SATURATION for W-refined LR-coefficients – II

15/10/20

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$\boxed{G = GL_n\mathbb{C}}$

- λ, μ : partitions into $\leq n$ parts.

- $$V(\lambda) \otimes V(\mu) = \bigoplus_{\gamma} c^{\gamma}_{\lambda\mu} V(\gamma)$$

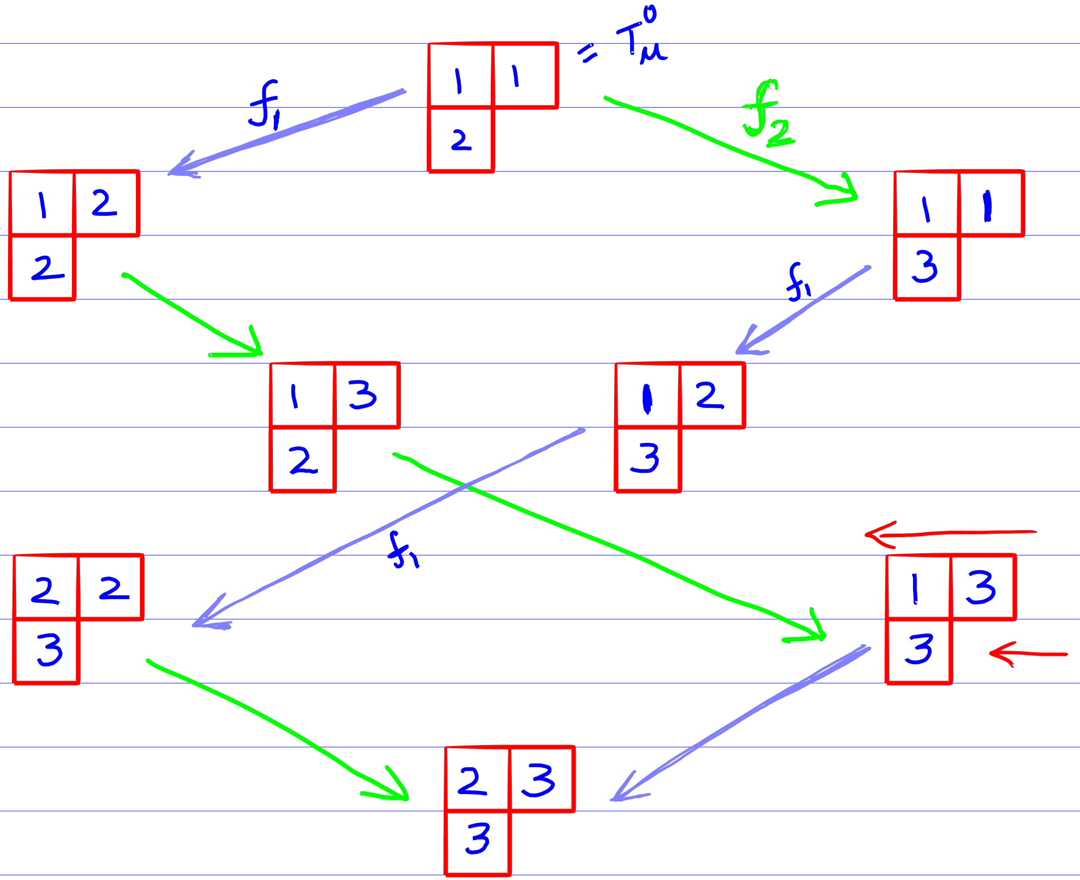
- LR coefficient $c^{\gamma}_{\lambda\mu}$ = number of T such that

$\boxed{\text{Tableaux Model for LR}}$

(i) $T \in SSYT(\mu)$ //

(ii) $\lambda * \text{word}(T)$ is dominant (ballot sequence)

(iii) $\lambda + wt(T) = \gamma$



$\mu = (2,1,0)$

$n = 3$

<u>$SSYT(\mu)$</u>

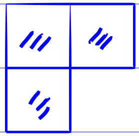
- $\mathrm{word}\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array}\right) = \underline{313}$

- $\underline{X}^{\mathrm{wt}\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array}\right)} = x_1^1 x_3^2$

Example

dominant

λ: $\boxed{1}\boxed{1}$

μ: 

$b^0_\lambda = 11$

11112

11212

11113

11312

11213

11223

11313

11323

For $\sigma \in S_n$ (the Weyl group), define the σ-refined LR coefficient:

$c^{\gamma}_{\lambda\mu}(\sigma)$ = no. of $T \in \mathrm{Dem}(\mu;\sigma) \subseteq \mathrm{SSYT}(\mu)$ satisfying

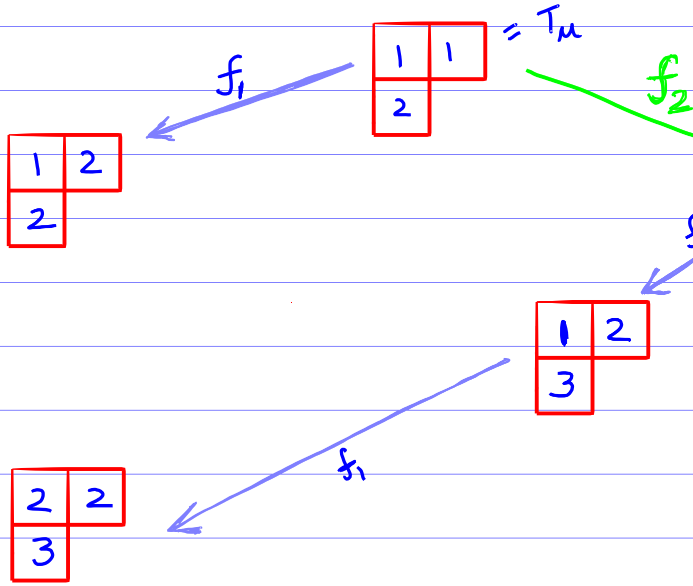
(ii) and (iii)

Theorem (Kushwaha, Raghavan, —):

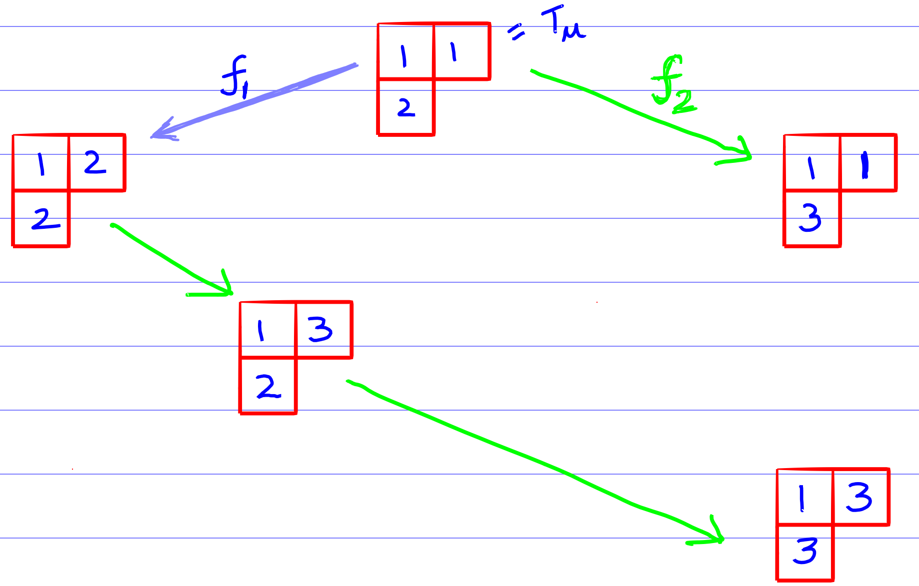
Let λ, μ, γ be partitions w/ $\leq n$ parts. Let $\sigma \in S_n$

be either 312-avoiding or 231-avoiding.

If $\exists\, k \geq 1$ such that $c^{k\gamma}_{k\lambda, k\mu}(\sigma) > 0$, then $c^{\gamma}_{\lambda\mu}(\sigma) > 0$.



$$\mathrm{Dem}(\mu;\ \underbrace{s_1 s_2})$$



$$\mathrm{Dem}(\mu;\ \underbrace{s_2 s_1})$$

Plan

1. The Gelfand-Tsetlin (GT) model for SSYT(μ).

2. (Dual) Kogan faces and a GT-model for Dem$(\mu;\sigma)$.

3. The hive models for a) SSYT(μ) and (a′) Dem$(\mu;\sigma)$
   b) $c^{\gamma}_{\lambda\mu}$ (b′) $c^{\gamma}_{\lambda\mu}(\sigma)$

4. The key steps in Knutson-Tao's proof of the saturation conjecture for $c^{\gamma}_{\lambda\mu}$

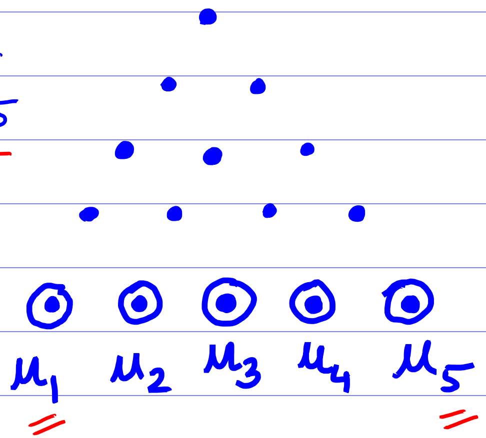
5. Extension to $c^{\gamma}_{\lambda\mu}(\sigma)$: σ = 312- or 231-avoiding.

1. The GT-model for SSYT(μ).

$\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$

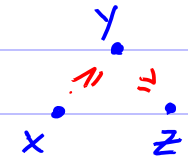
GT pattern of shape μ

GT
$n=5$



Triangular array of real numbers:

- bottom row $\mu_1, \mu_2, \ldots, \mu_n$
- Inequalities:



$x \geq y \geq z.$

- Polytope:

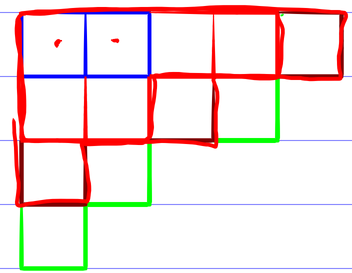
$$GT(\mu) = \{ \text{GT patterns of shape } \mu \}$$

$$GT_{\mathbb{Z}}(\mu) = \{ P \in GT(\mu) : P \text{ has integer entries} \}$$

<u>FACT</u>: $GT_{\mathbb{Z}}(\mu) \xrightarrow{\sim} SSYT(\mu)$

(eg)

$$\begin{array}{ccccccc} & & & 2 & & & \\ & & 4 & & 2 & & \\ & 5 & & 3 & & 1 & \\ 5 & & 4 & & 2 & & 1 \end{array}$$



$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 & 4 \\ \cline{1-4} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

- Last row of GT pattern $= \mu =$ shape of the SSYT.

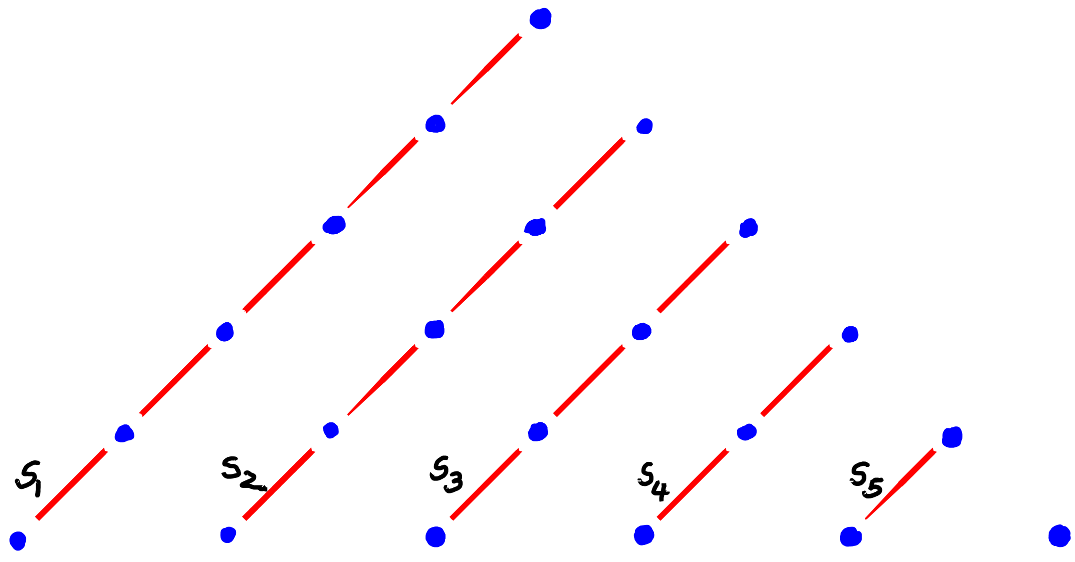
(eg)

$$\begin{array}{ccccccc} & & & 5 & & & \\ & & 5 & & 4 & & \\ & 5 & & 4 & & 2 & \\ 5 & & 4 & & 2 & & 1 \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & \\ \cline{1-4} 3 & 3 & & & \\ \cline{1-2} 4 & & & & \\ \cline{1-1} \end{array} = T_\mu^0$$

superstandard

2. Demazure crystals via GT



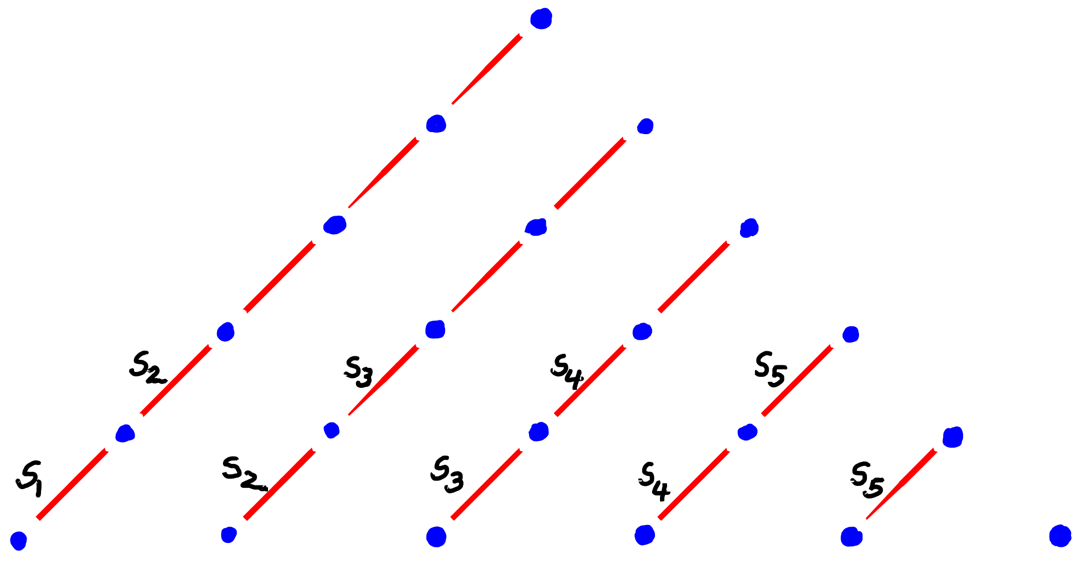
$n = 6$

Weyl group elements $\rightsquigarrow$ GT faces

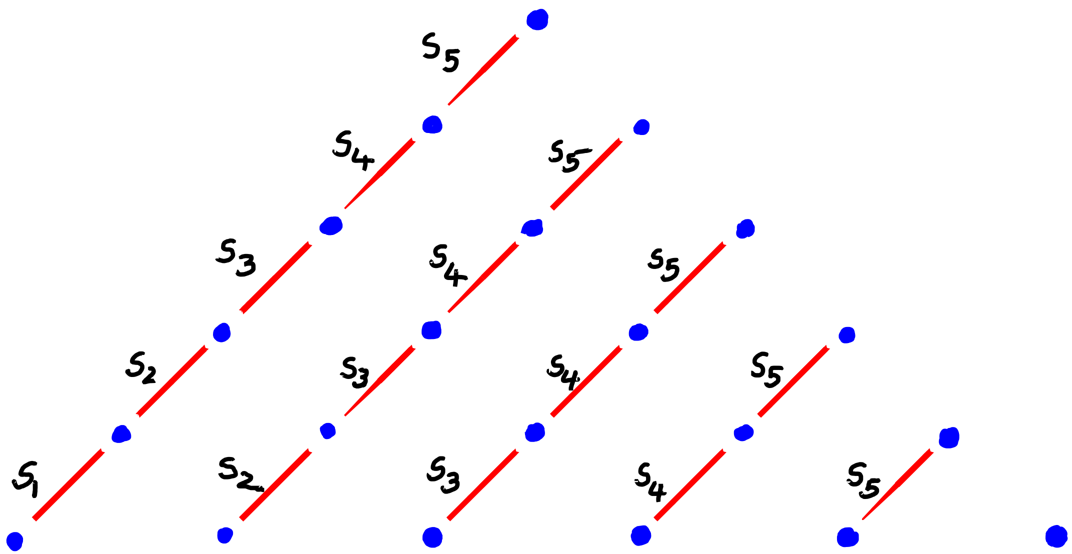
Each northeast edge is labelled by a simple reflection.

$s_i = (i \;\; i+1)$

$\in S_n$.

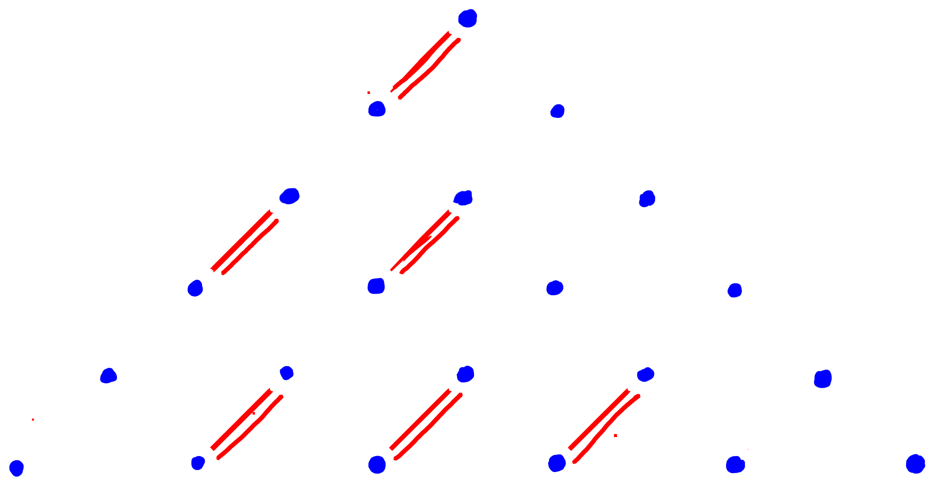


Each northeast edge is labelled by a simple reflection.



Each northeast edge is labelled by a simple reflection.

Let F be a subset of NE edges.



<u>Consider</u> [M. Kogan, 2000]

$$K_\mu(F) = \{ P \in GT(\mu) : \text{the edges in } F \text{ are satisfied with equality in } P \}$$

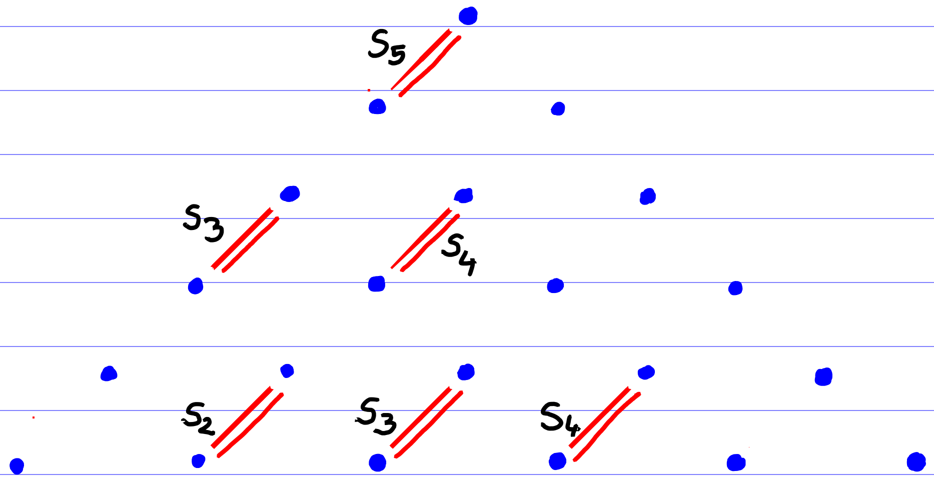
in $GT(\mu)$

$\geqslant \rightsquigarrow =$

(other edges may or may not be $=$)

$K_\mu(F)$: a dual Kogan face of $GT(\mu)$.

(Kogan face : $\diagdown\!\!\diagdown$ SE edges)



$\underline{F} \longmapsto \omega(F) \in S_n.$

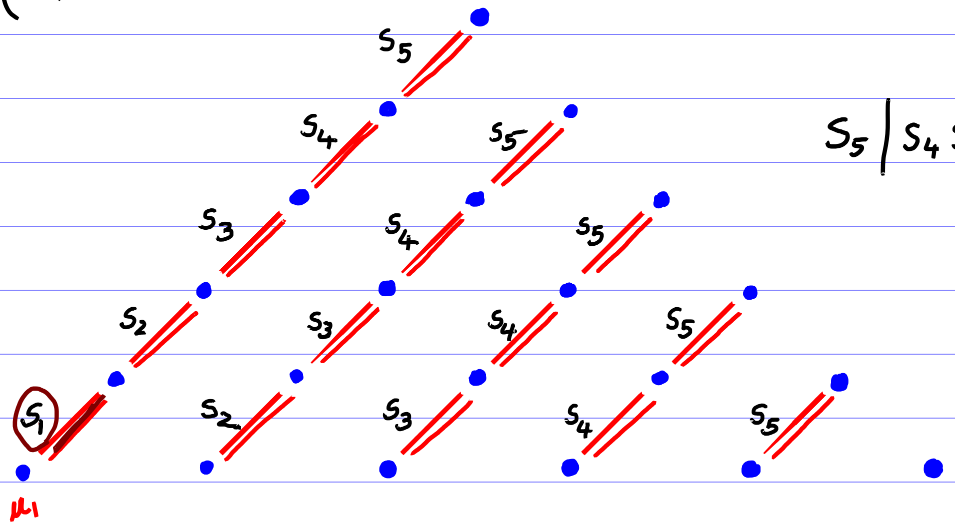
$= \; s_5 \mid s_3 s_4 \mid s_2 s_3 s_4$

Read: Left to Right & Top to Bottom.

If this word is reduced, we call $K_\mu(F)$ a reduced (dual Kogan) face.

("F is reduced")

(eg)



$$s_5 \mid s_4 s_5 \mid s_3 s_4 s_5 \mid s_2 s_3 s_4 s_5 \mid s_1 s_2 s_3 s_4 s_5$$

$$\parallel$$

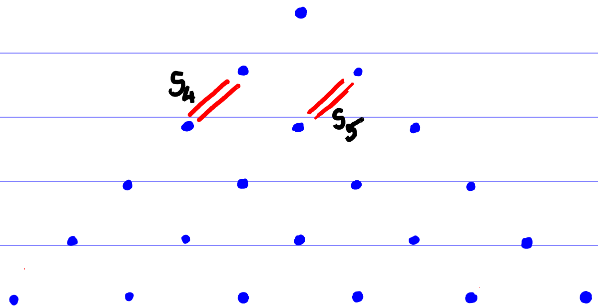
$$\omega_0$$

Longest element

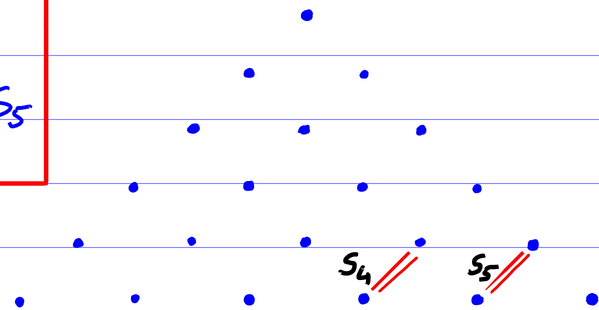
$n \quad n-1 \quad \cdots \quad 2 \quad 1$

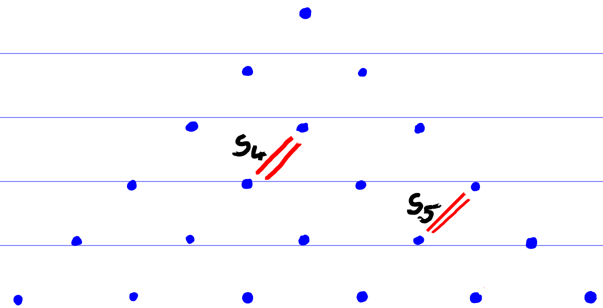
(in one-line notation)

- $F = \{\text{all NE edges}\}$ $\qquad K_\mu(F) = \{T^0_\mu\}$ $\qquad \omega(F) = \omega_0$
- $F = \emptyset$ $\qquad K_\mu(F) = GT(\mu)$ $\qquad \omega(F) = 1$



$\boxed{\text{All these } F\text{'s are reduced \& } \omega = s_4 s_5}$





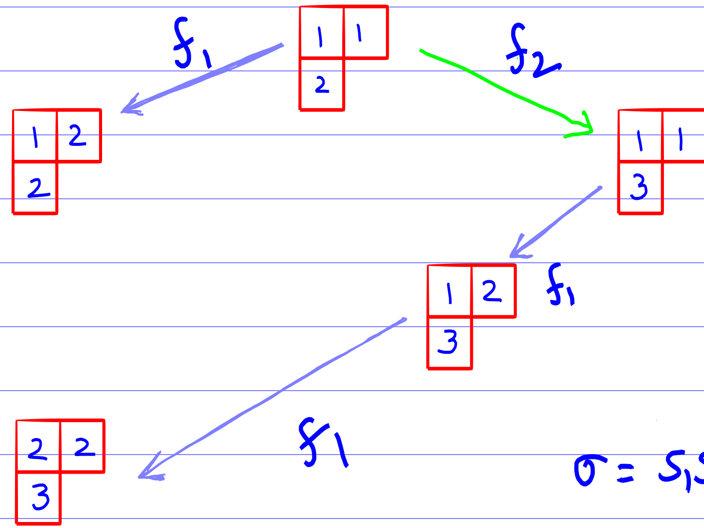
Let $K(\mu;\sigma) = \bigcup K_\mu(F)$ → union over reduced F with $\omega(F) = \sigma$.

- $\sigma \geq \tau$ (Bruhat) $\Rightarrow$ $K(\mu;\sigma) \subseteq K(\mu;\tau)$

Recall Demazure crystal: Fix reduced expression $\sigma = s_{i_1} \cdots s_{i_k}$

$$\underline{\mathrm{Dem}(\mu;\sigma)} := \left\{ \underline{f_{i_1}^{m_1} f_{i_2}^{m_2} \cdots f_{i_k}^{m_k}} (T^0_{\underline{\mu}}) : m_j \geq 0 \ \forall j \right\}$$

- $$\mathrm{char}\, V_\sigma(\mu) = \sum_{T \in \mathrm{Dem}(\mu;\sigma)} x^T$$



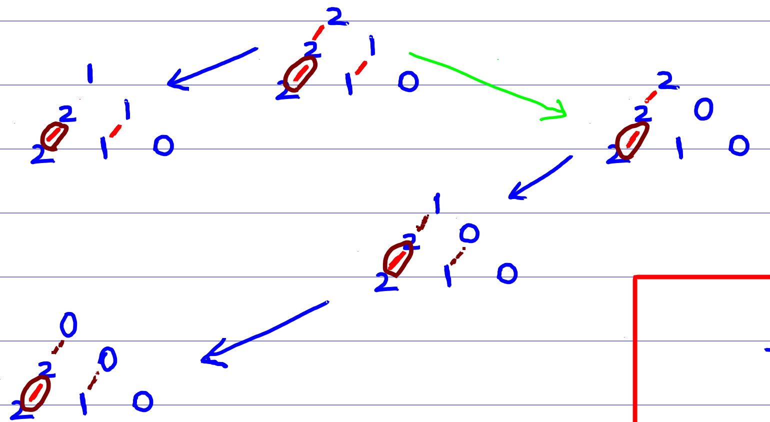
$\sigma = s_1 \underline{s_2}$

Theorem (Fujita 2020): Let $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$ and $\sigma \in S_n$.

The set $\mathrm{Dem}(\mu;\sigma) \subseteq GT_{\mathbb{Z}}(\mu)$ coincides with

$$\mathcal{K}(\mu; w_0\sigma) \cap GT_{\mathbb{Z}}(\mu).$$

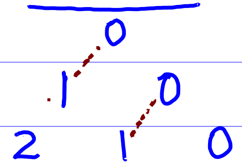
"SSYT(μ)"



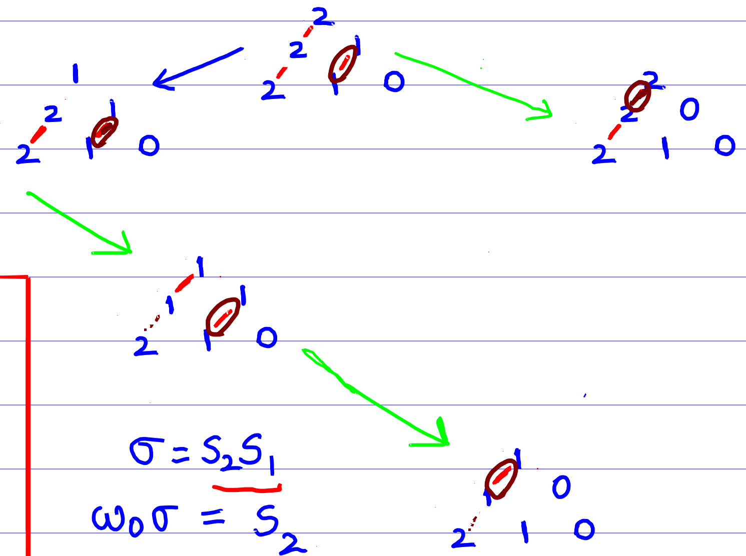
$\sigma = s_1 s_2$

$w_0\sigma = s_1$

Left over



$=$



$\sigma = s_2 s_1$

$w_0\sigma = s_2$

$\sigma = s_1 s_2$ $\sigma = s_2 s_1$

- Kiritchenko - Smirnov - Timorin : 2011
    - character of Demazure module in terms of dual Kogan faces
    - Regular dominant μ
- Postnikov - Stanley: 2004
    - When σ is 312-avoiding (or 231-avoiding), they obtained the Demazure character in terms of the dual Kogan face.
    - σ avoids 312 $\Longleftrightarrow$ $w_0\sigma$ avoids 132

      $\Longleftrightarrow$ $\exists!$ reduced F such that $w(F) = w_0\sigma$

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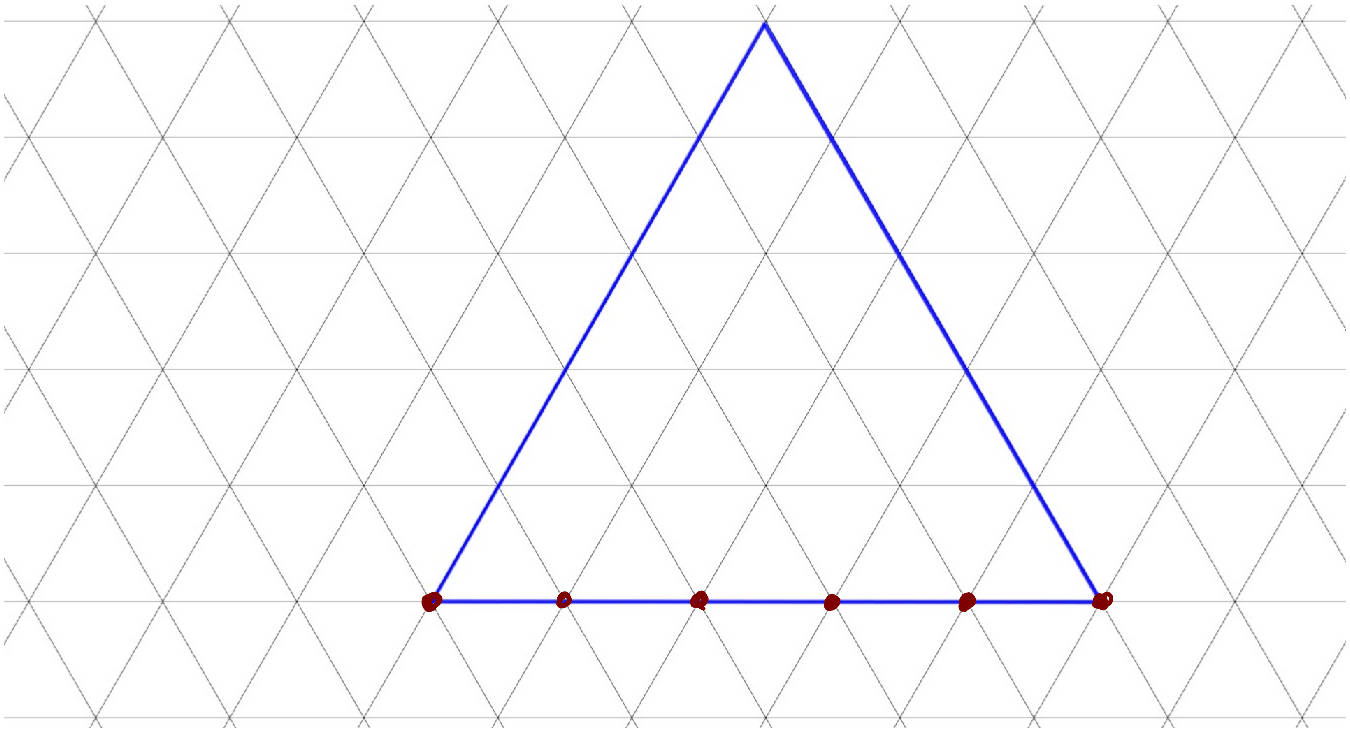
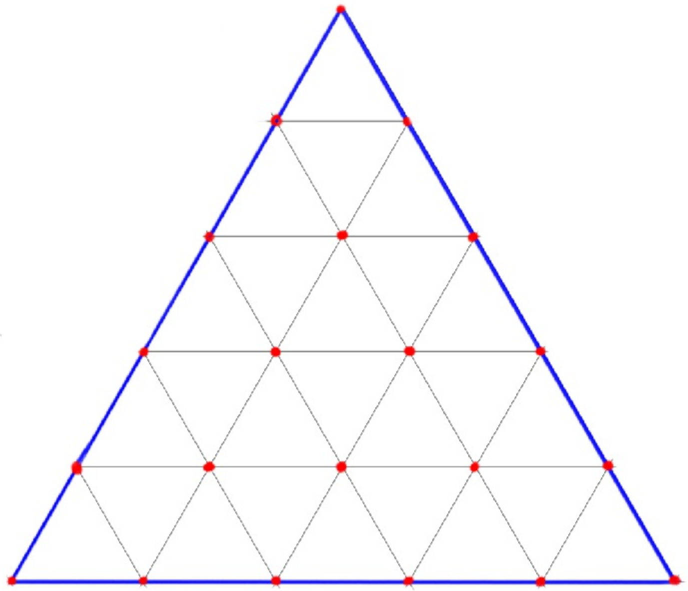


BIG HIVE TRIANGLE $(n \overset{eg}{=} 5)$

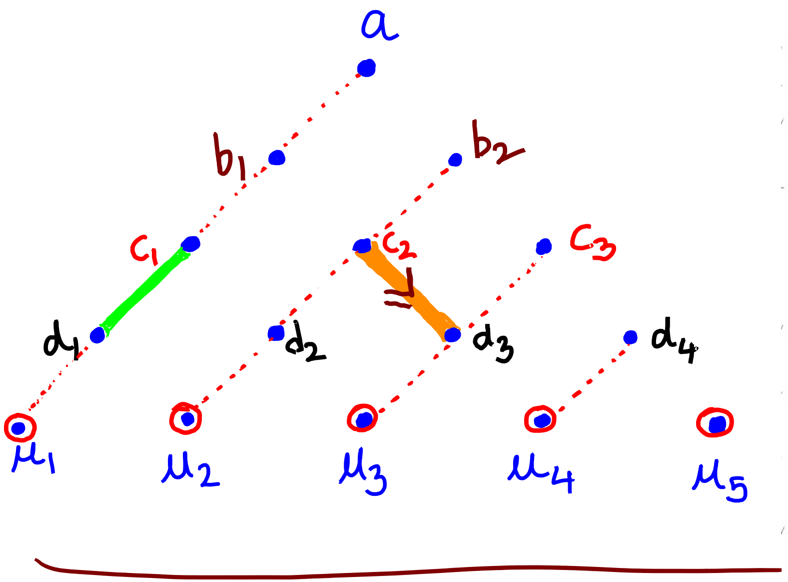


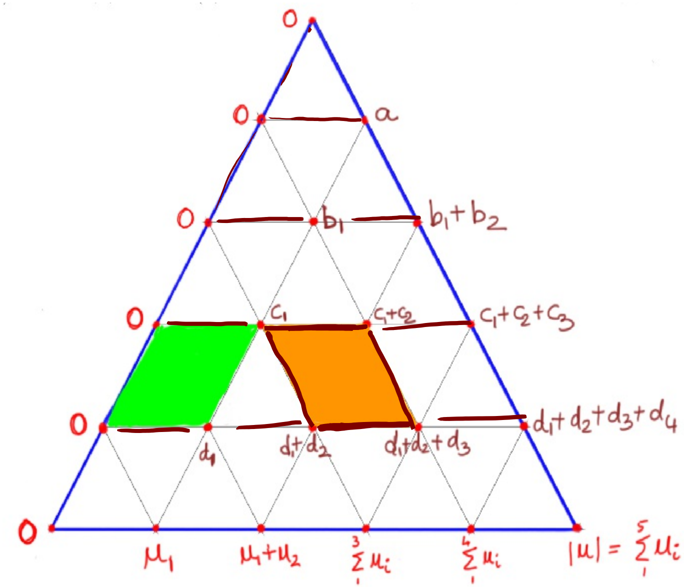
Label each red dot with a real number

"Hive"

(more conditions later)

$\in \mathbb{R}^{\frac{n(n+1)}{2}}$

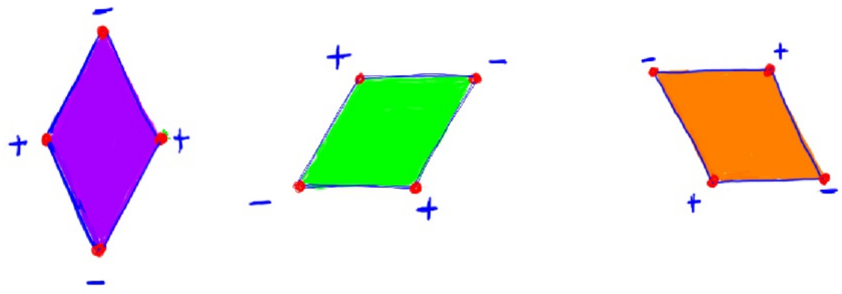




$$P \supseteq GT(\mu)$$

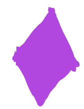
$P \xrightarrow{\text{row-wise partial sums}}$ Hive w/ boundaries 0/μ & satisfying  and  rhombus inequalities.

Rhombus inequalities :



$$\text{OBTUSE} - \text{ACUTE} \geq 0$$

Three kinds of rhombi :

for $GT(\mu)$ ignore vertical rhombus  inequalities.

$$\boxed{\begin{gathered} GT_{\mathbb{Z}}(\mu) \\ \wr\! \mid \\ SSYT(\mu) \end{gathered}}$$

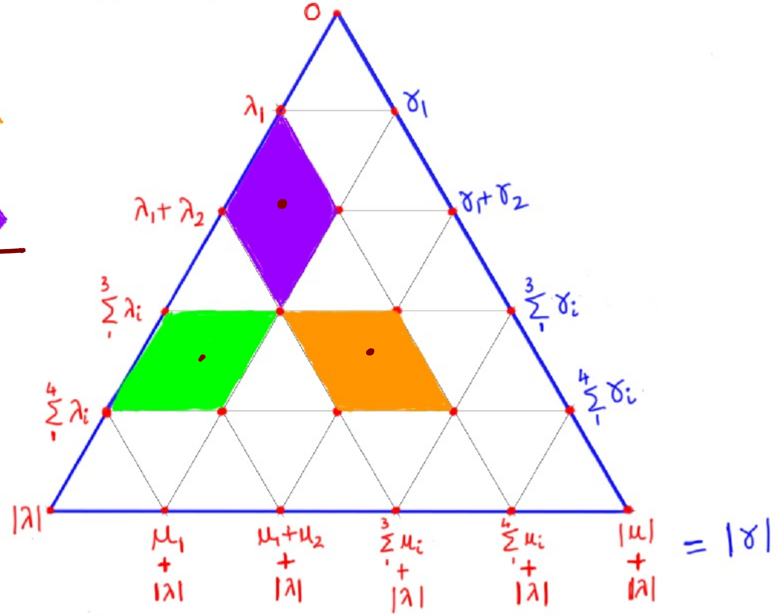
LR coeffs

SSYT

shape $= \mu$

weight $= \gamma - \lambda$

λ-shifted dominant



$Hive_{\mathbb{R}}(\lambda, \mu, \gamma) :=$ {$\mathbb{R}$-hives with given boundary labels & satisfying all 3 types of rhombus inequalities}

$$\boxed{c^{\gamma}_{\lambda\mu} = \left| Hive_{\mathbb{Z}}(\lambda, \mu, \gamma) \right|}$$

4. Key steps in the Knutson-Tao proof of saturation via hives:

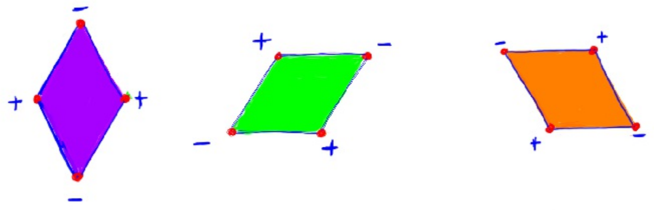
A If $c^{k\gamma}_{k\lambda,\, k\mu} > 0$ for some $k \geq 1$, then

$$\mathrm{Hive}_{\mathbb{Z}}(k\lambda, k\mu, k\gamma) \neq \emptyset$$

$$\Rightarrow \underline{\underline{\mathrm{Hive}_{\mathbb{R}}(\lambda, \mu, \gamma)}} = \frac{1}{k}\, \underline{\mathrm{Hive}_{\mathbb{R}}(k\lambda, k\mu, k\gamma)} \neq \emptyset$$

B <u>Proposition</u>: If H is a <u>vertex</u> of the hive polytope $\mathrm{Hive}_{\mathbb{R}}(\lambda, \mu, \gamma)$ such that the <u>flat rhombi</u> of H are <u>non-overlapping</u>, then $\boxed{H \in \mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \gamma).}$

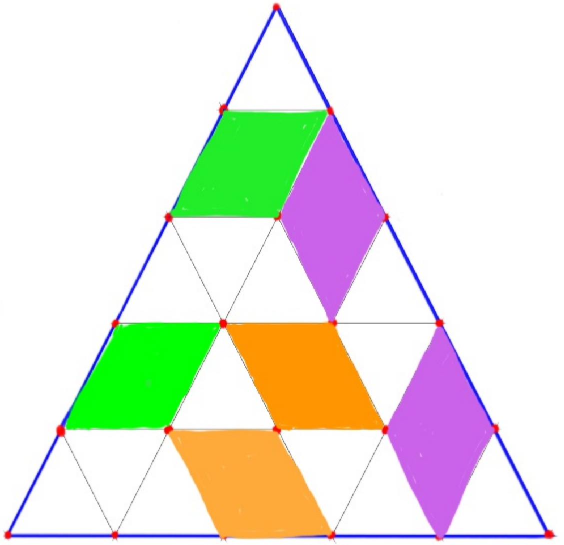
A flat rhombus



OBTUSE − ACUTE = 0

Flat rhombus

Inequality is satisfied by equality.



Non-overlapping rhombi.

(C) <u>Proposition:</u>

Such a vertex $H \in \mathrm{Hive}_{\mathbb{R}}(\lambda, \mu, \gamma)$ exists if λ, μ, γ are regular dominant.

<u>Proof</u>: Let Γ be a generic positive linear functional

$$\underline{\underline{\Gamma}}(h) = \sum_{v \in \{\text{red dots}\}} c_v \, h(v)$$

w/ $h(v)$ = label of h @ v

$\underline{\underline{c_v}} \in \underline{\underline{\mathbb{R}_{>0}}}$

Let H be the <u>unique</u> vertex of $\mathrm{Hive}_{\mathbb{R}}(\lambda, \mu, \gamma)$ which maximizes Γ.

Key property (by maximality)

- H has no increasable subsets.

A subset of interior hive vertices, whose H-labels can be simultaneously increased by some $\epsilon > 0$ to give another element $H' \in \mathrm{Hive}_{\mathbb{R}}(\lambda, \mu, \nu)$.

Prop: If $h \in \mathrm{Hive}_{\mathbb{R}}(\lambda, \mu, \nu)$ has no increasable subsets, then the flat rhombi of h are non-overlapping.

analysis of flatspaces

(λ, μ, ν regular) is needed

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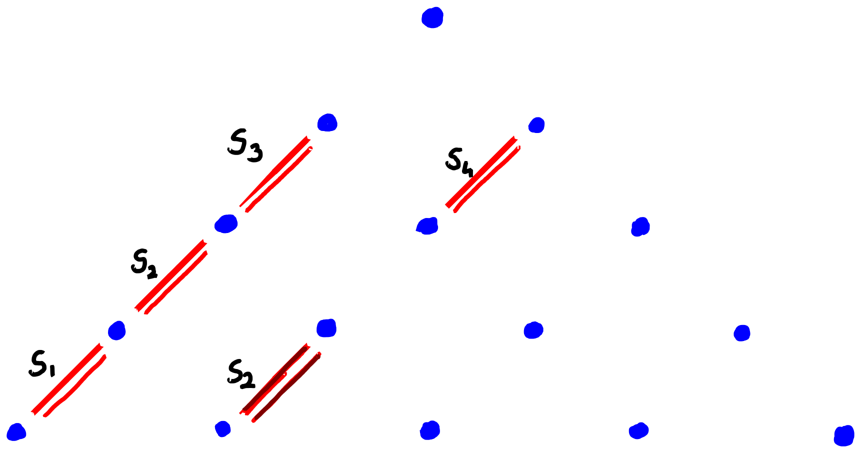
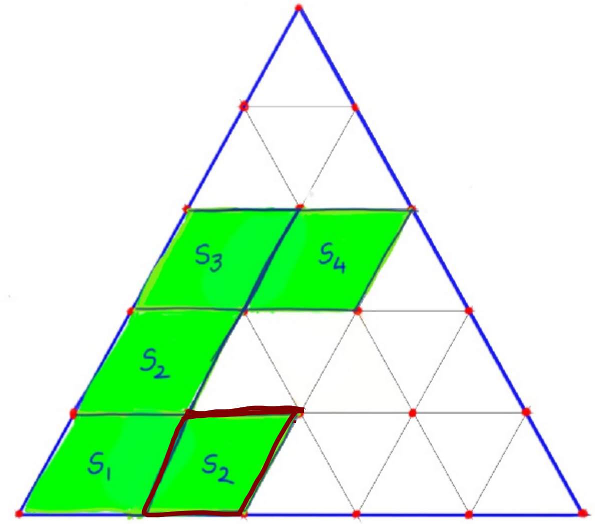
   b) $c_{\lambda\mu}^{\gamma}$ (b') $c_{\lambda\mu}^{\gamma}(\sigma)$

4. The key steps in Knutson-Tao's proof of the saturation conjecture for $c_{\lambda\mu}^{\gamma}$

5. Extension to $c_{\lambda\mu}^{\gamma}(\sigma)$: σ = 312- or 231-avoiding.





Dual Kogan face of GT polytope

Dual Kogan face of hive polytope : some subset of  rhombi are flat ($\geq \rightsquigarrow =$)

Let $\mathcal{KH}(b;\sigma) = \bigcup \mathcal{KH}_b(F)$ union over reduced F with $\omega(F) = \sigma$.

$b =$ boundary (λ, μ, γ)

Prop^n : $C_{\lambda\mu}^{\gamma}(\sigma) = \left| \mathcal{KH}_{\mathbb{Z}}(b\,;\,w_0\sigma) \right|$

Theorem (Fujita 2020): Let $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$ and $\sigma \in S_n$.

The set $Dem(\mu;\sigma) \subseteq GT_{\mathbb{Z}}(\mu)$ coincides with

$\mathcal{K}(\mu;\, w_0\sigma) \cap GT_{\mathbb{Z}}(\mu)$.

//

5. Saturation for $c^{\gamma}_{\lambda\mu}(\sigma)$ when σ is 312- or 231- avoiding

$\boxed{\text{FACT:}}$ σ 312- avoiding $\Longrightarrow$ $\underline{\underline{w_0\sigma}}$ is 132- avoiding //

(Kogan; Postnikov - Stanley)

$\Rightarrow \exists!\ F \subseteq \{$ rhombi $\}$ such that

$$\underline{\omega(F) = w_0\sigma}$$

Further, F has a special form !

σ is $\dot{3}\dot{1}\dot{2}$-avoiding

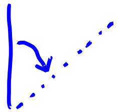
$i_1\ i_2\ \ldots\ i_n$

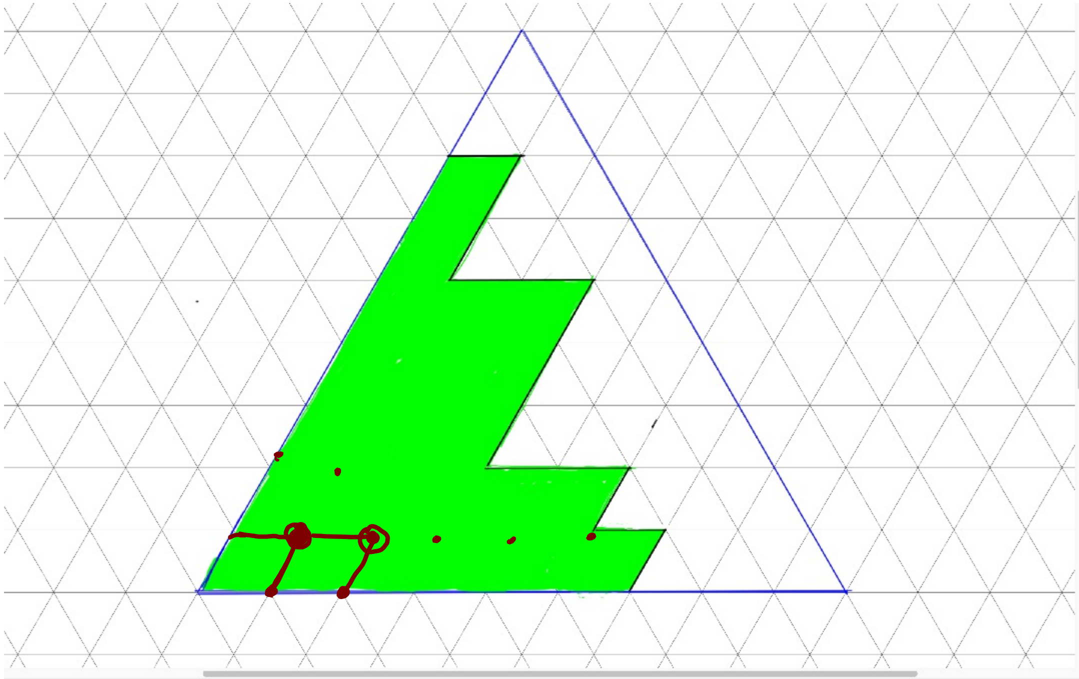
$\nexists$ c a b

c' a' b' st

$a < b < c$

F = Young diagram of a (French) partition subjected to a 30° compression





Dyck path view.

\# 132-avoiding perms in S_n

$$= \frac{1}{n+1}\binom{2n}{n}$$

(catalan number)

Key observation : Let $H \in$ the face F of $\mathrm{Hive}_{\mathbb{R}}(\lambda, \mu, \nu)$.

which maximizes a generic positive functional Γ on F.

Then, H has no increasable subsets relative to F (i.e) cannot increase any subset of labels by $\varepsilon > 0$ while staying within F.

But the special form of F ensures H has no increasable subsets, i.e, ↑ labels will result in leaving the hive polytope.

($\Rightarrow$ KT's argument applies ; H has non-overlapping flats ... QED) ∎