Filtering Grassmannian Cohomology via k-Schur Functions

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Overview

The Ring $\Lambda^{(k)}$

Definition

A k-**bounded partition** is a partition $\lambda = (\lambda_1 \geq ... \geq \lambda_d)$ where $\lambda_1 \leq k$. We denote the set of all k-bounded partitions by $\mathcal{P}^k$.

Definition

Write $\Lambda^{(k)}$ to denote the k-**bounded symmetric function ring**

$$\Lambda^{(k)} := \mathbb{Q}[h_1, \cdots, h_k] = \mathbb{Q}[e_1, \cdots, e_k].$$

Remark

The ring $\Lambda^{(k)} := \mathbb{Q}[h_1, \cdots, h_k] = \mathbb{Q}[e_1, \cdots, e_k]$ is a subring of the symmetric function ring $\Lambda := \mathbb{Q}[h_1, h_2, h_3, \cdots] = \mathbb{Q}[e_1, e_2, e_3, \cdots]$, and both $\{h_\lambda \mid \lambda \in \mathcal{P}^k\}$ and $\{e_\lambda \mid \lambda \in \mathcal{P}^k\}$ are bases of $\Lambda^{(k)}$.

The Ring $R^{\ell,k}$

We are interested in the subalgebras of $H^*(Gr(\ell, \mathbb{C}^{\ell+k}); \mathbb{Q})$. The following theorem gives a concrete presentation of this ring.

Theorem

The cohomology ring of the complex Grassmannian $Gr(\ell, \mathbb{C}^{\ell+k})$ with coefficients in $\mathbb{Q}$ can be interpreted as the graded vector space:

$$R^{\ell,k} \cong \mathbb{Q}[e_1, e_2, \ldots, e_\ell, h_1, h_2, \ldots, h_k] \ / \ \left(\sum_{i=0}^{d} (-1)^i e_i h_{d-i} \right)_{d=1,2,\ldots,k+\ell}$$

$$\cong \mathbb{Q}[h_1, h_2, \ldots, h_k]/(e_{\ell+1}, \cdots, e_{\ell+k}) = \Lambda^{(k)}/(e_{\ell+1}, \cdots, e_{\ell+k})$$

where $deg(e_i) = deg(h_i) = i$, and the e_i's in the second expression are the ith **Jacobi-Trudi determinants** $\det \begin{pmatrix} h_1 & h_2 & \cdots & & \\ 1 & h_1 & \cdots & & \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & 1 & h_1 & h_2 \\ 0 & \cdots & 0 & 1 & h_1 \end{pmatrix}$.

Hilbert Series and q-Binomial Coefficients

Definition

Given any graded vector space $R = \bigoplus_{d=0}^{\infty} R_d$ over $\mathbb{Q}$, the **Hilbert series** of R is $\mathrm{Hilb}(R,q) = \sum_{d=0}^{\infty} \dim_{\mathbb{Q}}(R_d) q^d$.

Definition

The q-**analogue** of a positive integer is $[n]_q := 1 + q + \ldots + q^{n-1}$. Write $[n]!_q := [n]_q[n-1]_q \ldots [1]_q$. The q-**binomial coefficients** are defined as $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q}$. Equivalently $\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = \sum_{\lambda \subset (k^\ell)} q^{|\lambda|}$.

Theorem

By decomposing $Gr(\ell, \mathbb{C}^{\ell+k})$ into Schubert cells indexed by partitions $\lambda \subseteq (k^\ell)$, $\mathbb{G}(\ell, \mathbb{C}^{k+\ell}) = \bigsqcup_\lambda X_\lambda$, we can show $\mathrm{Hilb}(R^{\ell,k}, q) = \begin{bmatrix} \ell+k \\ \ell \end{bmatrix}_q$.

The R-T Conjecture

Write $R^{\ell,k,m}$ to denote the subalgebra of $R^{\ell,k}$ generated by $h_1, \ldots, h_m$. See that $\mathbb{Q} = R^{\ell,k,0} \subset R^{\ell,k,1} \subset R^{\ell,k,2} \subset \cdots \subset R^{\ell,k,m} \subset \cdots \subset R^{\ell,k}$.

Conjecture (Reiner & Tudose, 2003 [5])

For each $m = 0, 1, 2, \ldots, \min(k, \ell)$,

$$\mathrm{Hilb}(R^{\ell,k,m}, q) = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q . \tag{2.1}$$

Equivalently, for each $m = 1, 2, \ldots, \min(k, \ell)$,

$$\begin{aligned} &\mathrm{Hilb}(R^{k,\ell,m}, q) - \mathrm{Hilb}(R^{k,\ell,m-1}, q) \quad \left(= \mathrm{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1}, q)\right) \\ &= q^m \begin{bmatrix} k \\ m \end{bmatrix}_q \sum_{j=0}^{k-m} q^{j(\ell-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q . \end{aligned} \tag{2.2}$$

Visualization of the R-T Conjecture

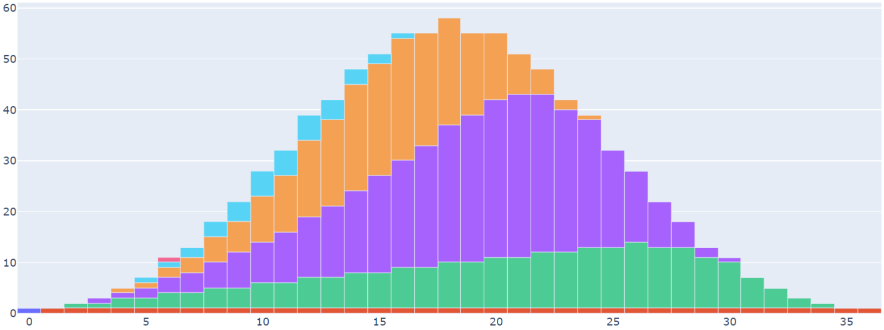


Figure: A bar graph visualization of coefficients of the Hilbert series of the various subalgebras of $R^{6,6}$, as predicted by the R-T conjecture.

The R-T Conjecture: Boundary Cases

Remarks

- One can check that for $m = 1$, this conjecture reduces to $\mathrm{Hilb}(R^{\ell,k,1}, q) = 1 + q + \ldots + q^{k\ell}$, which can be deduced from either Schubert calculus or the hard Lefschetz theorem.
- For $m = \min(k, \ell)$, this conjecture must be consistent with $\mathrm{Hilb}(R^{\ell,k}, q) = \begin{bmatrix} \ell + k \\ \ell \end{bmatrix}_q$. We can verify that the RHS of the R-T Conjecture reduces to this q-binomial coefficient via a combinatorial interpretation of the R-T conjecture involving the notion of i-vacant partitions.

i-vacant Partitions

Definition

A k-bounded partition $\lambda \in \mathcal{P}^k$ is ***i*-vacant** if i is the largest integer for which the complementary *skew diagram* $(k^{\ell(\lambda)})/\lambda$ contains an $i \times (i-1)$ rectangle in its southeast corner.
We will call $(k^{\ell(\lambda)})$ the **ambient** k**-rectangle** of λ.

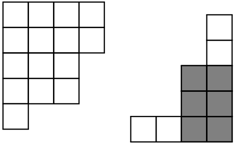
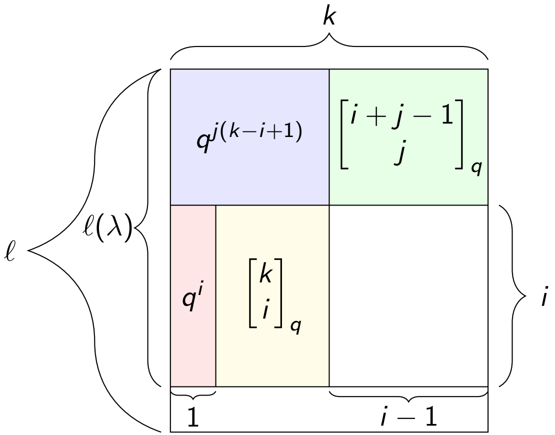


Figure: The 5-bounded partition $\lambda = (4,4,3,3,1) \in \mathcal{P}^5$ has (5^5) as its ambient 5-rectangle and is 3-vacant because its complementary skew diagram $(5^5)/\lambda$ contains a 3×2 rectangle in its southeast corner but not a 4×3 rectangle.

Interpreting the R-T Conjecture via i-vacant Partitions

$$\mathrm{Hilb}(R^{\ell,k,m}, q) = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q$$



A Combinatorial Interpretation of the R-T Conjecture

This interpretation was implicit in the Reiner & Tudose 2003 paper.

Theorem

For each $m = 0, 1, 2, \ldots, \min(k, \ell)$,

$$\sum_{\substack{i-\text{vacant},\ i \le m \\ \lambda \subseteq (k^\ell)}} q^{|\lambda|} = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q .$$

Equivalently, for each $m = 1, 2, \ldots, \min(k, \ell)$,

$$\sum_{\substack{m-\text{vacant} \\ \lambda \subseteq (k^\ell)}} q^{|\lambda|} = q^m \begin{bmatrix} k \\ m \end{bmatrix}_q \sum_{j=0}^{\ell-m} q^{j(k-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q .$$

When $m = \min(k, \ell)$, $\sum_{\substack{i-\text{vacant},\ i \le m \\ \lambda \subseteq (k^\ell)}} q^{|\lambda|} = \sum_{\lambda \subset (k^\ell)} q^{|\lambda|} = \begin{bmatrix} \ell + k \\ \ell \end{bmatrix}_q$.

k-Bounded Partitions and $(k+1)$-Cores

Definition

A k-**bounded partition** is a partition $\lambda = (\lambda_1 \geq ... \geq \lambda_d)$ where $\lambda_1 \leq k$. We denote the set of all k-bounded partition by $\mathcal{P}^k$.

Definition

The **hook length** of a box b in the Ferrer's diagram of a partition λ is the number of boxes weakly to the right and weakly below b.

Definition

A $(k+1)$-**core** is a partition $\lambda = (\lambda_1 \geq ... \geq \lambda_d)$ where no box has hook-length equal to $k+1$. We denote the set of all $(k+1)$-core by $\mathcal{C}^{k+1}$.

The partition $\lambda = (4, 3, 1, 1)$ is 4-bounded, and it is also a 6-core.

7	4	3	1
5	2	1	
2			
1			

k-Bounded Partitions and $(k+1)$-Cores

Lemma

There exists a bijection between $\mathcal{P}^k$ and $\mathcal{C}^{k+1}$ described as follows:

- $\mathfrak{c} : \mathcal{P}^k \to \mathcal{C}^{k+1}$ *slides rows of the Ferrer's diagram of λ rightward so that all boxes with hook-length greater than k have hook-length less than k; boxes introduced will have hook length greater than $k+1$.*
- $\mathfrak{p} : \mathcal{C}^{k+1} \to \mathcal{P}^k$ *removes from λ all boxes with hook-length greater than $k+1$ and slides rows leftward to obtain a k-bounded partition.*

Considering again the 4-bounded partition $\lambda = (4,3,1,1)$ and we apply the map $\mathfrak{c}$ to obtain a 5-core $\mathfrak{c}(\lambda)$. Considering the 3-bounded partition $\mu = (2,2,1)$ and we apply the map $\mathfrak{c}$ to obtain a 4-core $\mathfrak{c}(\mu)$.

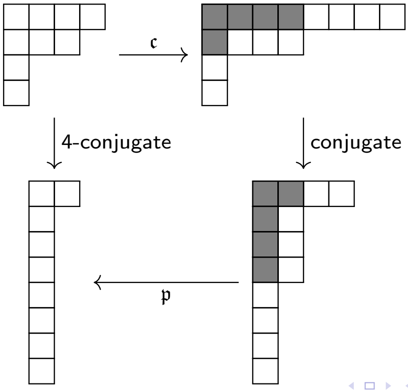
$$\begin{array}{|c|c|c|c|}\hline 7&4&3&1\\ \hline 5&2&1\\ \cline{1-3} 2\\ \cline{1-1} 1\\ \cline{1-1}\end{array} \quad\to\quad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 11&8&7&6&4&3&2&1\\ \hline 6&3&2&1\\ \cline{1-4} 2\\ \cline{1-1} 1\\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 4&2\\ \hline 3&1\\ \hline 1\\ \cline{1-1}\end{array} \quad\to\quad \begin{array}{|c|c|c|}\hline 5&3&1\\ \hline 3&1\\ \cline{1-2} 1\\ \cline{1-1}\end{array}$$

k-Conjugation

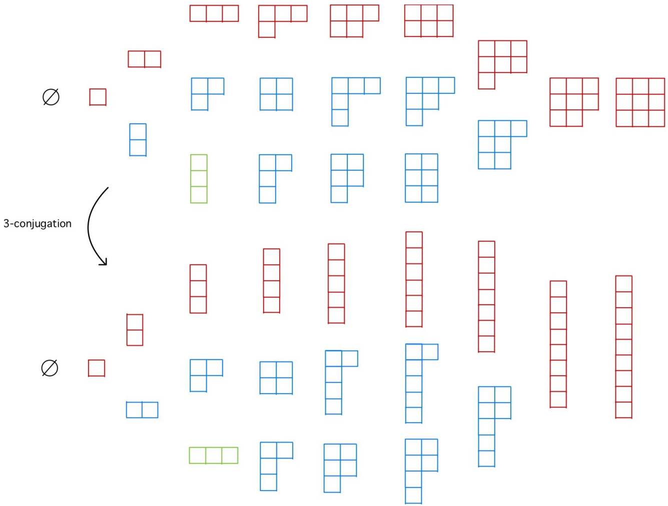
Theorem (Lapointe, Lascoux and Morse, 2003 [3])

The map $\omega(k) : \mathcal{P}^k \to \mathcal{P}^k$ defined by sending a k-bounded partition λ to $\lambda^{\omega(k)} := \mathfrak{p}(\mathfrak{c}(\lambda)')$ is an involution, where $(-)'$ denotes usual conjugation.

We shall call $\lambda^{\omega(k)}$ the k-**conjugate** of λ.



Example: $\ell = 3$, $k = 3$



Suggests: A k-bounded partition λ is i-vacant $\iff$ $\mu := \lambda^{(k)}$ has $\mu_1 = i$.

The Sets $P^{\ell,k}$ and $P^{\ell,k,m}$

Theorem

A k-bounded partition λ is i-vacant if and only if $\mu := \lambda^{(k)}$ has $\mu_1 = i$.

From now on, we shall denote:

$$P^{\ell,k} =: \{\lambda \mid \lambda^{\omega(k)} \subset (k^\ell)\}$$

$$P^{\ell,k,m} =: \{\lambda \mid \lambda_1 \leq m,\ \lambda^{\omega(k)} \subset (k^\ell)\}.$$

Theorem

A k-bounded partition λ is i-vacant for some $i \leq m$ if and only if

$$\lambda^{(k)} \in P^{\ell,k,m}.$$

A k-bounded partition λ is m-vacant if and only if

$$\lambda^{(k)} \in P^{\ell,k,m} \setminus P^{\ell,k,m-1}.$$

New Combinatorial Interpretation of the RT Conjecture

Theorem

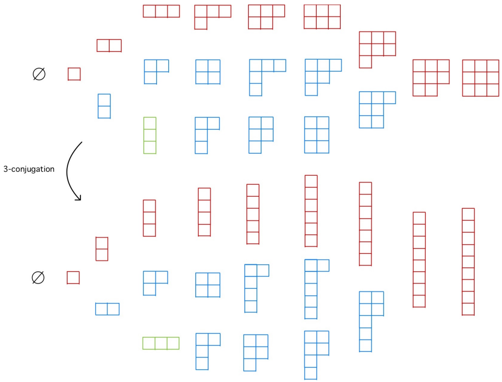
For each $m = 0, 1, 2, ..., min(k, \ell)$,

$$\sum_{\lambda \in P^{\ell,k,m}} q^{|\lambda|} = \sum_{\substack{i\text{-vacant},\ i \leq m \\ \lambda \subset (k^\ell)}} q^{|\lambda|} = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q .$$

Equivalently, for each $m = 1, 2, ..., min(k, \ell)$,

$$\sum_{\lambda \in P^{\ell,k,m} \setminus P^{\ell,k,m-1}} q^{|\lambda|} = \sum_{\substack{m-\text{vacant} \\ \lambda \subseteq (k^\ell)}} q^{|\lambda|} = q^m \begin{bmatrix} k \\ m \end{bmatrix}_q \sum_{j=0}^{\ell-m} q^{j(k-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q .$$

In other words, the q-binomial expression in the RHS of R-T conjecture counts the number of partitions inside (k^ℓ) whose k-conjugate is m-bounded.



$$\mathrm{Hilb}(R^{3,3,0}, q) = 1$$

$$\mathrm{Hilb}(R^{3,3,1}/R^{3,3,0}, q) = q + q^2 + q^3 + q^5 + q^6 + q^7 + q^8 + q^9$$

$$\mathrm{Hilb}(R^{3,3,2}/R^{3,3,1}, q) = q^2 + q^3 + 2q^4 + 2q^5 + 2q^6 + q^7$$

$$\mathrm{Hilb}(R^{3,3,3}/R^{3,3,2}, q) = q^3$$

k-Schur Functions

Recall the k-bounded symmetric function ring $\Lambda^{(k)}$ has bases

$$\{h_\lambda \mid \lambda \subset (k^\ell)\} \text{ and } \{e_\lambda \mid \lambda \subset (k^\ell)\}.$$

Definition

The k-**Schur functions**, indexed by k-bounded partitions, form another basis of $\Lambda^{(k)}$ and are defined by inverting the unitriangular system in $\Lambda^{(k)}$:

$$h_\lambda = s_\lambda^{(k)} + \sum_{\mu : \mu \rhd \lambda} K_{\mu\lambda}^{(k)} s_\mu^{(k)} \text{ for all } \lambda_1, \mu_1 \leq k$$

Remarks:

- $\mu \rhd \lambda$ is the **dominance partial ordering** on partitions of a fixed size n defined by the condition $\mu_1 + \cdots + \mu_i > \lambda_1 + \cdots + \lambda_i$ for some i and $\mu_1 + \cdots + \mu_j = \lambda_1 + \cdots + \lambda_j$ for all $j < i$
- $K_{\mu\lambda}^{(k)}$ are the k-**Kostka numbers**, which are defined as the number of k-**tableaux** of shape $\mathfrak{c}(\mu)$ and k-**weight** λ.

k-tableaux and k-weight

Definition

Let $\mathfrak{c}(\lambda)$ be a $(k+1)$-core and let $\mu = (\mu_1, \ldots, \mu_r)$ be a composition of $|\lambda|$. A k-**tableau** of shape $\mathfrak{c}(\lambda)$ and k-**weight** μ is a filling of $\mathfrak{c}(\lambda)$ with integers $1, 2, \ldots, r$ such that

- rows are weakly increasing and columns are strictly increasing
- the collection of cells filled with letter i are labeled by exactly μ_i distinct $(k+1)$-residues.

Example

The 3-tableaux of 3-weight $(1,3,1,2,1,1)$ and shape $(8,5,2,1)$ are:

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&2&3&4&4&6\\\hline 2&3&4&4&6\\\cline{1-5} 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&2&3&4&4&5\\\hline 2&3&4&4&5\\\cline{1-5} 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&2&4&4&5&6\\\hline 2&4&4&5&6\\\cline{1-5} 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

The Involution ω

Recall that $\Lambda^{(k)}$ has an algebra involution ω that takes e_λ to h_λ. This induces an algebra involution, also denoted as ω, between $R^{k,\ell}$ and $R^{\ell,k}$ that takes e_λ to h_λ. These two involutions are related in the following commutative diagram:

$$\begin{array}{ccc} \Lambda^{(k)} & \xrightarrow{\omega} & \Lambda^{(k)} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ R^{k,\ell} & \xrightarrow{\omega} & R^{\ell,k} \end{array}$$

Hence $R^{\ell,k} \cong R^{k,\ell}$.

Theorem

The set $\{s_\lambda^{(k)} \mid \lambda \subseteq (k^\ell)\}$ forms a basis of $R^{\ell,k}$, where $s_\lambda^{(k)}$ denotes its image under the canonical surjection $\Lambda^{(k)} \to R^{\ell,k}$.

The action of ω on k-schur functions

The commutative diagram from the previous slide:

$$\begin{array}{ccc} \Lambda^{(k)} & \xrightarrow{\omega} & \Lambda^{(k)} \\ \downarrow \phi & & \downarrow \psi \\ R^{k,\ell} & \xrightarrow{\omega} & R^{\ell,k} \end{array}$$

Theorem

The involution $\omega : \Lambda^{(k)} \to \Lambda^{(k)}$ has the following action on the k-Schur basis

$$\omega(s_\lambda^{(k)}) = s_{\lambda^{\omega(k)}}^{(k)}$$

Combining the two theorems in the previous two slides, we get

Corollary

The set $\{s^{(k)}_{\lambda^{\omega(k)}} \mid \lambda \subseteq (k^\ell)\}$ is a basis of $R^{k,\ell}$.

Recall that we denote:

$$P^{\ell,k} =: \{\lambda \mid \lambda^{\omega(k)} \subset (k^\ell)\}$$
$$P^{\ell,k,m} =: \{\lambda \mid \lambda_1 \leq m,\ \lambda^{\omega(k)} \subset (k^\ell)\}.$$

So the corollary says that $\{s^{(k)}_\lambda \mid \lambda \in P^{k,\ell}\}$ is a basis of $R^{\ell,k}$.

Filtered Bases Conjectures

We conjecture that

Conjectures (Existence of Filtered Bases)

1. (a) The set $\{h_\lambda \mid \lambda \in P^{k,\ell}\}$ is a basis of $R^{\ell,k}$, and more strongly
   (b) the set $\{h_\lambda \mid \lambda \in P^{k,\ell,m}\}$ is a basis of $R^{\ell,k,m}$ for each $m = 1, 2, \ldots$.
2. (a) The set $\{s_\lambda^{(i)} \mid \lambda \in P^{k,\ell}, i = \lambda_1\}$ is a basis of $R^{\ell,k}$, and more strongly
   (b) the set $\{s_\lambda^{(i)} \mid \lambda \in P^{k,\ell,m}, i = \lambda_1\}$ is a basis of $R^{\ell,k,m}$ for each $m = 1, 2, \ldots$.

Remarks:

- Part (b) of either conjecture implies the full R-T conjecture due to the new combinatorial interpretation.
- Part (a) of either conjecture will show half of the R-T conjecture (LHS $\geq$ RHS), which is sufficient to greatly simplify the proof of Hoffman's theorem.

Hoffman's Theorem

Theorem (Hoffman)

For $\ell \neq k$, every graded algebra endomorphism $\phi : R^{\ell,k} \to R^{\ell,k}$ which does not annihilate $R_1^{\ell,k}$ is of the form $\phi_\alpha(x) = \alpha^{\deg(x)}x$ for some $\alpha \in \mathbb{Q}^\times$. For $\ell = k$, any such endomorphism is either of the form ϕ_α or $\omega \circ \phi_\alpha$, where $\omega : R^{k,\ell} \to R^{\ell,k}$ is the algebra involution that takes any h_i to e_i.

The R-T conjecture simplifies Hoffman's original proof of this theorem. we observed conj. 1(a) or 1(b) from the previous slide suffices to imply that $Gr(k, \mathbb{C}^{k+\ell})$ has the *fixed point property* if and only if $k\ell$ is odd. We refer the interested readers to Reiner and Tudose's paper *Conjectures on the cohomology of the Grassmannian* [5].

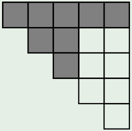
Lagrangian Analogue

Definition

For a strictly decreasing partition $\lambda = (\lambda_1 > \cdots > \lambda_\ell)$, we define its **shifted Young diagram** to be a diagram with λ_i boxes in row i with each row shifted one unit right of the previous one. An **ambient triangle of size** n, denoted as Δ_n, is a shifted Young diagram $\lambda = (n > n-1 > \cdots > 1)$.

Example

An ambient triangle Δ_5 and a shifted Young diagram $\lambda = (5, 2, 1)$ colored gray are illustrated below.



Lagrangian Analogue (Continued)

In Lie type C, we replace $Gr(\ell, \mathbb{C}^{k+\ell})$ by the Lagrangian Grassmannian $\mathbb{LG}(n, \mathbb{C}^{2n})$ and define the graded ring $R^n_{\mathbb{LG}} := H^*(\mathbb{LG}(n, 2n); \mathbb{Q})$.
The ring has a nice presentation due to Borel:
$R^n_{\mathbb{LG}} \cong \mathbb{Q}[e_1, e_2, \ldots, e_n] \ / \ \left(e_i^2 + 2\sum_{k=1}^{n-i}(-1)^k e_{i+k}e_{i-k}\right)_{i=1,2,\ldots,n}$

Theorem

$$\mathrm{Hilb}(R^n_{\mathbb{LG}}, q) = \sum_{\lambda \subset \Delta^n} q^{|\lambda|} = (1+q)(1+q^2)(1+q^3)\cdots(1+q^n).$$

The R-T Conjecture (Type C Analogue)

For each $m = 0, 1, \cdots, n$,

$$\mathrm{Hilb}(R^{n,m}_{\mathbb{LG}}, q) = 1 + \sum_{\substack{1 \le i \le m \\ i \text{ odd}}} q^i \sum_{j=0}^{n-i} q^{\binom{j+1}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q . \qquad (5.1)$$

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