Generating functions for the powers in $\mathrm{GL}(n, q)$

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- If C is a conjugacy class of G, such that $C \subset G^M$, then we call C, a M^{th} power conjugacy class.
- In this talk, we will consider $G = \mathrm{GL}(n, q)$, which is the group of all invertible matrices over the finite field $\mathbb{F}_q$.

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- Can one count the number of M^{th} power conjugacy classes in $\mathrm{GL}(n,q)$?

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Results on powers in S_n

- For $r \geq 2$, consider $S_n^r = \{\pi^r | \pi \in S_n\}$ be the set of r^{th} power permutations in S_n. Let, $p_r(n) := \frac{|S_n^r|}{n!}$.

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- **J.Blum(1974)**: Using generating functions, he proved that $p_2(2n+1) = p_2(2n)$ for $n \geq 1$. Further, showed that $p_2(n) \sim K\sqrt{\frac{2}{\pi}} n^{-\frac{1}{2}}$, where $K = \prod_{k=1}^{\infty} \cosh(\frac{1}{2k})$.

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- **Bona et.al.(2001)**: Studied r^{th} powers in S_n for a prime r, and showed $p_r(n+1) = p_r(n)$ where $n \not\equiv -1(\text{mod } r)$. Moreover they showed that $p_r(n)$ is decreasing sequence in n, and $\lim_{n\to\infty} p_r(n) = 0$.

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- **N.Pouyanne (2002)** For $r \geq 2$,

$$p_r(n) \sim_{n\to\infty} \frac{\pi_r}{n^{1-\varphi(r)/r}}$$

where φ denotes the Euler's phi function and π_r, an explicit constant.

Statistical Properties in Finite Classical Groups

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- **C.Praeger, P.Neumann, Fulman (2005)**: extended the results to other finite classical groups like $\mathrm{Sp}(2n, q), \mathrm{U}(n, q)$ etc. using a generating function approach.

Certain related Questions

- Let $M \geq 2$. Suppose, $\mathrm{GL}(n,q)^M_{rg}, \mathrm{GL}(n,q)^M_{ss}, \mathrm{GL}(n,q)^M_{rs}$ be the set of M^{th} power regular, semisimple and, regular semisimple elements in $\mathrm{GL}(n,q)$.

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- Obtain generating function for the proportion of M^{th} power regular, semisimple, and regular semisimple elements in $\mathrm{GL}(n,q)$, and hence find estimates and asymptotics of these proportions?
- Enumerate the number of M^{th} power regular, semisimple and regular semisimple classes.

Some notations

- A Partition is a collection of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$, such that $\lambda_1 \geq \lambda_2 \geq \ldots$ and $|\lambda| = \sum_i \lambda_i$ is finite.

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- Let λ be a partition. For $i \geq 1$, let m_i denote the number of times i occur as a part in λ. Then, in power notation, we write $\lambda = 1^{m_1} 2^{m_2} \ldots$. Ex- $(3, 1, 1, 1) \vdash 6$ which is written as $1^3.3 \vdash 6$ in power notation.

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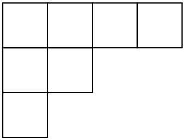
- Let Λ denote the set of all partitions, which also includes the empty partition of 0.

Some notations

- The Young Diagram corresponding to a partition is an arrangement of square boxes in rows and columns, where the i^{th} row consists of λ_i number of boxes. Example- The Young diagram of the partition $(4, 2, 1) \vdash 7$ is,



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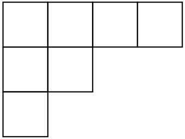

- The conjugate transpose of a partition $\lambda \vdash n$, denoted by λ' is also a partition of n obtained by transposing the rows and columns of the Young diagram of λ.

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  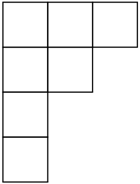

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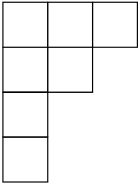

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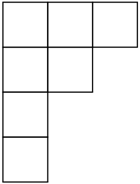

  Thus $\lambda^{'} = (3,2,1,1) \vdash 7$.
- Let Φ denote the number of monic non-constant irreducible polynomials over $\mathbb{F}_q$ except the linear polynomial x.
- Let $N(q,d)$ denote the number of polynomial of degree d in Φ.

$$N(q,d) = \frac{1}{d} \sum_{r|d} \mu(r)(q^{d/r} - 1)$$

  where μ is the Möbius function.

A Combinatorial parametrization of the conjugacy classes in GL(n, q)

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- Consider a map from $\Phi \to \Lambda$. Such a function, attaches to each $f \in \Phi$, a partition λ of $|\lambda|$. We say, $f \mapsto \lambda_f \vdash |\lambda_f|$

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- The conjugacy classes in $\mathrm{GL}(n, q)$ are in one-one correspondence with functions from $\Phi \to \Lambda$ satisfying the relation $\sum_{f \in \Phi} \deg(f)|\lambda_f| = n$.
- Let C be a conjugacy class of $\mathrm{GL}(n, q)$, and $\alpha \in C$. Recall that the $\mathbb{F}_q[x]$-module V_α has the decomposition,

$$V_\alpha = V \cong N_1 \oplus N_2 \oplus \ldots \oplus N_t$$

where N_i is the f_i-primary component of V, where f_i is a monic, non-constant, irreducible polynomial. Thus,

$$N_i = \frac{\mathbb{F}_q[x]}{(f_i(x)^{\lambda_{i_1}})} \oplus \frac{\mathbb{F}_q[x]}{(f_i(x)^{\lambda_{i_2}})} \oplus \cdots \oplus \frac{\mathbb{F}_q[x]}{(f_i(x)^{\lambda_{i_r}})}$$

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- Thus, mapping f_i to the partition $(\lambda_{i_1}, \lambda_{i_2}, \ldots) = \lambda_{f_i}$, for each $i = 1, 2, \ldots, t$ and mapping all other polynomials in Φ to the empty partition of 0, we get a function from $\Phi \to \Lambda$ satisfying $\sum_{f \in \Phi} \deg(f)|\lambda_f| = n$(since, $\mathrm{Dim}(N_i) = \deg(f_i)|\lambda_{f_i}|$ for all $1 \leq i \leq t$).

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- Conversely a function $\Phi \to \Lambda$ satisfying the condition $\sum_{f \in \Phi} \deg(f)|\lambda_f| = n$ defines a conjugacy class, say C, in $\mathrm{GL}(n, q)$ uniquely, where each pair (f, λ_f)(such that $|\lambda_f| > 0$), defines the f-primary component of an element in C. This correspondence is bijective since a conjugacy class in $\mathrm{GL}(n, q)$ is uniquely defined by a collection of admissible elementary divisors.

Cycle index in $\mathrm{GL}(n, q)$

- Let $x_{f,\lambda}$ be a variable associated to a pair (f, λ), where f is a monic non-constant irreducible polynomial and λ a partition. The **cycle index** is defined to be

$$Z_{\mathrm{GL}(n,q)} = \frac{1}{|\mathrm{GL}(n,q)|} \sum_{\alpha \in \mathrm{GL}(n,q)} \prod_{\substack{f \in \Phi \\ |\lambda_f(\alpha)|>0}} x_{f,\lambda_f(\alpha)}.$$

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- Consider a monomial of the form $x_{f_1,\lambda_{f_1}} x_{f_2,\lambda_{f_2}} \dots x_{f_l,\lambda_{f_l}}$. Suppose, $\sum_{i=1}^{l} \deg(f_i)|\lambda_{f_i}| = n$.

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- Consider a monomial of the form $x_{f_1,\lambda_{f_1}} x_{f_2,\lambda_{f_2}} \ldots x_{f_l,\lambda_{f_l}}$. Suppose, $\sum_{i=1}^{l} \deg(f_i)|\lambda_{f_i}| = n$.
- The coefficient of the above monomial is $\frac{|Cl(\alpha)|}{|\mathrm{GL}(n,q)|}$, where α is such that, the combinatorial data of $Cl(\alpha)$ is given by the function $\Phi \to \Lambda$ defined by $f_i \mapsto \lambda_{f_i}$ for $1 \leq i \leq l$.

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- Thus, the coefficient is equal to $\frac{1}{|\mathcal{Z}(\alpha)|}$, where $\mathcal{Z}(\alpha)$ denote the centralizer of α in $\mathrm{GL}(n, q)$.

Cycle index generating function

The fact that, for $\alpha \in \mathrm{GL}(n, q)$,

$$|\mathcal{Z}(\alpha)| = \prod_{\substack{f \in \Phi \\ |\lambda_f(\alpha)|>0}} q^{\deg(f).\sum_j \lambda_{i_j}'^2} \prod_{t \geq 1} \left(\frac{1}{q^{\deg f}}\right)_{m_t(\lambda_f)}$$

where, $\left(\frac{u}{q}\right)_i = (1 - \frac{u}{q})(1 - \frac{u}{q^2}) \ldots (1 - \frac{u}{q^i})$, gives the factorization,

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Theorem 3.1 (Cycle index generating function)

$$1 + \sum_{n=1}^{\infty} Z_{GL(n,q)} u^n =$$

$$\prod_{f \in \Phi} \left(1 + \sum_{j \geq 1} \sum_{\lambda \vdash j} x_{f,\lambda} \frac{u^{j.\deg(f)}}{q^{\deg(f).\sum_i (\lambda_i')^2} \prod_{t \geq 1} \left(\frac{1}{q^{\deg(f)}}\right)_{m_t(\lambda)}} \right)$$

Generating functions for the proportion of regular, semisimple, and regular semisimple matrices

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Therefore, in terms of the combinatorial parametrization of an element (or, the conjugacy class in which it is present), we have,

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Therefore, in terms of the combinatorial parametrization of an element (or, the conjugacy class in which it is present), we have,

1. α is semisimple $\iff \lambda_f(\alpha) = (1, 1, \ldots, 1) \vdash |\lambda_f(\alpha)|$, for each $f \in \Phi$ such that $|\lambda_f(\alpha)| > 0$.
2. α is regular $\iff \lambda_f(\alpha) = (|\lambda_f(\alpha)|) \vdash |\lambda_f(\alpha)|$, for each $f \in \Phi$ such that $|\lambda_f(\alpha)| > 0$.
3. α is regular semisimple $\iff |\lambda_f(\alpha)| = 1$, for each $f \in \Phi$ such that $|\lambda_f(\alpha)| > 0$.

Generating functions for the proportion of regular, semisimple, and regular semisimple matrices

1. The generating function for the regular semisimple elements is,

$$1 + \sum_{n=1}^{\infty} \frac{|\mathrm{GL}(n,q)_{rs}|}{|\mathrm{GL}(n,q)|} u^n = \prod_{d \geq 1} \left(1 + \frac{u^d}{q^d - 1}\right)^{N(q,d)}$$

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2. The generating function for the regular elements is,

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M-power polynomials

- Let $M \geq 2$ be an integer. For a polynomial, $f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0 \in \mathbb{F}_q[x]$, we denote the composed polynomial,

$$f(x^M) = x^{Md} + a_{d-1}x^{M(d-1)} + \ldots + a_1 x^M + a_0.$$

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Definition 4.1 (M-power polynomial)

A non-constant, irreducible, monic polynomial $f(x) \in \mathbb{F}_q[x]$ is said to be an **M-power polynomial** if $f(x^M)$ has an irreducible factor of degree $\deg(f)$. In general, a non-constant, monic polynomial f is said to be an **M-power polynomial** if each irreducible factor of f is an M-power polynomial.

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- Let Φ^M be the set all $f \in \Phi$ which are M-power polynomials.

M-power polynomials

- Let $N_M(q, d)$ denote the number of monic non-constant M-power polynomials of degree d except x.

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Suppose $(M,q)=1$. Suppose $M=r^a$, where r is a prime. Suppose $f(x)$ is an irreducible polynomial of degree d over $\mathbb{F}_q$. Then either of the two cases occur:

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1. *the polynomial $f(x^M)$ has an irreducible factor of degree d, that is, f is an M-power polynomial.*
2. *the polynomial $f(x^M)$ factors as a product of r^{a-i} irreducible polynomials each of degree dr^i for some $1 \leq i \leq a$.*

How does a M^{th} power element in $\mathrm{GL}(n, q)$ look like?

A Notation - Suppose $(M, q) = 1$ and $M = r^a$, where r is a prime. For $1 \leq i \leq a$, let $\Phi_{M,i}$ be the set of all $f \in \Phi$ such that $f(x^M)$ factors as a product irreducible polynomials each of degree equal to $r^i \deg(f)$. We have,

$$\Phi = \Phi^M \cup \cup_{i=1}^{a} \Phi_{M,i}$$

Theorem 4.3

Let $M = r^a$, where r is a prime and $(q, M) = 1$. Let $\alpha \in GL(n, q)$. Then, $X^M = \alpha$ has a solution in $GL(n, q)$ if and only if for each $f \in \Phi$ such that $|\lambda_f(\alpha)| > 0$, one of the following holds:

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1. *$f \in \Phi^M$, that is, f is M-power.*
2. *$f \in \Phi_{M,b}$ for some b, $1 \leq b \leq a$, and $r^b \mid m_j(\lambda_f)$ for all $j \geq 1$.*

How does a M^{th} power element in GL(n, q) look like?

Corollary 4.4

Let $M = r^a$ where r is a prime and $(q, M) = 1$. Let $\alpha \in GL(n, q)$ be semisimple. Then, $X^M = \alpha$ has a solution in $GL(n, q)$ if and only if for each $f \in \Phi$ such that $|\lambda_f(\alpha)| > 0$, one of the following holds:

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Proposition 4.5

Suppose $M \geq 2$ be an integer and $(q, M) = 1$. Let $\alpha \in GL(n, q)$ be a regular (or, regular semisimple) element . Then, α is a M^{th} power iff f is a M-power polynomial, for all f such that $|\lambda_f(\alpha)| > 0$.

Sketch of Proof.

Step 1: Suppose $\alpha \in \mathrm{GL}(n, q)$ such that there exists a single polynomial $f \in \Phi$ of degree k, such that $|\lambda_f| > 0$. Let $\lambda_f = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. We get a combinatorial parametrization of the matrix α^M.

It turns out that the combinatorial data of α^M consists of a single polynomial f_M of degree $d (\leq k)$, which is the minimal polynomial of the M^{th} power of a root of f, and the partition λ_{f_M} is given by

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{\frac{k}{d} \text{ times}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{\frac{k}{d} \text{ times}}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{\frac{k}{d} \text{ times}})$$

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Step 3: Suppose $\beta^M = \alpha$. Thus, the combinatorial data of β^M must match with that of α. Matching these data we obtain the necessary condition. The sufficient condition is obtained by explicitly finding a $\beta \in \mathrm{GL}(n, q)$ such that $\beta^M = \alpha$. □

Theorem 4.6

Let $M \geq 2$ be an integer and $(q, M) = 1$. Then,

Generating functions for the powers in GL(n, q)

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1. *The generating function for the proportion of regular semisimple elements which are M^{th} powers in $GL(n, q)$ is*

$$1 + \sum_{n=1}^{\infty} \frac{|GL(n,q)_{rs}^{M}|}{|GL(n,q)|} u^n = \prod_{d \geq 1} \left(1 + \frac{u^d}{q^d - 1}\right)^{N_M(q,d)}.$$

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2. *The generating function for the proportion of regular elements which are M^{th} powers is*

$$1 + \sum_{n=1}^{\infty} \frac{|GL(n, q)_{rg}^{M}|}{|GL(n, q)|} u^n = \prod_{d \geq 1} \left(1 + \sum_{j=1}^{\infty} \frac{u^{jd}}{q^{(j-1)d}(q^d - 1)}\right)^{N_M(q,d)}$$

Generating functions for the powers in GL(n, q)

Theorem 4.7

Let $M \geq 2$ be a prime and $(q, M) = 1$. The generating function for the proportion of M^{th} power semisimple elements in $GL(n, q)$ is,

$$1+\sum_{n=1}^{\infty} \frac{|GL(n,q)_{ss}^{M}|}{|GL(n,q)|} u^n = \prod_{d \geq 1} \left(1 + \sum_{j=1}^{\infty} \frac{u^{jd}}{q^{\frac{j(j-1)d}{2}} \prod_{t=1}^{j} (q^{td} - 1)} \right)^{N_M(q,d)}$$

$$\times \prod_{d \geq 1} \left(1 + \sum_{j=1}^{\infty} \frac{u^{Mjd}}{q^{\frac{Mj(Mj-1)d}{2}} \prod_{t=1}^{Mj} (q^{td} - 1)} \right)^{\widehat{N}(q,d)}$$

where $\widehat{N}(q, d) = N(q, d) - N_M(q, d)$.

The M^{th}-power probability in lower ranks

Lemma 4.8 (Surjectivity)

Let $M \geq 2$ be a prime. Then the power map ω_M on $GL(n, q)$ is surjective iff $(M, q) = 1$ and $n < o(q)$, where $o(q)$ is the order of q in $(\mathbb{Z}/M\mathbb{Z})^\times$.

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Theorem 4.9

Let M be a prime. Assume $(M, q) = 1$, and let t denote the order of q in $(\mathbb{Z}/M\mathbb{Z})^\times$. Then,

$$\frac{|GL(n,q)^M|}{|GL(n,q)|} = \sum_{\substack{\lambda \vdash n \\ \lambda = 1^{m_1} \dots i^{m_i} \dots}} \frac{1}{M^{\pi_t(\lambda)}} \prod_i \frac{1}{m_i! i^{m_i}}$$

if $n < Mt$, where $\pi_t(\lambda)$ denotes the number of parts of $\lambda \vdash n$ divisible by t.

The M^{th}-power probability in lower ranks

Example 4.10

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