Total variation cutoff for random walks on some finite groups

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Brief overview of the presentation.

- Background and motivation.
- Our models.
- The warp-transpose top with random shuffle.
- Background theory to study the warp-transpose top with random shuffle.
- Spectrum of the transition matrix.
- Order of the mixing time.
- Main theorem on cutoff.

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- This has application in many subjects including mathematics, computer science, statistical physics and biology.
- This theory took a new direction in 1981, when Diaconis and Shahshahani introduced the use of non-commutative Fourier analysis techniques.
- Our models are mainly inspired by the *transpose top with random shuffle* studied by Flatto, Odlyzko and Wales in 1985.

Our models.

- Transpose top-2 with random shuffle: Random walk on the alternating group A_n generated by 3-cycles of the form $(i, n, n-1)$ and $(i, n-1, n)$. We have obtained sharp mixing time for this shuffle at $\left(n - \frac{3}{2}\right) \log n$.

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- Flip-transpose top with random shuffle: Random walk on the hyperoctahedral group B_n generated by the signed permutations of the form (i, n) and $(-i, n)$ for $1 \leq i \leq n$. This shuffle exhibits cutoff phenomenon with cutoff time $n \log n$. Moreover a similar random walk on the demihyperoctahedral group D_n generated by the signed permutations of the form (i, n) and $(-i, n)$ for $1 \leq i < n$ has a cutoff at $\left(n - \frac{1}{2}\right)\log n$.

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- Warp-transpose top with random shuffle: This is a generalization of the flip-transpose top with random shuffle, when number of orientations of each card is more than 2 (also it can depend on n).

A combinatorial description of $G \wr S_n$.

- G: Finite group, S_n: symmetric group on n letters. The complete monomial group $G \wr S_n$ is the *wreath product* of G with S_n. The elements of $G \wr S_n$ are $(n+1)$-tuples $(g_1, g_2, \ldots, g_n; \pi)$ where $g_i \in G$ and $\pi \in S_n$. The multiplication in $G \wr S_n$ is given by

$$(g_1, \ldots, g_n; \pi) \cdot (h_1, \ldots, h_n; \eta) = (g_1 h_{\pi^{-1}(1)}, \ldots, g_n h_{\pi^{-1}(n)}; \pi\eta).$$

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- $\mathcal{A}_n(G)$ denotes the set of all arrangements of n coloured cards in a row such that the colours of the cards are indexed by the set G.
- Elements of $G \wr S_n$ can be identified with the elements of $\mathcal{A}_n(G)$ as follows: $(g_1, \ldots, g_n; \pi) \in G \wr S_n$ is identified with the arrangement in $\mathcal{A}_n(G)$ such that the label of the i^{th} card is $\pi(i)$ and its colour is $g_{\pi(i)}$, for each $i \in \{1, \ldots, n\}$.

The warp-transpose top with random shuffle on $G_n \wr S_n$.

- Let $G_1 \subseteq G_2 \subseteq \cdots$ be a sequence of finite groups such that $|G_1| > 2$. Then the *warp-transpose top with random shuffle on* $G_n \wr S_n$ is a shuffle on $\mathcal{A}_n(G_n)$.
- For $x, y \in G$, updating the colour x using colour y means the colour x is being updated to colour $x \cdot y$.

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- For $x, y \in G$, updating the colour x using colour y means the colour x is being updated to colour $x \cdot y$.
- Given an arrangement of coloured cards in $\mathcal{A}_n(G_n)$, the shuffling scheme is the following: Choose a positive integer i uniformly from the set $\{1, 2, \ldots, n\}$ and choose a colour g uniformly from G_n, independent of the choice of the integer i.
    - If $i = n$: update the colour of the n^{th} card using colour g.
    - If $i < n$: first transpose the i^{th} and n^{th} cards. Then simultaneously update the colour of the n^{th} card using colour g and update the colour of the i^{th} card using colour g^{-1}.

Example: Typical transition for this shuffle on $\mathbb{Z}_3 \wr S_9$

Assume $\mathbb{Z}_3$, the additive group of integers modulo 3 consists of the colours red, green and blue such that red represents the identity element. Then a typical transition for the warp-transpose top with random shuffle on $\mathbb{Z}_3 \wr S_9$ is given as follows.

123459786

12345 6 789 $\longrightarrow$ 123459786

123459786

> Recall shuffling scheme: Choose a positive integer i uniformly from the set $\{1, 2, \ldots, n\}$ and choose a colour g uniformly from G_n, independent of the choice of the integer i.
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Aim

Study the mixing time for the warp-transpose top with random shuffle on $G_n \wr S_n$.

Discrete time Markov chain on finite state space.

- Markov chain: Sequence of random variables $\{X_0, X_1, \dots\}$ satisfying

$$\mathbb{P}\left(X_{t+1} = y | X_t = x, X_{t-1} = x_{t-1}, \dots, X_0 = x_0\right) = \mathbb{P}\left(X_{t+1} = y | X_t = x\right),$$

for all $x_0, \dots, x_{t-1}, x, y \in \Omega$ and $\mathbb{P}\left(X_t = x, X_{t-1} = x_{t-1}, \dots, X_0 = x_0\right) \neq 0$.

- Transition matrix: For any t, $\quad M = \left(\mathbb{P}\left(X_{t+1} = y | X_t = x\right)\right)_{x,y\in\Omega}$.
- If distribution of X_0 is Π_0 then distribution of X_t is $\Pi_0 M^t$.

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- If distribution of X_0 is Π_0 then distribution of X_t is $\Pi_0 M^t$.
- Irreducibility: For any $x, y \in \Omega$ there exists an integer t such that

$$\mathbb{P}(X_t = y \mid X_0 = x) = M^t(x, y) > 0.$$

- Stationary distribution: Probability distribution Π on Ω satisfying $\Pi M = \Pi$.
- Irreducible Markov chain always has a unique stationary distribution.

Convergence and mixing time.

- Period of state x $(\in \Omega)$: The greatest common divisor of $\tau(x)$,

$$\tau(x) := \{t \geq 1 \mid \mathbb{P}(X_t = x \mid X_0 = x) > 0\}.$$

- Aperiodicity: All states have period 1.
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- Total variation distance: $||\mu - \nu||_{\mathrm{TV}} := \sup_{A \subset \Omega} |\mu(A) - \nu(A)|$.
- Mixing time: The ε-mixing time $(0 < \varepsilon < 1)$ is defined as follows,

$$t_{\mathrm{mix}}(\varepsilon) := \min\{t : d(t) < \varepsilon\}, \text{ where } d(t) = \max_{x \in \Omega} ||M^t(x, \cdot) - \Pi||_{\mathrm{TV}}.$$

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- Cutoff phenomenon: Let $\{X^{(n)}\}_n$ be a sequence of Markov chains and $t^{(n)}_{\mathrm{mix}}(\varepsilon)$ denote the ε-mixing time for $X^{(n)}$. Then the sequence is said to satisfy the cutoff phenomenon if
$$\lim_{n\to\infty} t^{(n)}_{\mathrm{mix}}(\varepsilon) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{t^{(n)}_{\mathrm{mix}}(\varepsilon)}{t^{(n)}_{\mathrm{mix}}(1-\varepsilon)} = 1 \text{ for all } 0 < \varepsilon < 1.$$

Representation theory background

- Linear representation of a finite group: $\rho : G \xrightarrow{\text{Hom.}} GL(V)$, V is a finite-dimensional vector space and $GL(V)$ is the set of all invertible linear maps from V to itself. The vector space V is called a G-module in this case.
- Character: Trace of the matrix $\rho(g)$, denoted by $\chi^{\rho}(g)$.
- Trivial representation: $\mathbb{1} : G \longrightarrow \mathbb{C}^{\times}$ defined by $\mathbb{1}(g) = 1$ for all $g \in G$.
- Right regular representation: $R : G \xrightarrow{\text{Hom.}} GL\left(\mathbb{C}[G]\right)$ defined by,

$$g \mapsto \left(\sum_{h \in G} C_h h \mapsto \sum_{h \in G} C_h hg\right), \; C_h \in \mathbb{C}, \; g \in G.$$

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- Decomposition $\mathbb{C}[G]$ into irreducible G-modules:
$$\mathbb{C}[G] \cong \bigoplus_{\sigma \in \widehat{G}} \dim(V^{\sigma})\, V^{\sigma}.$$

Non-commutative Fourier analysis techniques.

- Convolution of probability measures on G: $\quad (p * q)(x) = \sum_{y \in G} p(xy^{-1})q(y).$
- Fourier transformation of p at a representation ρ: $\quad \widehat{p}(\rho) = \sum_{x \in G} p(x)\rho(x).$

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- Fourier transformation of p at a representation ρ: $\widehat{p}(\rho) = \sum_{x \in G} p(x)\rho(x)$.
- Random walk on G driven by p: Markov chain on G with transition probabilities $M_p(x, y) = \mathbb{P}\left(X_1 = y | X_0 = x\right) := p(x^{-1}y),\ x, y \in G$.

$$M_p = \left(M_p(x, y)\right)_{x,y \in G} = \left(\widehat{p}(R)\right)^T .$$

- The distribution after k^{th} transition will be p^{*k}, more precisely $\mathbb{P}\left(X_k = y | X_0 = x\right) = p^{*k}(x^{-1}y)$.
- Irreducible if and only if the support of p generates G. In that case the stationary distribution is the uniform distribution on G.

Our model as random walk on $G_n \wr S_n$.

- The warp-transpose top with random shuffle on $G_n \wr S_n$ is the random walk on $G_n \wr S_n$ driven by P, defined on $G_n \wr S_n$ by

$$P(x) = \begin{cases} \frac{1}{n|G_n|} & \text{if } x = (\mathrm{e}, \dots, \mathrm{e}, g; \mathrm{id}) \text{ for } g \in G_n, \\ \frac{1}{n|G_n|} & \text{if } x = (\mathrm{e}, \dots, \mathrm{e}, g^{-1}, \mathrm{e}, \dots, \mathrm{e}, g; (i, n)) \text{ for } g \in G_n,\ 1 \leq i < n, \\ 0 & \text{otherwise.} \end{cases}$$

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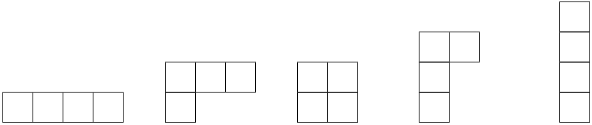
- Transition matrix: $\widehat{P}(R)$.
- Notation: $d_n(k) := ||P^{*k} - U||_{\mathrm{TV}}$.
- Eigenvalues of the transition matrix are useful to bound $d_n(k)$.

Definitions of useful combinatorial objects.

- Partition: $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ if $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ and $|\lambda| := \sum_{i=1}^{\ell} \lambda_i = n$.
- Young diagrams of shape λ:



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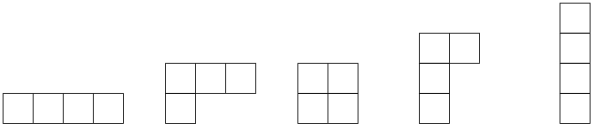

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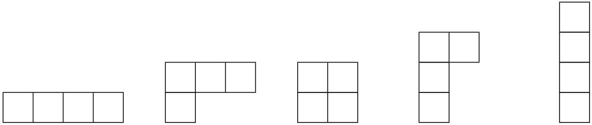

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- $\mathcal{Y}_n(\widehat{G})$: set of all Young G-diagram with n boxes, mappings μ from $\widehat{G}$ to the set of all Young diagrams such that $\sum_{\sigma \in \widehat{G}} |\mu(\sigma)| = n$.
- $\mathrm{tab}_G(n, \mu)$: set of all standard Young G-tableaux of shape μ.

- $T \in \mathrm{tab}_G(n,\mu)$. If i appear in the Young diagram $\mu(\sigma)$, $\sigma \in \widehat{G}$, then we write $r_T(i) = \sigma$. Also let $b_T(i)$ denote the box in $\mu(\sigma)$, with the number i resides and $c(b_T(i))$ denote the content of $b_T(i)$.

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- Take $G = \mathbb{Z}_{10}$ and assume $\widehat{\mathbb{Z}}_{10} = \{\sigma_1, \sigma_2, \ldots, \sigma_{10}\}$. Let $\mu \in \mathcal{Y}_{10}(\widehat{\mathbb{Z}}_{10})$ be such that

$$\mu(\sigma_1) = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\;,\quad \mu(\sigma_2) = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\;,\quad \mu(\sigma_8) = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\;,\quad \mu(\sigma_{10}) = \begin{array}{|c|}\hline \\ \hline\end{array}$$

and $\mu(\sigma_i) = \phi$ for all $i \in \{3,4,5,6,7,9\}$. Then for the element T of $\mathrm{tab}_{\mathbb{Z}_{10}}(10, \mu)$ given by

$$\mu(\sigma_1) \rightsquigarrow \begin{array}{|c|c|c|}\hline 4 & 6 & 9 \\ \hline 7 & 10 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\;,\quad \mu(\sigma_2) \rightsquigarrow \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\;,\quad \mu(\sigma_8) \rightsquigarrow \begin{array}{|c|}\hline 3 \\ \hline 8 \\ \hline\end{array}\;,\quad \mu(\sigma_{10}) \rightsquigarrow \begin{array}{|c|}\hline 5 \\ \hline\end{array}$$

and $\mu(\sigma_i) \rightsquigarrow \phi$ for $i \in \{3,4,5,6,7,9\}$, we have the following:

$$r_T(8) = \sigma_8,\ r_T(9) = \sigma_1 \text{ and } c(b_T(8)) = -1,\ c(b_T(9)) = 2.$$

Spectrum of the transition matrix.

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Theorem

For each $\mu \in \mathcal{Y}_n(\widehat{G}_n)$, let $\widehat{P}(R)\big|_{V^\mu}$ denote the restriction of $\widehat{P}(R)$ to the irreducible $G_n \wr S_n$-module V^μ. Also let χ^σ denote the character of the irreducible representation of G_n indexed by $\sigma (\in \widehat{G}_n)$. Then the eigenvalues of $\widehat{P}(R)\big|_{V^\mu}$ are given by,

$$\frac{1}{n \dim(W^{r_T(n)})} \left(c(b_T(n)) + \langle \chi^{r_T(n)}, \chi^{\mathbb{1}} \rangle \right),$$

with multiplicity $d_1^{m_1} \cdots d_t^{m_t}$, for each $T \in \mathrm{tab}_{G_n}(n, \mu)$.

Illustration.

- Consider $G_n = \mathbb{Z}_3$ for all n. Focus on the warp-transpose top with random shuffle on $\mathbb{Z}_3 \wr S_9$.

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- Let $\widehat{\mathbb{Z}_3} = \{\sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_1 = \mathbb{1}$ and $\mu \in \mathcal{Y}_9(\widehat{\mathbb{Z}_3})$ be such that,

$$\mu(\sigma_1) = \ydiagram{4}\ , \ \mu(\sigma_2) = \ydiagram{2,2} \ \text{ and } \mu(\sigma_3) = \ydiagram{1}\ .$$

Illustration.

- Consider $G_n = \mathbb{Z}_3$ for all n. Focus on the warp-transpose top with random shuffle on $\mathbb{Z}_3 \wr S_9$.
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- The eigenvalue for the standard Young $\mathbb{Z}_3$-tableaux of the form

$$\mu(\sigma_1) \rightsquigarrow \begin{array}{|c|c|c|c|}\hline * & * & * & 9 \\ \hline \end{array}\ , \ \mu(\sigma_2) \rightsquigarrow \begin{array}{|c|c|}\hline * & * \\ \hline * & * \\ \hline \end{array} \ \text{ and } \mu(\sigma_3) \rightsquigarrow \begin{array}{|c|}\hline * \\ \hline \end{array}$$

is $\frac{1}{9\times 1}(3+1) = \frac{4}{9}$. There are $\binom{8}{3\ 4\ 1} \times 2 = 560$ such standard Young $\mathbb{Z}_3$-tableaux. A typical example is given below

$$\mu(\sigma_1) \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 4 & 6 & 7 & 9 \\ \hline \end{array}\ , \ \mu(\sigma_2) \rightsquigarrow \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 8 \\ \hline \end{array} \ \text{ and } \mu(\sigma_3) \rightsquigarrow \begin{array}{|c|}\hline 2 \\ \hline \end{array}\ .$$

Recall: Let $\mu \in \mathcal{Y}_n(\widehat{G_n})$. For each $T \in \mathrm{tab}_{G_n}(n, \mu)$, eigenvalues of $\widehat{P}(R)\big|_{V^\mu}$ are given by,

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is $\frac{1}{9\times 1}(0+0) = 0$. There are $\binom{8}{4} \times 2 = 140$ such standard Young $\mathbb{Z}_3$-tableaux. A typical example is given below

$$\mu(\sigma_1) \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 2 & 4 & 6 & 7 \\ \hline\end{array}\ , \ \mu(\sigma_2) \rightsquigarrow \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 8 \\ \hline\end{array} \ \text{ and } \mu(\sigma_3) \rightsquigarrow \begin{array}{|c|}\hline 9 \\ \hline\end{array}\ .$$

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Proof idea of the theorem on spectrum.

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- A multiplicity free chain of finite groups $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \cdots \subseteq \mathcal{G}_n$. Irreducible $\mathcal{G}_n$-modules V has a canonical decomposition into irreducible $\mathcal{G}_1$-modules. This decomposition is known as Gelfand-Tsetlin decomposition and the irreducible $\mathcal{G}_1$-modules are known as the Gelfand-Tsetlin subspaces of V.
- GT_n is a maximal commuting subalgebra of $\mathbb{C}[\mathcal{G}_n]$ generated by $\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n$, where $\mathcal{Z}_i$ denotes the center of $\mathbb{C}[\mathcal{G}_i]$. GT_n is known as the Gelfand-Tsetlin subalgebra of $\mathbb{C}[\mathcal{G}_n]$. Elements of GT_n act by scalars on the Gelfand-Tsetlin subspaces of all irreducible representations of $\mathcal{G}_n$.

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- The chain $\{\mathrm{id}\} = S_1 \subseteq \cdots \subseteq S_n$ is multiplicity free. The Gelfand-Tsetlin subalgebra for this chain is generated by the Young-Jucys-Murphy elements $Y_1, \ldots, Y_n$. The Young-Jucys-Murphy elements are defined as follows: $Y_1 = 0$ and $Y_i = (1, i) + \cdots + (i-1, i) \in \mathbb{C}[S_i]$ for all $2 \leq i \leq n$. (Work of Vershik and Okounkov (2005)).

Upper bound for $||P^{*k} - U||_{\mathrm{TV}}$.

Theorem

For the random walk on $G_n \wr S_n$ driven by P, we have the following:

1. Let $C > 1$. Then for $k \geq n \log n + Cn \log(|G_n| - 1)$, we have

$$||P^{*k} - U||_{\mathrm{TV}} < \sqrt{\frac{1+2e}{2}}\, 2^{-C} + o(1).$$

2. Let $a > \frac{1}{2}$ and $k = n \log n + \frac{1}{2} n \log(|G_n| - 1) + an \log(|\widehat{G}_n| - 1)$. Then

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Key inequality: For all $k \geq n\log n$,

$$\begin{aligned} 4\,||P^{*k} - U||_{\mathrm{TV}}^2 \;<\; & \left(e^{n^2 e^{-\frac{2k}{n}}} - 1\right) + e\left(e^{n^2(|G_n|-1)e^{-\frac{2k}{n}}} - 1\right) \\ & + (|\widehat{G}_n| - 1)\left(e^{-\frac{2k}{n}} e^{n^2 e^{-\frac{2k}{n}}} + \frac{e}{n^2}\left(e^{n^2(|G_n|-1)e^{-\frac{2k}{n}}} - 1\right)\right). \end{aligned}$$

Recall: $t_{\mathrm{mix}}^{(n)}(\varepsilon) := \min\{k : ||P^{*k} - U||_{\mathrm{TV}} < \varepsilon\}$, $0 < \varepsilon < 1$.

Lower bound for $||P^{*k} - U||_{\mathrm{TV}}$

Theorem

Let $c \ll 0$. Then for $k = n\log n + cn$, we have

$$||P^{*k} - U||_{\mathrm{TV}} > 1 - \frac{2\left(2 + \frac{1}{|G_n|}\right)\left(e^{-c} + \frac{1}{|G_n|}\right) + o(1)(1 + e^{-c} + e^{-2c})}{\left(\frac{1}{|G_n|} + (1 + o(1))e^{-c}\right)^2}.$$

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Sketch of proof:

- Let $V = \mathbb{C}[G_n \times \{1, \ldots, n\}]$ be the complex vector space of all formal linear combinations of elements of $G_n \times \{1, \ldots, n\}$.
- Define the representation $\mathcal{R} : G_n \wr S_n \longrightarrow GL(V)$ on the basis elements of V by
$$\mathcal{R}(g_1, \ldots, g_n; \pi)\left((h, i)\right) = \left(g_{\pi(i)} h, \pi(i)\right).$$
- The random variable X counts the number of fixed points of the action of $\mathcal{R}$ i.e. X is the character $\chi^{\mathcal{R}}$ of $\mathcal{R}$.

Lower bound for $||P^{*k} - U||_{\text{TV}}$ (proof sketch continued).

Lemma

We have $E_U(X) = 1$ and for $k > 1$, the following hold:

$$E_k(X) \approx 1 + ((n-1)|G_n| - 1)\, e^{-\frac{k}{n}},$$

$$\mathrm{Var}_k(X) \approx |G_n| + \left((n-1)|G_n|^2 - |G_n|\right) e^{-\frac{k}{n}} - (n-1)|G_n|^2 e^{-\frac{2k}{n}} + c_n \left(1 + (|G_n| - 1)^2\right),$$

where $c_n \to 0$ as $n \to \infty$.

- Using $E_U(X) = 1$, Chebychev's and Markov's inequality inequality, we have
$$||P^{*k} - U||_{\text{TV}} \geq 1 - \frac{4\,\mathrm{Var}_k(X)}{(E_k(X))^2} - \frac{2}{E_k(X)}.$$
- Now use the values of $E_k(X)$ and $\mathrm{Var}_k(X)$. □

Cutoff.

Theorem

The warp-transpose top with random shuffle on $G_n \wr S_n$ exhibits cutoff phenomenon with cutoff time $n \log n$ if $|G_n| = o(n^{\delta})$ for all $\delta > 0$.

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Sketch of proof: (The condition implies cutoff)

- $|G_n| = o(n^{\delta})$ for all $\delta > 0 \implies \lim\limits_{n\to\infty} \frac{\log(|G_n|-1)}{\log n} = 0$. Let $\varepsilon \in (0, 1)$.

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Sketch of proof: (The condition implies cutoff)

- $|G_n| = o(n^\delta)$ for all $\delta > 0 \implies \lim\limits_{n\to\infty} \frac{\log(|G_n|-1)}{\log n} = 0$. Let $\varepsilon \in (0,1)$.
- For appropriate choice of a positive integer N_0, $C > 1$ and $c \ll 0$, we have

$$n \log n + cn \leq t_{\text{mix}}^{(n)}(\varepsilon) \leq n \log n + Cn \log(|G_n| - 1), \text{ for all } n \geq N_0$$

$$\implies \lim_{n\to\infty} \frac{t_{\text{mix}}^{(n)}(\varepsilon)}{n \log n} = 1 \implies \text{Cutoff at } n \log n.$$

Recall: $t_{\text{mix}}^{(n)}(\varepsilon) := \min\{k : ||P^{*k} - U||_{\text{TV}} < \varepsilon\}$, $0 < \varepsilon < 1$.

- $||P^{*k} - U||_{\text{TV}} < \sqrt{\frac{1+2e}{2}}\, 2^{-C} + o(1)$ for $k \geq n \log n + Cn \log(|G_n| - 1)$, $C > 1$.
- $||P^{*k} - U||_{\text{TV}} > 1 - \frac{2\left(2e^c + \frac{e^c}{|G_n|}\right)\left(1 + \frac{e^c}{|G_n|}\right) + o(1)(1+e^c+e^{2c})}{\left(\frac{e^c}{|G_n|} + (1+o(1))\right)^2}$ for $k = n \log n + cn$, $c \ll 0$.

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Thank You!