{n+1}(\ell, m) - p_n(\ell, m) \sim \frac{d}{m} p_n(\ell, m) \sim \frac{d e^{m \left[ cA + 2dB - \log(1 - e^{-c-d}) \right]}}{2\pi m^3 \sqrt{D}}.$$
Beyond explicit formulas: LR and Kronecker

?? Values of:

**Littlewood-Richardson coefficients** $c^\lambda_{\mu\nu}$ of $GL(V)$:

$$V_\mu \otimes V_\nu = \bigoplus_\lambda V_\lambda^{c^\lambda_{\mu\nu}}$$

**Kronecker coefficients** $g(\lambda, \mu, \nu)$ of $S_n$:

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Question: [Stanley] For which $\lambda, \mu, \nu$ are these maxima achieved?
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**Theorem [Stanley]**

$$\max_{\lambda \vdash n} \max_{\mu \vdash n} \max_{\nu \vdash n} g(\lambda, \mu, \nu) = \sqrt{n}! \ e^{-O(\sqrt{n})},$$

$$\max_{0 \leq k \leq n} \max_{\lambda \vdash n} \max_{\mu \vdash n-k} \max_{\nu \vdash n-k} c_{\mu\nu}^\lambda = 2^{n/2-O(\sqrt{n})}.$$
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**Question:** [Stanley] For which $\lambda, \mu, \nu$ are these maxima achieved?
Maximal $D(n) := \max_{\lambda \vdash n} f^\lambda$ 

**Theorem** [Vershik-Kerov, McKay]: The maximal dimension $f^\lambda$ for $\lambda \vdash n$, denoted $D(n)$ satisfies

$$\sqrt{n!} \, e^{-c_1 \sqrt{n}(1+o(1))} \leq D(n) \leq \sqrt{n!} \, e^{-c_2 \sqrt{n}(1+o(1))}$$

for some $c_1 > c_2 > 0$. Moreover $\lambda^{(n)}$ satisfies $f^{\lambda^{(n)}} \geq \sqrt{n!} e^{-a \sqrt{n}}$ for some $a$ iff it has the:

**Vershik–Kerov–Logan–Shepp (VKLS) shape** if

$$\left| \frac{1}{\sqrt{n}} \left[ \lambda^{(n)} \right] - \varphi \right| < C n^{1/6} \quad \text{for some } \ C > 0,$$

where $\varphi : [0, 2] \to [0, 2]$ is the $135^\circ$ rotation of $(x, y(x))$:

$$y(x) := \frac{2}{\pi} \left( x \arcsin \frac{x}{\sqrt{2}} + \sqrt{2 - x^2} \right), \quad -\sqrt{2} \leq x \leq \sqrt{2}$$

$$D(n) \sim e^{\frac{1}{2} n \log(n) - \frac{1}{2} n + O(\sqrt{n})}$$
Largest Kroneckers

Identities:

\[
\sum_{\lambda,\mu,\nu \vdash n} g(\lambda, \mu, \nu)^2 = \sum_{\alpha \vdash n} z_\alpha \geq z_1^n = n!,
\]

where \(z_\alpha = 1^{m_1} m_1! 2^{m_2} m_2! \cdots\) when \(\alpha = (1^{m_1} 2^{m_2} \ldots)\),
so \(\max g(\lambda, \mu, \nu) \geq \sqrt{n!/p(n)^3} = e^{\frac{1}{2} n \log(n) - \frac{1}{2} n - O(\sqrt{n})}.\)
Largest Kroneckers

Identities:
\[
\sum_{\lambda,\mu,\nu \vdash n} g(\lambda, \mu, \nu)^2 = \sum_{\alpha \vdash n} z_{\alpha} \geq z_1 n = n!,
\]

where \( z_{\alpha} = 1^{m_1} m_1! 2^{m_2} m_2! \cdots \) when \( \alpha = (1^{m_1} 2^{m_2} \ldots) \),
so \( \max g(\lambda, \mu, \nu) \geq \sqrt{n!} / p(n)^3 = e^{\frac{1}{2} n \log(n) - \frac{1}{2} n - O(\sqrt{n})}. \)

**Theorem (Pak-Panova-Yeliussizov)**

Let \( \{\lambda^{(n)} \vdash n\}, \{\mu^{(n)} \vdash n\}, \{\nu^{(n)} \vdash n\} \) be three partitions sequences, such that

\[
g(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}) = \sqrt{n!} e^{-O(\sqrt{n})}.
\]

Then all three partition sequences are Plancherel (i.e. VKLS shape). Conversely, for every two Plancherel partition sequences \( \{\lambda^{(n)} \vdash n\} \) and \( \{\mu^{(n)} \vdash n\} \), there exists a Plancherel partition sequence \( \{\nu^{(n)} \vdash n\} \), s.t. \( (*) \) holds.
Largest Kroneckers

Identities:
\[ \sum_{\lambda, \mu, \nu \vdash n} g(\lambda, \mu, \nu)^2 = \sum_{\alpha \vdash n} z_\alpha \geq z_1^n = n! , \]
where \( z_\alpha = 1^{m_1} m_1 ! 2^{m_2} m_2 ! \cdots \) when \( \alpha = (1^{m_1} 2^{m_2} \cdots) \), so \( \max g(\lambda, \mu, \nu) \geq \sqrt{n!} / p(n)^3 = e^{\frac{1}{2} n \log(n) - \frac{1}{2} n - O(\sqrt{n})} \).

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**Proof:**
\( g(\lambda, \mu, \nu) \leq \min\{ f^\lambda, f^\mu, f^\nu \} \), so if \( g(\lambda, \mu, \nu) \) is large then all \( f^\lambda, f^\mu, f^\nu \) are large, i.e. VKLS shape.
Largest Kroneckers

Theorem (Pak-Panova-Yeliussizov)

Let \( \{\lambda(n) \vdash n\} \), \( \{\mu(n) \vdash n\} \), \( \{\nu(n) \vdash n\} \) be three partitions sequences, such that

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\[
\sum_{\nu} g(\lambda, \mu, \nu)f^\nu = f^\lambda f^\mu
\]

at least one of the summands on the LHS \( \sim n! e^{-O(\sqrt{n})} \), this is \( \nu(n) \).
The mystery
Partitions
Inequalities and Asymptotics
Hardness
Reduced Kroneckers

Largest Kroneckers

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at least one of the summands on the LHS \( \sim n! e^{-O(\sqrt{n})} \), this is \( \nu(n) \).

Theorem[PPY]: Let \( \mu, \nu \vdash n \). Suppose \( f^\mu, f^\nu \geq D(n)/a \) for some \( a \geq 1 \). Then there exist \( \lambda \vdash n \), s.t.

\[
f^\lambda \geq \frac{D(n)}{a \sqrt{p(n)}} \quad \text{and} \quad g(\lambda, \mu, \nu) \geq \frac{D(n)}{a^2 p(n)}.
\]
Small number of rows

Theorem (Pak-P’20)
Let $\lambda, \mu, \nu \vdash n$ such that $\ell(\lambda) = \ell, \ell(\mu) = m, \text{ and } \ell(\nu) = r$. Then:

$$g(\lambda, \mu, \nu) \leq \left(1 + \frac{\ell mr}{n}\right)^n \left(1 + \frac{n}{\ell mr}\right)^{\ell mr}.$$ 

Corollary
Let $\lambda = (\ell^2)^\ell$, where $\ell = \sqrt[3]{n}$, then

$$g(\lambda, \lambda, \lambda) \leq 4^n.$$ 

(Compare to $f^\lambda = \exp\left[\frac{1}{3} n \log(n) + O(n^{2/3})\right]$ and $D(n) = \exp\left[\frac{1}{2} n \log n + O(\sqrt{n})\right]$)
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Proof via contingency arrays:

$$T(\lambda, \mu, \nu) = \#\{ (X_{i,j,k}) \in \mathbb{Z}_{\geq 0}^{\ell mr} : \sum_{j=1, k=1}^{m, r} X_{i,j,k} = \lambda_i, \sum_{i=1, k=1}^{\ell, r} X_{i,j,k} = \mu_j, \sum_{i=1, j=1}^{\ell, m} X_{i,j,k} = \nu_k \},$$

$$\sum_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) s_\lambda(x)s_\mu(y)s_\nu(z) = \sum_{\alpha, \beta, \gamma} T(\alpha, \beta, \gamma) x^\alpha y^\beta z^\gamma.$$ 

$$\implies g(\lambda, \mu, \nu) \leq T(\lambda, \mu, \nu),$$
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Let \( \lambda = (\ell^2)^\ell \), where \( \ell = \sqrt[3]{n} \), then

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\]

\[
\implies g(\lambda, \mu, \nu) \leq T(\lambda, \mu, \nu),
\]

[Barvinok]: The number of 3d contingency tables with marginals \((\alpha, \beta, \gamma)\) is

\[
\leq \exp\left(\max_{Z \in P(\alpha, \beta, \gamma)} \sum_{i,j,k} (Z_{ijk} + 1) \log(Z_{ijk} + 1) - Z_{ijk} \log(Z_{ijk})\right)
\]
The elusive lower bound

\[ g(\lambda, \lambda, \lambda) \geq 1 \quad \text{for } \lambda = \lambda' \]

(from \( g(\lambda, \lambda, \lambda) \geq |\chi^\lambda(2\lambda_1 - 1, 2\lambda_2 - 3, \ldots)| = 1 \))
The elusive lower bound

$\text{Pyr}(\alpha, \beta, \gamma) := \# \text{ of pyramids (solid partitions, plane partitions) with marginals } \alpha, \beta, \gamma.$

**Theorem** [Manivel, Vallejo]

$$g(\lambda, \mu, \nu) \geq \text{Pyr}(\lambda', \mu', \nu')$$
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**Proposition** For some \( \alpha, \beta, \gamma \vdash n \) we have

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**Proposition** [Pak-P, '20]: For an explicit \( \lambda \vdash n \) we have \( g(\lambda, \lambda, \lambda) \geq \exp \Theta(n^{2/3}) \):

\[ \lambda = \left( 3\binom{s}{2} + 7, 3\binom{s}{2} + 3, 3\binom{s}{2} + 2, 3\binom{s-1}{2} + 7, \ldots, 7, 3, 2 \right) \]
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**Conjecture** [Pak-P]:

\[ \sum_{\lambda \vdash n, \lambda = \lambda'} g(\lambda, \lambda, \lambda) = \exp \left( \frac{1}{2} n \log n + O(n) \right). \]
Computational Complexity

Input: \( l \), size(\( l \)) = \( n \) (bits)
Computational Complexity

Input: $I$, $\text{size}(I) = n$ (bits)

Decision problems: is there...
... an object $X$, s.t. $X \in C(I)$?
Is $C(I) \neq \emptyset$?
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\( \mathsf{P} \) = yes/no answer in time \( O(n^d) \) some fixed \( d \).
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Some NP-complete problems: 3-colorability, 3-SAT, Hamiltonian cycle, knapsack.
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\begin{align*}
\text{P} & = \text{yes/no answer in time } O(n^d) \text{ some fixed } d. \\
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& \text{Given } X, \text{ is } X \in C(I)? \text{ Answer in } O(n^d).
\end{align*}

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Counting problems:
Compute \( F(I) = ? \)
Computational Complexity

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Counting problems: Compute \( F(I) =? \)

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\text{FP} = F(I) \text{ can be found in } O(n^d) \text{ time.}
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Counting problems:

Compute $F(I) = ?$

$\textbf{FP}$ = $F(I)$ can be found in $O(n^d)$ time.

$\#\textbf{P}$ = $|C(I)|$, number of objects in $C(I)$ for an NP problem.

Some $\textbf{NP}$-complete problems: 3-colorability, 3-SAT, Hamiltonian cycle, knapsack.
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**Counting problems:**
Compute \( F(I) =? \)

- **FP** = \( F(I) \) can be found in \( O(n^d) \) time.
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**The P vs NP Millennium Problem:**
Is \( P = NP \)?
**Complexity of Computing Multiplicities**

**Dimension** of irreducible representations:

$$\dim S_\lambda = f^\lambda = \#SYT(\lambda) = \frac{n!}{\prod_{(i,j) \in \lambda} (\lambda_i - i + \lambda'_j - j + 1)}.$$
Complexity of Computing Multiplicities

**Dimension** of irreducible representations:

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\]

\[
f(3,2) = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & \\
4 & 5 &
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & \\
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\end{array}
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**Kostka numbers:** \( K_{\lambda \mu} = \# SSYT(\text{shape} = \lambda, \text{content} = \mu), \)

\( (= \text{mult}_\lambda M_\mu = \dim \text{of } \mu\text{-weight space in } V_\lambda) \)

\( K_{(4,2,1)(3,2,2)} = 2 \)

\[
\begin{array}{cccc}
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\( \in \mathbb{P} \)
Complexity of Computing Multiplicities

**Dimension** of irreducible representations: \( \dim S_\lambda = f^\lambda = \#SYT(\lambda) = \frac{n!}{\prod_{(i,j) \in \lambda} (\lambda_i - i + \lambda_j' - j + 1)}. \)

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\( K_{(4,2,1)(3,2,2)} = 2 \)

KostkaPos: Input \( \lambda, \mu \), Output: Is \( K_{\lambda \mu} > 0. \)

Answer: Iff \( \lambda \succ \mu \) (\( \sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i \) for every \( i \)).
Complexity of Computing Multiplicities

**Dimension** of irreducible representations:

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\dim S_\lambda = f^\lambda = \# \text{SYT}(\lambda) = \frac{n!}{\prod_{(i,j) \in \lambda} (\lambda_i - i + \lambda'_j - j + 1)}.
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**KOSTKAPOS:** Input \(\lambda, \mu,\) Output: Is \(K_{\lambda\mu} > 0.\)

**Answer:** Iff \(\lambda > \mu \ (\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i \text{ for every } i).\)

**KOSTKAPOS \(\in \text{P}**
Complexity of Computing Multiplicities

**Dimension** of irreducible representations:

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\dim S_\lambda = f_\lambda = \#SYT(\lambda) = \frac{n!}{\prod_{(i,j) \in \lambda} (\lambda_i - i + \lambda_j' - j + 1)}.
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Kostka numbers: \(K_{\lambda\mu} = \#SSYT\) (shape = \(\lambda\), content = \(\mu\)),
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**ComputeKostka**: Input \(\lambda, \mu\). Output: the integer \(K_{\lambda\mu}\).
Complexity of Computing Multiplicities

**Dimension** of irreducible representations:

$$\dim \mathbb{S}_\lambda = f_\lambda = \# \text{SYT}(\lambda) = \frac{n!}{\prod_{(i,j) \in \lambda} (\lambda_i - i + \lambda'_j - j + 1)}.$$ 

$$f(3,2) = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5$$

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2 4

**Kostka numbers:** $K_{\lambda\mu} = \# \text{SSYT}(\text{shape } = \lambda, \text{ content } = \mu)$,

$$= \text{ mult}_\lambda M_\mu = \dim \text{ of } \mu\text{-weight space in } V_\lambda$$

$K_{(4,2,1)(3,2,2)} = 2$

1 1 1 2
2 3
3

1 1 1 3
2 2
3

**KostkaPos:** Input $\lambda, \mu$, Output: Is $K_{\lambda\mu} > 0$.

**Answer:** Iff $\lambda \succ \mu$ ($\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$ for every $i$).

$KostkaPos \in P$

**ComputeKostka:** Input $\lambda, \mu$. Output: the integer $K_{\lambda\mu}$.

**Theorem (Narayanan’06)**

*ComputeKostka is \#P-complete. (input – binary)*

**Theorem (Pak-Panova’20+)**

*ComputeKostka is strongly \#P-complete. (input – unary)*
Littlewood-Richardson coefficients: $c^\lambda_{\mu \nu} = \text{mult}_\lambda V_\mu \otimes V_\nu = \#LR - \text{tableaux}$

$c^{(6,4,3)}_{(4,3,2)(3,1)} = 2$:

\[
\begin{array}{c}
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3
\end{array}
\quad \quad
\begin{array}{c}
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3
\end{array}
\]

LR-Pos:
Input: $\lambda, \mu$,
Output: Is $c^\lambda_{\mu \nu} > 0$. 

Theorem (cor. to Knutson-Tao'01): LR-Pos is in P.

Theorem (Pak-Panova'20+): ComputeLR is strongly #P-hard.
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**Littlewood-Richardson** coefficients: $c^\lambda_{\mu \nu} = \text{mult}_{\lambda} V_\mu \otimes V_\nu = \# \text{LR-tableaux}$

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LR-Pos *is in P.*
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\]

LR-Pos:
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Output: Is \( c_{\mu \nu}^{\lambda} > 0. \)

**Theorem (cor. to Knutson-Tao’01)**

LR-Pos is in \( P. \)

**Theorem (Pak-Panova’20+)**

\texttt{ComputeLR} is strongly \( \#P \)-hard.
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\[ \chi^\lambda(\alpha) = \text{character of } S_\lambda \text{ at permutation of cycle type } \alpha. \]

**CHARPos:**

**Input:** \( \lambda, \alpha \).

**Output:** is \( \chi^\lambda(\alpha) \neq 0 \)?
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**ComputeChar:**

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**Output:** the integer \( |\chi^\lambda(\alpha)| \).
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**Theorem (Pak-Panova'20+)**

*CharPos* is strongly \( \text{NP} \)-hard. *ComputeChar* is strongly \( \text{#P} \)-hard.
Complexity of Computing ...

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CharPos is strongly NP-hard. ComputeChar is strongly \#P-hard.

and the worst...

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KronPos is NP-hard.
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ComputeKron is in GapP (\( = F - G, \text{ where } F, G \in \#P \)).
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**Conjecture**

ComputeKron *is in \#P*. 

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**The reduced (but not simpler!) Kronecker coefficients**

\[ \bar{g}(\alpha, \beta, \gamma) := \lim_{n \to \infty} g(\alpha[n], \beta[n], \gamma[n]), \quad \alpha[n] := (n - |\alpha|, \alpha_1, \alpha_2, \ldots), \quad n \geq |\alpha| + \alpha_1, \]

\[ \bar{g}(\alpha, \beta, \gamma) = c_{\beta \gamma}^\alpha \quad \text{for} \quad |\alpha| = |\beta| + |\gamma|, \]
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**Conjecture (Kirillov, Klyachko)**

The reduced Kronecker coefficients satisfy the saturation property:

$$\overline{g}(N\alpha, N\beta, N\gamma) > 0 \text{ for some } N \geq 1 \implies \overline{g}(\alpha, \beta, \gamma) > 0.$$
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The reduced (but not simpler!) Kronecker coefficients

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Theorem (Pak-P, '20)

*For all } k \geq 3, \text{ the triple of partitions } (1^{k^2-1}, 1^{k^2-1}, k^{k-1}) \text{ is a counterexample to the Conjecture. For every partition } \gamma \text{ s.t. } \gamma_2 \geq 3, \text{ there are infinitely many pairs } (a, b) \in \mathbb{N}^2 \text{ s.t. } (a^b, a^b, \gamma) \text{ is a counterexample.} \]
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**Example**: $\alpha = 1^5$, $\gamma = 3^2$. Then $\bar{g}(\alpha, \alpha, \gamma) = 0$, but $\bar{g}(2\alpha, 2\alpha, 2\gamma) > 0$. 

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The reduced (but not simpler!) Kronecker coefficients

\[ g(\alpha, \beta, \gamma) := \lim_{n \to \infty} g(\alpha[n], \beta[n], \gamma[n]), \quad \alpha[n] := (n - |\alpha|, \alpha_1, \alpha_2, \ldots), \quad n \geq |\alpha| + \alpha_1, \]

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**Example:** \( \alpha = 1^5, \gamma = 3^2 \). Then \( g(\alpha, \alpha, \gamma) = 0 \), but \( g(2\alpha, 2\alpha, 2\gamma) > 0 \).

**Proof sketch:**

[Dvir]: If \( g(\lambda, \mu, \nu) > 0 \) then \( d(\lambda) \leq 2d(\mu)d(\nu) \). (\( d \)– Durfee square size)
The reduced (but not simpler!) Kronecker coefficients

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Theorem (Pak-P, '20)

For all \( k \geq 3 \), the triple of partitions \((1^{k^2-1}, 1^{k^2-1}, k^{k-1})\) is a counterexample to the Conjecture. For every partition \( \gamma \) s.t. \( \gamma_2 \geq 3 \), there are infinitely many pairs \((a, b) \in \mathbb{N}^2\) s.t. \((a^b, a^b, \gamma)\) is a counterexample.

Example: \( \alpha = 1^5, \gamma = 3^2 \). Then \( \bar{g}(\alpha, \alpha, \gamma) = 0 \), but \( \bar{g}(2\alpha, 2\alpha, 2\gamma) > 0 \).

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[Dvir]: If \( g(\lambda, \mu, \nu) > 0 \) then \( d(\lambda) \leq 2d(\mu)d(\nu) \). \((d-\text{Durfee square size})\)

Let \( a, b \) be s.t. \( b \geq \max\{3\ell(\gamma)^{3/2}, |\gamma|/(3\sqrt{2d(\gamma[n]) - 6})\} \) and \( |\gamma|/(6b) \leq a \leq \sqrt{d(\gamma[n])}/2 \). By [Ikenmeyer-P, Adv.Math'16] for \( N \geq 3\ell^2/a \):

\[ \bar{g}(N\alpha, N\alpha, \gamma) \geq g((Na)^{b+1}, (Na)^{b+1}, N\gamma[n]) > 0. \]
Bounds

Theorem

We have:

\[
\max_{a+b+c \leq 3n} \max_{\alpha \vdash a} \max_{\beta \vdash b} \max_{\gamma \vdash c} \bar{g}(\alpha, \beta, \gamma) = \sqrt{n!} \ e^{O(n)}.
\]
**Bounds**

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We have:

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\]

**Proof:**

[Bowman-DeVisscher-Orellana]:

\[
\bar{g}(\alpha, \beta, \gamma) = \sum_{m=0}^{[k/2]} \sum_{\pi \vdash q+m-b} \sum_{\rho \vdash q+m-a} \sum_{\sigma \vdash m} \sum_{\lambda, \mu, \nu \vdash k-2m} c^{\alpha}_{\nu \pi \rho} c^{\beta}_{\mu \pi \sigma} c^{\gamma}_{\lambda \rho \sigma} g(\lambda, \mu, \nu),
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Bounds

**Theorem**

We have:

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\]

\[
\bar{g}(\alpha, \beta, \gamma) \leq (3n/2) \cdot p(3n)^6 \cdot 3^{3n/2} \cdot \sqrt{n!} = \sqrt{n!} e^{O(n)}.
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$$\bar{g}(\alpha, \beta, \gamma) \leq (3n/2) \cdot p(3n)^6 \cdot 3^{3n/2} \cdot \sqrt{n}! = \sqrt{n}! \ e^{O(n)}.$$

Lower bound: \(\bar{g}(\alpha, \beta, \gamma) \geq g(\alpha, \beta, \gamma)\)

[Stanley:] \(\max g(\alpha, \beta, \gamma) \geq \sqrt{n}! e^{O(n)}\)
Complexity

Theorem

*Computing the reduced Kronecker coefficients $\bar{g}(\alpha, \beta, \gamma)$ is strongly \#P-hard.*
Complexity

Theorem

Computing the reduced Kronecker coefficients $\bar{g}(\alpha, \beta, \gamma)$ is strongly $\#P$-hard.

Proof:

[Briand-Orellana-Rosas]:

$$g(\lambda, \mu, \nu) = \ell(\mu) \ell(\nu) \sum_{i=1}^{\ell(\mu) \ell(\nu)} (-1)^i \bar{g}(\lambda^i, \mu^1, \nu^1),$$

where $\lambda^i = (\lambda_1 + 1, \ldots, \lambda_{i-1} + 1, \lambda_{i+1}, \ldots)$. 
Complexity

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Computing the reduced Kronecker coefficients $\overline{g}(\alpha, \beta, \gamma)$ is strongly $\#P$-hard.

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[IIkenmeyer-Mulmuley-Walter]: $\text{COMPUTEKRON}$ is strongly $\#P$-hard.
Thank you!