A Pieri rule for key polynomials

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Outline

1. Polynomials
   - Schur polynomials
   - Key polynomials
   - Combinatorial equivalence
2. Pieri rules
   - Inserting boxes into tableaux
   - Inserting bubbles into diagrams
   - RSK on diagrams
3. Positivity
   - Top and bottom insertion
   - Schubert polynomials
   - Schubert Pieri rule

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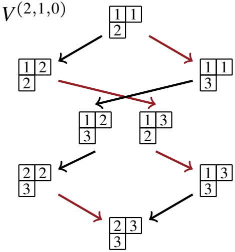
Characters of irreducible GL_n modules

Partitions are nonincreasing sequences of nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$.

Definition (Schur 1901)

Schur modules V^λ indexed by partitions are the irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$. Schur polynomials are their characters, $\mathrm{char}(V^\lambda) = s_\lambda(x_1, \ldots, x_n)$



$$V^\mu \otimes V^\nu \cong \bigoplus_\lambda \left(V^\lambda\right)^{\oplus c^\lambda_{\mu,\nu}}$$

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$$

Example (Schur modules)

- $V^{(m)} = \mathrm{Sym}^m(\mathbb{C})$ is symmetric product
- $V^{(1,\ldots,1)} = \bigwedge^m(\mathbb{C})$ is exterior product

The irreducible characters are a basis of the ring of symmetric polynomials, indexed by partitions.

Example (Schur polynomials)

- $s_{(m)}(x_1, \ldots, x_n) = \sum_{a_1+\cdots+a_n=m} x_1^{a_1} \cdots x_n^{a_n}$
- $s_{(1,\ldots,1)}(x_1, \ldots, x_n) = \sum_{i_1<\cdots<i_m} x_{i_1} \cdots x_{i_m}$

Semistandard Young tableaux

Definition

A semistandard Young tableau is a map $T : \lambda \to \mathbb{N}$ s.t.

- $T(c) \leq T(d) \leq n$ if c, d same row with c left of d;
- $T(c) < T(d)$ if c, d same column with c above of d.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 4 \\ \hline 3 & 3 & 4 & 4 & \\ \cline{1-4} 4 & 5 & 5 & 5 & \\ \cline{1-4} 5 & & & & \\ \cline{1-1} \end{array}$$

$$x^{\mathbf{wt}(T)} = x_1^2\, x_2\, x_3^3\, x_4^4\, x_5^4$$

Example (Enumerating $\mathrm{SSYT}_3(2,1)$ to compute $s_{(2,1)}(x_1, x_2, x_3)$)

$$\mathrm{SSYT}_3(2,1) = \begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\ \hline 3\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\ \hline 3\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\ \hline 3\\ \cline{1-1}\end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + 2x_1x_2x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

Theorem (Schur polynomial indexed by the partition λ)

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} x_1^{\mathbf{wt}(T)_1} \cdots x_n^{\mathbf{wt}(T)_n}$$

Here the weight of an SSYT T is the weak composition $\mathbf{wt}(T)$

$$\mathbf{wt}(T)_i = \#\ i\text{'s in } T$$

Schur polynomials $\{s_\lambda\}_{\ell(\lambda) \leq n}$ form a basis of $\mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n} \subset \mathbb{Z}[x_1, x_2, \ldots, x_n]$.

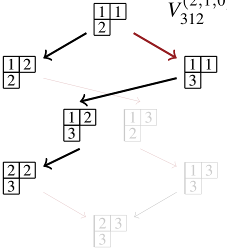
Characters of GL_n Demazure modules

Finite dimensional irred. representations of $\mathfrak{g}$ decompose into weight spaces $V^\lambda = \bigoplus_{\mathbf{a}} V^\lambda_{\mathbf{a}}$.
The Weyl group acts on extremal weight spaces $\{V^\lambda_{w\cdot\lambda} \mid w \in W\}$, which are all 1-dimensional.

Definition (Demazure 1974)

The Demazure module V^λ_w is the $\mathfrak{b}$-submodule of the irreducible $\mathfrak{g}$-representation V^λ generated by the extremal weight space $V^\lambda_{w\cdot\lambda}$. Demazure characters are $\mathrm{char}(V^\lambda_w) = \kappa_{w\cdot\lambda}$

$V^{(2,1,0)}_{312}$



$$\{u_\lambda\} = V^\lambda_{123} \subset V^\lambda_{132} \subset V^\lambda_{312} \subset V^\lambda_{321} = V^\lambda$$

Key polynomials are a basis for $\mathbb{Z}[x_1, \ldots, x_n]$ with negative structure constants.

Example (Demazure modules)

- $V^\lambda_{\mathrm{id}} = V^\lambda_\lambda$ is the 1-dim highest wt space
- $V^\lambda_{w_0} = V^\lambda_{\mathrm{rev}(\lambda)} = V^\lambda$ is the full module

For $\mathfrak{gl}_n$, index Demazure modules by

$$(w, \lambda) \mapsto w \cdot \lambda \qquad \mathbf{a} \mapsto (w_{\mathbf{a}}, \mathrm{sort}(\mathbf{a}))$$

where $w_{\mathbf{a}}$ is the shortest s.t. $w_{\mathbf{a}} \cdot \mathbf{a} = \lambda$.

Example (Key polynomials)

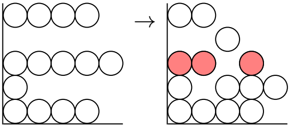
- $\kappa_{(\lambda_1,\ldots,\lambda_n)} = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$
- $\kappa_{(\lambda_n,\ldots,\lambda_1)} = s_\lambda(x_1, \ldots, x_n)$

Kohnert diagrams

Definition (Kohnert 1991)

A Kohnert diagram is a diagram obtained by a sequence of Kohnert moves on $\mathbf{a}$:

- select the rightmost cell c of a row
- move c to the first available position below
- jump over other cells in its way as needed.



$$x^{\mathbf{wt}(T)} = x_1^4\, x_2^4\, x_3^3\, x_4^1\, x_5^2$$

Example (Enumerating $\mathrm{KD}(0,2,1)$ to compute $\kappa_{(0,2,1)}$.)

$$\mathrm{KD}(0,2,1) =$$

$$\kappa_{(0,2,1)} = x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2$$

Theorem (Kohnert 1991)

$$\kappa_{\mathbf{a}} = \sum_{T \in \mathrm{KD}(\mathbf{a})} x_1^{\mathrm{wt}(T)_1} \cdots x_n^{\mathrm{wt}(T)_n}$$

Here the weight of a diagram T is the weak composition $\mathbf{wt}(T)$

$$\mathbf{wt}(T)_i = \#\text{ cells of } T \text{ in row } i$$

The key polynomials are not always symmetric. However $\{\kappa_{\mathbf{a}}\}$ are a basis of $\mathbb{Z}[x_1, x_2, \ldots, x_n]$.

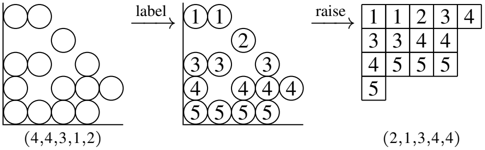
Young tableaux as Kohnert diagrams

Recall for λ a partition of length n, we have $\kappa_{(\lambda_n,\ldots,\lambda_1)} = s_\lambda(x_1,\ldots,x_n)$. Therefore we can relate the combinatorics of Kohnert diagrams and semistandard Young tableaux.

Definition (Assaf–Searles 2018)

Define a map φ on Kohnert diagrams by

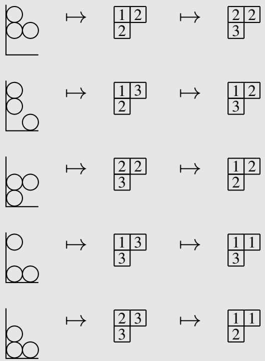
- label cells of row i with label $n-i+1$,
- raise cells to partition shape (same column order).



Theorem (Assaf–Searles 2018)

The map φ is a weight-reversing, injective map $\mathrm{KD}(\mathbf{a}) \hookrightarrow \mathrm{SSYT}_n(\mathrm{sort}(\mathbf{a}))$. Moreover, φ is a bijection if and only if $\mathbf{a}$ is weakly increasing, i.e. for $\mathbf{a} = (\lambda_n,\ldots,\lambda_1)$.

Example (The crystal map $\mathrm{KD}(0,2,1) \hookrightarrow \mathrm{SSYT}_3(2,1)$)



Notice T above lies in $\mathrm{KD}(4,1,5,0,4)$ and in $\mathrm{KD}(0,1,4,4,5)$ and in many other Kohnert sets.

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Adding boxes

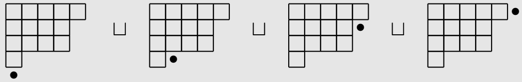
Since Schur functions are a basis of symmetric functions, their structure constants are well-defined. We know from representation theory they must be nonnegative.

Theorem (Pieri 1893)

For a partition λ,

$$s_\lambda s_{(1)} = \sum_{\substack{\mu \supset \lambda \\ |\mu/\lambda|=1}} s_\mu$$

Example (Computing the Schur expansion of $s_{(5,4,4,1)} s_{(1)}$)



$$s_{(5,4,4,1,1)} + s_{(5,4,4,2)} + s_{(5,5,4,1)} + s_{(6,4,4,1)}$$

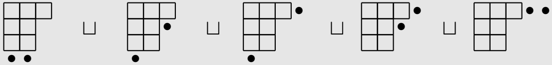
Pieri proved an elegant formula for the multiplicity-free expansion for a single row.

Theorem (Pieri 1893)

For a partition λ and $m > 0$,

$$s_\lambda s_{(m)} = \sum_{\substack{\mu \supset \lambda \\ \mu/\lambda \text{ hor. } m-\text{strip}}} s_\mu$$

Example (Computing the Schur expansion of $s_{(3,2,2)} s_{(2)}$)



$$s_{(3,2,2,2)} + s_{(3,3,2,1)} + s_{(4,2,2,1)} + s_{(4,3,2)} + s_{(5,2,2)}$$

We use the poset structure of Young's lattice defined on partitions by $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i.

RSK insertion

An elegant combinatorial proof of the Pieri rule for Schur polynomials uses the Robinson–Schensted–Knuth insertion algorithm on semistandard Young tableaux.

Definition (RSK Insertion)

To insert a letter x into $T \in \mathrm{SSYT}(\lambda)$,

- if $x \geq j$ for j in row r, append x to row r;
- else let j be the leftmost entry $j > x$,
- swap x, j and insert x into row $r + 1$.

Theorem (Schensted 1961)

RSK is a weight-preserving bijection

$$\mathrm{SSYT}_n(\lambda) \times \mathrm{SSYT}_n(1) \xrightarrow{\sim} \bigsqcup_{\substack{\mu \supset \lambda \\ |\mu/\lambda| = 1}} \mathrm{SSYT}_n(\mu)$$

1	1	2	**3**	4	← 2
3	3	4	4		
4	5	5	5		
5					

1	1	2	2	4	
3	3	**4**	4		← 3
4	5	5	5		
5					

1	1	2	2	4	
3	3	3	4		
4	**5**	5	5		← 4
5					

1	1	2	2	4	
3	3	3	4		
4	4	5	5		
5					← 5

1	1	2	2	4
3	3	3	4	
4	4	5	5	
5	5			

Row Bumping Lemma (Schensted 1961)

For $x \leq y$, the added box of $T \leftarrow x$ is strictly left of and weakly below the added box of $T \leftarrow y$.

Each term corresponds to a set of SSYT, and the sets in the formula are pairwise disjoint, so taking generating polynomials proves the Pieri formula for Schur polynomials.

Adding bubbles

Since $s_\lambda s_{(1)} = \sum s_{\lambda+\mathbf{e}_j}$ for certain j, we might try to write $\kappa_{\mathbf{a}}\kappa_{\mathbf{e}_k} = \sum \kappa_{\mathbf{a}+\mathbf{e}_j}$ for certain j.

$$\kappa_{(4,1,5,0,4)}\kappa_{(0,0,1,0,0)} = \underbrace{\kappa_{(4,2,5,0,4)}}_{\mathrm{KD}(4,2,5,0,4)} + \underbrace{\kappa_{(5,1,5,0,4)} + \kappa_{(4,5,5,0,1)} - \kappa_{(5,4,5,0,1)}}_{\mathrm{KD}(5,1,5,0,4)\cup\mathrm{KD}(4,5,5,0,1)} + \underbrace{\kappa_{(4,1,6,0,4)}}_{\mathrm{KD}(4,1,6,0,4)}$$

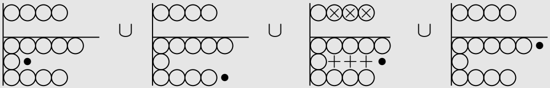
The row of added cell must be weakly below row k.

We can drop cells down to support the added cell.

Some sets are not disjoint:

$\mathrm{KD}(5,1,5,0,4)\cap\mathrm{KD}(4,5,5,0,1)$
$=\mathrm{KD}(5,4,5,0,1)$

Example (Computing the key expansion of $\kappa_{(4,1,5,0,4)}\kappa_{(0,0,1)}$)



$\mathrm{KD}(4,2,5,0,4) \;\cup\; \mathrm{KD}(5,1,5,0,4) \;\cup\; \mathrm{KD}(4,5,5,0,1) \;\cup\; \mathrm{KD}(4,1,6,0,4)$

This gives rise to a negative term in the key expansion when taking generating polynomials.

Theorem (Assaf–Quijada 2019+)

Given $\mathbf{a}$ *and* $k \le n$*, we have a weight-preserving bijection*

$$\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_k) \xrightarrow{\sim} \bigcup_{\substack{\mathbf{b}\preceq\mathbf{a}\\ 1\le j\le k}} \mathrm{KD}(\mathbf{b}+\mathbf{e}_j)$$

Unlike in the Schur case, the union on the right hand side is not disjoint. Thus the negative signs arise from inclusion–exclusion when taking intersections.

Horizontal strips

Theorem (Assaf–Quijada 2019+)

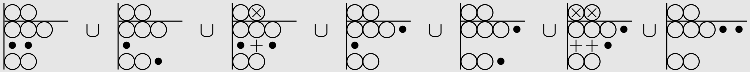
Given $\mathbf{a}$*,* $k \leq n$ *and* m*, we have a weight-preserving bijection*

$$\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(m\mathbf{e}_k) \xrightarrow{\sim} \bigcup_{\mathbf{b}} \mathrm{KD}(\mathbf{b})$$

over all $\mathbf{b}$ *obtainable from a sequence* $\mathbf{b}^{(0)}, \mathbf{b}^{(1)}, \cdots, \mathbf{b}^{(m)} = \mathbf{b}$ *satisfying:*

- $\mathbf{b}^{(0)} \preceq \mathbf{a}$ *and for* $i > 0$ *we have* $\mathbf{b}^{(i)} \preceq \mathbf{b}^{(i-1)} + \mathbf{e}_{j_i}$ *with* $j_i \leq k$*;*
- *the columns of the added cells* $\mathbf{b}^{(1)}_{j_1}, \ldots, \mathbf{b}^{(m)}_{j_m}$ *are all distinct.*

Example (Computing the key expansion of $\kappa_{(2,0,3,2)}\kappa_{(0,0,2)}$ using Kohnert diagrams)



$$\mathrm{KD}(2,2,3,2) \cup \mathrm{KD}(3,1,3,2) \cup \mathrm{KD}(2,3,3,1) \cup \mathrm{KD}(2,1,4,2) \cup \mathrm{KD}(3,0,4,2) \cup \mathrm{KD}(2,3,4) \cup \mathrm{KD}(2,0,5,2)$$

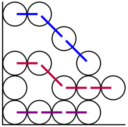
$$\kappa_{(2,2,3,2)} + \kappa_{(3,1,3,2)} + \kappa_{(2,3,3,1)} - \kappa_{(2,2,3,1)} + \kappa_{(2,1,4,2)} + \kappa_{(3,0,4,2)} + \kappa_{(2,3,4)} - \kappa_{(3,2,4)} + \kappa_{(2,0,5,2)}$$

Rectication of diagrams

Lemma (Assaf–Searles 2018)

A diagram T is a Kohnert diagram if and only if for every position (c, r)

$$\#\{(c-1, s) \in T \mid s \geq r\} \geq \#\{(c, s) \in T \mid s \geq r\}$$



Definition (Rectification)

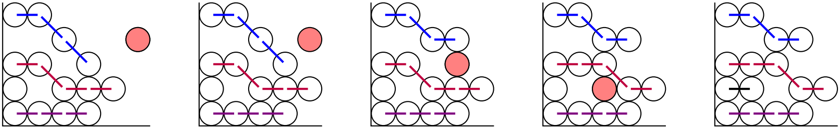
To insert a cell into a Kohnert diagram,

- if T is a Kohnert diagram, then stop;
- else x such that $T \setminus x$ Kohnert,
- move x left one position, rectify result.

Theorem (Assaf–Quijada 2019+)

For $T \in \mathrm{SSYT}$ and $\mathbb{D}(T)$ its diagram,

$$\mathbb{D}(\mathrm{RSK}(T, j)) = \mathrm{rectify}\left(\mathbb{D}(T) \sqcup (c+1, n+1-j)\right)$$



Rectification gives the bijection whenever $k \geq n$, but does not work for the general case.
In general, decompose $U \in \cup_{\mathbf{b} \preceq \mathbf{a}} \mathrm{KD}(\mathbf{b} + \mathbf{e}_j)$ and rectify a piece to recover $T \in \mathrm{KD}(\mathbf{a})$ and j.

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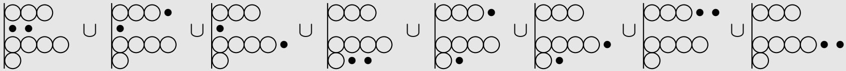
Inserting above row n or at row 1

Example (Computing the key expansion of $\kappa_{(1,4,0,3)}s_{(2)}(x_1,\cdots,x_4)$ using Kohnert diagrams)



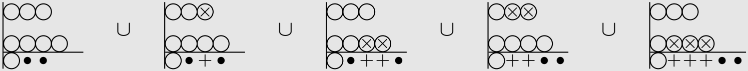
$\kappa_{(1,4,2,3)} + \kappa_{(1,4,1,4)} + \kappa_{(1,5,1,3)} + \kappa_{(3,4,0,3)} + \kappa_{(2,4,0,4)} + \kappa_{(2,5,0,3)} + \kappa_{(1,4,0,5)} + \kappa_{(1,6,0,3)}$

Theorem (Assaf–Quijada 2019+)

For $\ell(\mathbf{a}) \leq k$, *we have* $\kappa_{\mathbf{a}} \cdot s_{(m)}(x_1 \dots x_k) = \sum_{\substack{c_1 < \cdots < c_m \\ c_i - 1 \in \{a_1,\dots,a_n,c_{i-1}\}}} \kappa_{\mathbf{a}+\mathbf{e}_{j_1}+\cdots+\mathbf{e}_{j_m}}$, *where* j_i *is* $\max$ *s.t.* $b_{j_i} = c_i - 1$.

We have $\kappa_{\mathbf{a}} \cdot s_{(m)}(x_1) = \sum_{\substack{a_1 \leq c_1 < \cdots < c_m \\ c_i - 1 \in \{a_1,\dots,a_n,c_{i-1}\}}} \kappa_{\mathbf{a}^{(m)}}$, *where* $\mathbf{a}^{(0)} = \mathbf{a}$ *and* $\mathbf{a}^{(i)} = \mathbf{supp}^{(c_i,1)}_{\mathbf{a}^{(i-1)}} + \mathbf{e}_1$.

Example (Computing the key expansion of $\kappa_{(1,4,0,3)}s_{(2)}(x_1)$ using Kohnert diagrams)



$\kappa_{(3,4,0,3)} + \kappa_{(4,4,0,2)} + \kappa_{(5,2,0,3)} + \kappa_{(5,4,0,1)} + \kappa_{(6,1,0,3)}$

Insertion into the bottom row is not equivalent to rectification insertion that works at the top.

Vexillary permutations

Schubert polynomials $\mathfrak{S}_w$ of Lascoux and Schützenberger (1982) are polynomials representatives of Schubert classes for the complete flag manifold whose structure constants $c_{u,v}^{w}$ count points in triple intersections of Schubert varieties, and so $c_{u,v}^{w} \in \mathbb{N}$.

Definition

- A Grassmannian permutation w has a unique index k with $w_k > w_{k+1}$.
- A vexillary permutation w never has $w_b < w_a < w_d < w_c$ for $a < b < c < d$.

Theorem (Lascoux–Schützenberger)

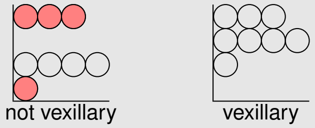
- *For w grassmannian, $\mathfrak{S}_w = s_\lambda(x_1 \ldots x_k)$ where $\lambda_{k-i+1} = w_i - i$, then*
- *For w vexillary, $\mathfrak{S}_w = \kappa_{\mathbf{L}(w)}$, where $\mathbf{L}(w)_i = \#\{j > i \mid w_i > w_j\}$.*

Definition (Macdonald 1991)

We have $\mathbf{a}$ is vexillary if and only if for $i < k$

- if $a_i > a_k$, then $\#\{i<j<k \mid a_j < a_k\} \leq a_i - a_k$
- if $a_i \leq a_k$, then $a_j \geq a_i$ for $i < j < k$

Example (Vexillary compositions)



Proposition (Macdonald 1991)

A permutation w is vexillary if and only if $\mathbf{L}(w)$ is vexillary, and in this case $\mathfrak{S}_w = \kappa_{\mathbf{L}(w)}$.

Vexillary Pieri rule

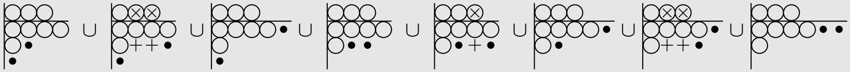
Theorem (Assaf–Quijada 2019+)

For w vexillary with $\mathbf{L}(w) = \mathbf{a}$ and $v((m),k)$ grassmannian corresponding to (m) and k,

$$\mathfrak{S}_w \mathfrak{S}_{v((m),k)} = \kappa_{\mathbf{a}} \cdot s_{(m)}(x_1,\ldots,x_k) = \sum_{\substack{\min(a_1,\ldots,a_k) < c_1 < \cdots < c_m \\ c_i - 1 \in \{a_1,\ldots,a_n,c_{i-1}\}}} \kappa_{\mathbf{a}^{(m)}}$$

where $\mathbf{a}^{(i)} = \mathbf{supp}_{\mathbf{a}^{(i-1)}}^{(c_i,r_i)} + \mathbf{e}_{r_i}$ for $r_i \leq k$ max such that $\mathbf{a}_r^{(i-1)} \leq c_i - 1$ (equal if possible).

Example (Computing the key expansion of $\kappa_{(0,1,4,3)} s_{(2)}(x_1,\ldots,x_3)$ using Kohnert diagrams)



$$\kappa_{(1,2,4,3)} + \kappa_{(1,4,4,1)} + \kappa_{(1,1,5,3)} + \kappa_{(0,3,4,3)} + \kappa_{(0,4,4,2)} + \kappa_{(0,2,5,3)} + \kappa_{(0,4,5,1)} + \kappa_{(0,1,6,3)}$$

We hope to extend our methods to prove a nonnegative rule for $\mathfrak{S}_u \mathfrak{S}_v$ with u, v vexillary...

References on arXiv

- Sami Assaf and Danjoseph Quijada, *A Pieri rule for Demazure characters of the general linear group*. arXiv:1908.08502
- Sami Assaf and Danjoseph Quijada, *A Pieri rule for key polynomials*, Sém. Lothar. Combin. **80B** (2018).
- Sami Assaf and Dominic Searles, *Kohnert tableaux and a lifting of quasi-Schur functions*, J. Combin. Theory Ser. A **156** (2018).
- Sami Assaf, *Demazure crystals for Kohnert polynomials*. arXiv:2002.07107

Thank you!