

Plethystic inversion and representations of the symmetric group

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2 & 9 July 2020

*Algebraic Combinatorics Seminar
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For the symmetric group it corresponds to forming representations of wreath products $S_m[S_n]$, and inducing up to S_{mn} .

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- A symmetric function f of homogeneous degree n is *Schur positive* if it is a nonnegative integer combination of Schur functions, i.e. if it corresponds to a true S_n -module.

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$(1 - p_1)^{-1} = \sum_{n \geq 0} p_1^n$ is the $Gl(V)$ -character on the full tensor algebra $T(V)$.

Plethysm and the symmetric group

If $F = \sum_{j \geq 1} f_j$ where each f_j is the Frobenius characteristic ch of an S_j -module W_j , and λ is the partition of n with m_i parts equal to i , then:

$$H_\lambda[F] = \text{ch} \left(\bigotimes_i \mathbf{1}_{S_{m_i}}[W_i] \right) \uparrow^{S_n} = \prod_i h_{m_i}[f_i]$$

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The modules are induced from the subgroup $\prod_i S_{m_i}[S_i]$, the normaliser in S_n of the direct product of m_i copies of S_i , $i \geq 1$.

The regular representation of a finite group G

$$\text{Reg}_G := 1 \uparrow_e^G = \sum_{\chi \text{ irreducible repn of } G} (\dim \chi) \chi.$$

For the symmetric group S_n :

Theorem (Reg0)

$$\text{Reg}_{S_n} = \sum_{\lambda \vdash n} f^\lambda \chi^\lambda,$$

with Frobenius characteristic

$$\text{ch } \text{Reg}_{S_n} := \sum_{\lambda \vdash n} f^\lambda s_\lambda,$$

where λ is an integer partition of n ,

$f^\lambda = |\{\text{standard Young tableaux of shape } \lambda\}|$, and

s_λ is the Schur function indexed by λ , so $s_\lambda = \text{ch } \chi^\lambda$.

The regular representation of S_n — (I)

Let C_n be the cyclic subgroup of S_n generated by the long cycle $\sigma = (1\ 2\ \dots\ n)$, let ω_n be a primitive n th root of unity. For $1 \leq k \leq n$, $\sigma \mapsto \omega_n^k$ yields a representation of C_n , and these are all the distinct irreducibles, so

$$\text{Reg}_{C_n} = \sum_{k=1}^n (\omega_n^k).$$

This gives a second decomposition of the regular representation of S_n :

The regular representation of S_n — (I) (continued)

Theorem (Reg1)

$$\text{Reg}_{S_n} = \sum_{k=1}^n (\omega_n^k) \uparrow_{C_n}^{S_n}$$

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Proof: Induce the decomposition of the regular representation of C_n up to S_n . (Or take tensor products over $\mathbb{C}C_n$ with $\mathbb{C}S_n$.)

Definition

$$\text{Lie}_n := \omega_n \uparrow_{C_n}^{S_n}.$$

Definition

$$\text{Conj}_n := \omega_n^n \uparrow_{C_n}^{S_n} = 1 \uparrow_{C_n}^{S_n}.$$

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$$\text{Lie}_n := \omega_n \uparrow_{C_n}^{S_n}.$$

Definition

$$\text{Conj}_n := \omega_n^n \uparrow_{C_n}^{S_n} = 1 \uparrow_{C_n}^{S_n}.$$

Conj_n is the permutation representation of S_n by conjugation on the class of n -cycles, since the stabiliser of an n -cycle is the cyclic group C_n .

Ramanujan sums

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Let $\ell_n^{(k)} := \text{ch } \omega_n^k \upharpoonright_{C_n}^{S_n}$, $1 \leq k \leq n$.

Theorem (Foulkes, 1972)

$$\ell_n^{(k)} = \frac{1}{n} \sum_{d|n} \phi(d) \frac{\mu\left(\frac{d}{(d,k)}\right)}{\phi\left(\frac{d}{(d,k)}\right)} p_d^{\frac{n}{d}}.$$

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The quantity

$$\phi(d) \frac{\mu\left(\frac{d}{(d,k)}\right)}{\phi\left(\frac{d}{(d,k)}\right)}$$

is a *Ramanujan sum*; it equals the sum of the k th powers of all primitive d th roots of unity.

Two special cases: Lie_n and $Conj_n$

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$$\text{ch } Conj_n = \frac{1}{n} \sum_{d|n} \phi(d) p_d^{\frac{n}{d}}.$$

Let $\mathcal{R} := \mathbb{C}[x_1, \dots, x_n]/\mathcal{I}_n$ where $\mathcal{I}_n$ is the ideal generated by the non-constant symmetric polynomials.

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Theorem (Chevalley 1955)

The ring of coinvariants $\mathcal{R}$ affords the regular representation of S_n .

The ring $\mathcal{R}$ is graded (by degree):

$$\mathcal{R} = \bigoplus_{i \geq 0} \mathcal{R}_i.$$

Theorem (Kraskiewicz & Weyman (1987/2001))

For each $k = 1, \dots, n$,

$\bigoplus_{i \equiv k \pmod n} \mathcal{R}_i$ is isomorphic to $\omega_n^k \uparrow_{C_n}^{S_n}$.

Decomposition into irreducibles?

A famous theorem in Algebraic Combinatorics

Theorem (Stanley, see EC2 Ex. 7.88, Kraskiewicz & Weyman (1987/2001))

*The multiplicity of the irreducible χ^λ in $\omega_n^k \uparrow_{C_n}^{S_n}$ is the number of standard Young tableaux $\mathbf{t}$ of shape λ whose **major index** is congruent to $k \bmod n$.*

More specifically:

Theorem (Lusztig; Stanley 1979)

*The multiplicity of the irreducible χ^λ in the i th graded piece of the coinvariant ring $\mathcal{R}_i$ is the number of standard Young tableaux $\mathbf{t}$ of shape λ whose **major index** equals i .*

Plethysm with Lie (I)

Let $Lie = \sum_{n \geq 1} Lie_n = \sum_{n \geq 1} \text{ch } \omega_n \uparrow_{C_n}^{S_n}$. Then

Theorem (Thrall 1942)

$$H[Lie] = (1 - p_1)^{-1}.$$

Proof.

Use the identity $H = \exp(\sum_{i \geq 1} \frac{p_i}{i})$ and the following facts:

- ① $p_i[f] = f[p_i]$; $p_1[f] = f[p_1] = f$; $c[f] = c$ for constants c .
- ② $p_i[p_j] = p_{ij} = p_j[p_i]$.
- ③ $(f + g)[r] = f[r] + g[r]$;
- ④ $(fg)[r] = f[r]g[r]$; $(f/g)[r] = f[r]/g[r]$.



The regular representation of S_n — (II)

The plethystic identity

$$H[Lie] = (1 - p_1)^{-1}$$

is equivalent to

Theorem (Reg2 : Thrall 1942)

$$\text{ch } \text{Reg}_{S_n} = \sum_{\lambda \vdash n} H_\lambda[Lie] = \sum_{\lambda \vdash n} h_{m_1}[Lie_1] h_{m_2}[Lie_2] \dots,$$

where λ has m_i parts equal to i .

Plethystic Inverses

Definition

We say symmetric functions f and g are plethystic inverses if

$$f[g] = g[f] = p_1.$$

Associativity of plethysm implies that $f[g] = p_1 \iff g[f] = p_1$.

Theorem

If f is a symmetric function with no constant term and nonzero coefficient for p_1 , then f has a plethystic inverse.

Proof: Let $f = p_1 + \sum_{n \geq 2} f_n$ where f_n is of homogeneous degree n . Then

$$p_1 = f[g] = g + \left(\sum_{n \geq 2} f_n[g] \right).$$

Isolate by degree and recursively compute the terms of g .

- ① $(p_1 - 1)$ and $(p_1 + 1)$ are inverses.

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- ③ $\sum_{n \geq 1} p_n$ and $\sum_{n \geq 1} \mu(n)p_n$ are inverses.

The plethystic inverse of $H - 1 = \sum_{n \geq 1} h_n$

Theorem (INV1: Cadogan 1971)

$$\sum_{n \geq 1} h_n \left[\sum_{n \geq 1} (-1)^{n-1} \omega(Lie_n) \right] = p_1.$$

Plethystic Identities: A meta theorem

Fix $\psi : \mathbb{N}_+ \rightarrow \mathbb{R}$.

Let $f_n := \frac{1}{n} \sum_{d|n} \psi(d) p_d^{\frac{n}{d}}$, and $f_n(t) := \frac{1}{n} \sum_{d|n} \psi(d) t^{\frac{n}{d}}$.

Let $F := \sum_{n \geq 1} f_n$, $F^{alt} := \sum_{n \geq 1} (-1)^{n-1} f_n$ (symmetric functions).

Theorem (S 2017)

$$H[F] = \prod_{m \geq 1} (1 - p_m)^{-f_m(1)} \quad (1)$$

$$\iff E[F] = \prod_{m \geq 1} (1 - p_m)^{f_m(-1)} \quad (2)$$

$$\iff H[\omega(F)^{alt}] = \prod_{m \geq 1} (1 + p_m)^{f_m(1)} \quad (3)$$

$$\iff E[\omega(F)^{alt}] = \prod_{m \geq 1} (1 + p_m)^{-f_m(-1)} \quad (4)$$

Specialisations of $\psi(d)$ — I

Take $\psi(d) = \mu(d)$ to obtain, for $F = Lie$, since $f_m(1) = \delta_{m,1}$,

$$H[Lie] = (1 - p_1)^{-1} \iff H[\omega(Lie)^{alt}] = 1 + p_1.$$

So Thrall's theorem is equivalent to Cadogan's computation of the plethystic inverse of $\sum_{n \geq 1} h_n$, and also to:

Theorem (INV2 : Plethystic inverse of Lie)

(Orlik-Solomon, Lehrer-Solomon 1986)

$$\sum_{n \geq 1} (-1)^{n-1} e_n[Lie] = p_1$$

Specialisations of $\psi(d)$ — II

Recall $E = \sum_{n \geq 0} e_n$.

Definition (S 2018)

Let k_n be the 2-valuation of the positive integer n . Define

$$Lie_n^{(2)} := \omega_n^{k_n} \uparrow_{C_n}^{S_n}.$$

Notice: $Lie_n^{(2)} = \begin{cases} Lie_n & n \text{ odd,} \\ Conj_n & n \text{ a power of 2.} \end{cases}$

Meta theorem + the corresponding specialisation of $\psi(d)$ gives

Theorem (Reg3: S 2018)

$$E[\sum_{n \geq 1} Lie_n^{(2)}] = (1 - p_1)^{-1}.$$

$$\text{ch } Reg_{S_n} = \sum_{\lambda \vdash n} E_\lambda[Lie_n^{(2)}] = \sum_{\lambda \vdash n} e_{m_1}[Lie_1^{(2)}] e_{m_2}[Lie_2^{(2)}] \dots$$

The inverses of $E - 1 = \sum_{n \geq 1} e_n$ and $\sum_{n \geq 1} Lie_n^{(2)}$

The meta theorem implies

Corollary (S 2018)

$$(INV3) \quad \sum_{n \geq 1} e_n \left[\sum_{n \geq 1} (-1)^{n-1} \omega(Lie_n^{(2)}) \right] = p_1.$$

and

$$(INV4) \quad \sum_{n \geq 1} (-1)^{n-1} h_n \left[\sum_{n \geq 1} Lie_n^{(2)} \right] = p_1.$$

Question: Is there a more conceptual explanation for (Reg3), (Inv3), (Inv4)? An acyclic complex?

Theorem (S 2018)

$$Lie_n = Lie_n^{(2)} - Lie_{\frac{n}{2}}^{(2)}[p_2].$$

Proof: First note that $H[p_1 - p_2] = E$.

Also, $(p_1 - p_2)$ and $\sum_{k \geq 0} p_{2^k}$ are plethystic inverses.

$$\begin{aligned} (1 - p_1)^{-1} &= H[Lie] = H[(p_1 - p_2) \left[\sum_{k \geq 0} p_{2^k} \right]] [Lie] \\ &= (H[p_1 - p_2]) \left[\sum_{k \geq 0} p_{2^k} [Lie] \right] = E \left[\sum_{k \geq 0} p_{2^k} [Lie] \right] \\ \implies \sum_{n \geq 1} Lie_n^{(2)} &= \sum_{k \geq 0} p_{2^k} [Lie]. \end{aligned}$$

Recall that $\ell_n^{(k)} = \text{ch } \omega_n^k \uparrow_{C_n}^{S_n}$.

Theorem (S 2017)

Fix $k \geq 1$. Then

$$H\left[\sum_{m \geq 1} \ell_n^{(k)}\right] = 1 + \sum_{\substack{\lambda \\ \lambda_i | k}} p_\lambda.$$

This generalises Thrall's Theorem.

From this one can deduce

Corollary (S 2018)

$$\ell_n^{(k)} = \sum_{m \mid \gcd(n,k)} \text{Lie}_{\frac{n}{m}}[p_m],$$

confirming that $\ell_n^{(k)}$ and the representation $\omega_n^k \uparrow_{C_n}^{S_n}$ depend only on $\gcd(n, k)$.

The plethystic inverse of the odd Lie representations

Theorem (INV5: S 2020; conjectured by Richard Stanley)

The plethystic inverse of $\sum_{n \geq 1} \text{Lie}_{2n-1}$ is

$$\frac{e_1 + e_3 + \dots}{1 + e_2 + e_4 + \dots}.$$

Theorem (S 2020)

Let $\delta_n = (n-1, n-2, \dots, 1)$, $n \geq 2$. (Set $\delta_1 = \emptyset$.) Then

$$\begin{aligned} \frac{e_1 + e_3 + \dots}{1 + e_2 + e_4 + \dots} &= s_{(1)} + \sum_{n \geq 3} (-1)^n s_{\delta_n / \delta_{n-2}} \\ &= \tanh\left(\sum_{i \geq 1} \operatorname{arctanh} x_i\right). \end{aligned}$$

The plethystic inverse of the *alternating* odd Lie's

Theorem (INV6: S 2020)

The plethystic inverse of $\sum_{n \geq 0} (-1)^n \text{Lie}_{2n+1}$ is

$$\frac{e_1 - e_3 + e_5 - \dots}{1 - e_2 + e_4 - \dots}.$$

Theorem (Carlitz 1973)

Let $\delta_n = (n-1, n-2, \dots, 1)$, $n \geq 2$. (Set $\delta_1 = \emptyset$.) Then

$$\begin{aligned} \frac{e_1 - e_3 + e_5 - \dots}{1 - e_2 + e_4 - \dots} &= s_{(1)} + \sum_{n \geq 3} s_{\delta_n / \delta_{n-2}} \\ &= \tan\left(\sum_{i \geq 1} \arctan x_i\right). \end{aligned}$$

The regular representation — IV

Define Hk_n to be the Frobenius characteristic of the (multiplicity-free) sum of all irreducibles indexed by hooks $(n-r, 1^r)$, $r = 0, 1, \dots, n-1$; $Hk_1 = h_1$. Then one has (yet another) decomposition of the regular representation of S_n :

Theorem (Reg4: S 2020)

$$\text{ch } \text{Reg}_{S_n} = \sum_{n \geq 1} Hk_n \left[\sum_{n \geq 1} \text{Lie}_{2n-1} \right].$$

Question: Is there a more conceptual explanation?

From sums to alternating sums: another meta theorem

Fix $k \geq 2$. Let $F_{1,k} := \sum_{m \geq 0} f_{mk+1}$, $G_{1,k} := \sum_{m \geq 0} g_{mk+1}$ be two series of symmetric functions where the f_i, g_i are of homogeneous degree i , and $f_1 = g_1 = p_1$. Define $F_{1,k}^{alt} := \sum_{m \geq 0} (-1)^m f_{mk+1}$ and similarly $G_{1,k}^{alt}$.

Theorem (S 2020)

$$F_{1,k}[G_{1,k}] = p_1 \iff F_{1,k}^{alt}[G_{1,k}^{alt}] = p_1.$$

The free Lie algebra

As a vector space, Lie_n is the degree n multilinear component of the free Lie algebra on m generators. Its Gl_m -character was computed by Brandt (1944) to be

$$\text{ch } Lie_n = Lie_n(x_1, \dots, x_m) = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}}.$$

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Recall (Reg2):

$$\text{ch } Reg_{S_n} = \sum_{\lambda \vdash n} H_\lambda[Lie] = \sum_{\lambda \vdash n} h_{m_1}[Lie_1] h_{m_2}[Lie_2] \dots,$$

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In the Gl_m -context, the Poincaré-Birkhoff-Witt theorem for the free Lie algebra says its universal enveloping algebra is its symmetric algebra. Hence by Schur-Weyl duality, Thrall's theorem gives the decomposition of the full tensor algebra as a sum of symmetrised Lie modules.

Other series that have *nice* plethystic inverses: $1 \bmod k$

Let $\beta_n^{(j)}$, $0 \leq j \leq k-1$, be the Frobenius characteristic of the unique nonvanishing homology S_n -module of the subposet of partitions of n with block sizes congruent to $j \bmod k$.

Theorem (Calderbank-Hanlon-Robinson 1986)

Fix $k \geq 2$. The symmetric functions

$$(INV7) \quad \sum_{m \geq 0} h_{1+mk} \quad \text{and} \quad \sum_{m \geq 0} (-1)^m \beta_{1+mk}^{(1)}$$

are plethystic inverses.

From the meta theorem we obtain:

Theorem (S 2020)

Fix $k \geq 2$. The following pairs of symmetric functions are plethystic inverses:

$$(INV8) \quad \sum_{m \geq 0} (-1)^m h_{1+mk} \quad \text{and} \quad \sum_{m \geq 0} \beta_{1+mk}^{(1)}.$$

$$(INV9) \quad \sum_{m \geq 0} e_{1+mk} \quad \text{and} \quad \sum_{m \geq 0} (-1)^m \omega(\beta_{1+mk}^{(1)}), \text{ } k \text{ even.}$$

$$(INV10) \quad \sum_{m \geq 0} (-1)^m e_{1+mk} \quad \text{and} \quad \sum_{m \geq 0} \omega(\beta_{1+mk}^{(1)}), \text{ } k \text{ even.}$$

Recall

$$\text{ch } \text{Reg}_{S_n} = \sum_{\lambda \vdash n} H_\lambda[\text{Lie}] = \sum_{\lambda \vdash n} h_{m_1}[\text{Lie}_1] h_{m_2}[\text{Lie}_2] \dots$$

The modules $H_\lambda[\text{Lie}]$ are the *higher Lie* modules. Their irreducible decomposition is known only for the cases $\lambda = (n), (1^n)$ and (2^a) or $(2^a, 1)$.

Higher odd *Lie* modules and the free Jordan algebra

Let J_n be the degree n multilinear component of the free Jordan algebra on m generators. Instead of the Lie bracket, we have the bracket

$$[x, y] = x \otimes y + y \otimes x.$$

Schur-Weyl duality:

View J_n as an S_n -module, with Frobenius characteristic η_n . Set $\eta_0 = 1$.

Theorem (Calderbank-Hanlon-S 1994)

The Frobenius characteristic of the S_n -module on the free Jordan algebra satisfies

$$H\left[\sum_{n \geq 1} \text{Lie}_{2n-1}\right] = \sum_{n \geq 0} \eta_n.$$

Deformations of the free Lie algebra

Fix $p \geq 2$. Let α_p be a primitive p th root of unity. Consider the degree n multilinear component of the free algebra with bracket $[x, y] := x \otimes y - \alpha_p y \otimes x$. The representation $\eta_n(p)$ of S_n on this component is a sum of higher Lie modules:

Theorem (Calderbank-Hanlon-S 1994)

$$\text{ch } \eta_n(p) = \sum_{\substack{\lambda \vdash n \\ \lambda_i \not\equiv 0 \pmod p}} H_\lambda[\text{Lie}].$$

The higher conjugacy modules 1

The meta theorem can be used to prove:

Theorem (Solomon 1961)

$$H[\sum_{n \geq 1} \text{Conj}_n] = \sum_{\lambda} H_{\lambda}[\sum_{n \geq 1} \text{Conj}_n] = \sum_{\mu \vdash n} p_{\mu}.$$

The higher conjugacy modules 1

The meta theorem can be used to prove:

Theorem (Solomon 1961)

$$H[\sum_{n \geq 1} \text{Conj}_n] = \sum_{\lambda} H_{\lambda}[\sum_{n \geq 1} \text{Conj}_n] = \sum_{\mu \vdash n} p_{\mu}.$$

$H_{\lambda}[\sum_{n \geq 1} \text{Conj}_n]$ is the action by conjugation on the class indexed by λ . The character value on σ is the order of the centraliser of σ .

The higher conjugacy modules 2

Theorem (S 2017)

$$E\left[\sum_{n \geq 1} \text{Conj}_n\right] = \sum_{\lambda} E_{\lambda}\left[\sum_{n \geq 1} \text{Conj}_n\right] = \sum_{\substack{\mu \vdash n \\ \text{all parts odd}}} p_{\mu}.$$

The higher $Lie_n^{(2)}$ modules

Theorem (S 2018)

$$H\left[\sum_{n \geq 1} Lie_n^{(2)}\right] = \sum_{\lambda} H_{\lambda}\left[\sum_{n \geq 1} Lie_n^{(2)}\right] = \sum_{\substack{\mu \vdash n \\ \text{all parts a power of 2}}} p_{\mu}.$$

(♣) Why does the number of standard Young tableaux $\mathbf{t}$ of shape λ with major index $\equiv k \pmod n$ depend only on $\gcd(k, n)$?

Fix $n \geq 1$. What is the irreducible decomposition of the degree n term in

- 1 $H_\lambda[\text{Lie}]$? (Thrall's problem – 1942)
- 2 $H_\lambda[\text{Conj}]$?
- 3 More generally, for the degree n term in $H_\lambda[\sum_{n \geq 1} \ell_n^{(k)}]$?

Known for these cases:

- 1 $\lambda = (n)$: (Stanley, Kraskiewicz-Weyman)
- 2 $\lambda = (1^n)$ (the trivial module)
- 3 $\lambda = (2^a)$, $n = 2a$, or $(2^a, 1)$, $n = 2a + 1$.
- 4 The multiplicity of s_μ in $h_a[\text{Lie}_b]$, $ab = n$ for $\mu = (n - k, 1^k)$ (recent work of Hegedus-Roichman).

Proof that $H[\text{Lie}] = (1 - p_1)^{-1}$: use $H = \exp(\sum_{i \geq 1} \frac{p_i}{i})$

$$\begin{aligned}
 \log H[\text{Lie}] &= \sum_{i \geq 1} \frac{p_i}{i} \left[\sum_{n \geq 1} \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}} \right] \\
 &= \sum_{i \geq 1} \frac{1}{i} \sum_{n \geq 1} \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}} [p_i] \\
 &= \sum_{i \geq 1} \frac{1}{i} \sum_{j, d \geq 1} \frac{1}{j} \mu(d) (p_d [p_i])^j \\
 &= \sum_{i, j, d \geq 1} \frac{1}{j} \mu(d) p_{di}^j = \sum_{m \geq 1} \frac{1}{j} p_m^j \sum_{d \geq 1, d|m} \mu(d) \\
 &= \sum_{m \geq 1} \frac{1}{j} p_m^j \delta_{1,m} = \sum_{j \geq 1} p_1^j = \log(1 - p_1)^{-1}.
 \end{aligned}$$

Classically known plethysms (Littlewood)

(See Stanley, EC2 or Macdonald.)

$$h_n[h_2] = \sum_{\lambda \vdash 2n, \lambda_i \text{ even}} s_\lambda = h_n[\text{Conj}_2].$$

$$h_n[e_2] = \sum_{\lambda \vdash 2n, \lambda'_i \text{ even}} s_\lambda = h_n[\text{Lie}_2].$$

Wreath Products 1

Define $S_m[S_n]$ to be the following subgroup of S_{mn} :

- 1 Partition $[mn]$ into n equal intervals of length m : Let $I_k = \{(k-1)n + 1, \dots, kn\}$, $1 \leq k \leq m$.
- 2 Identify the Young subgroup $\underbrace{S_n \times \dots \times S_n}_m$ with the product $S_n^m = \times_{j=1}^m S_{I_j}$, a subgroup of S_{mn} .
- 3 Let $\hat{\sigma}_k$ be the permutation which exchanges the letters of I_1 with I_k . So $\hat{\sigma}_k^2 = 1$ and the map $(1, k) \mapsto \hat{\sigma}_k$ identifies S_m with a subgroup of S_{mn} .
- 4 Elements in S_m^n look like $\prod_{i=1}^m (\hat{\sigma}_i x_i \hat{\sigma}_i)$, $x_i \in S_n$.
- 5 Ex: Let $x_1 = (12)$, $x_2 = (123)$, $x_3 = (1)$, $x_4 = (1)$ be 4 elements in S_3 . The element in S_3^4 looks like **213** **564** **789** **101112** in one-line notation.

Wreath Products 2: $S_m[S_n]$ as a subgroup of S_{mn}

Lemma

Every element in the normaliser $N(S_m^n)$ has a unique expression of the form

$$\prod_{i=1}^m (\hat{\sigma}_i x_i \hat{\sigma}_i) \hat{\sigma},$$

where $x_i \in S_n$, $\sigma \in S_m$, and $\hat{\sigma}$ is the image of σ in S_{mn} under the map $(1, k) \mapsto \hat{\sigma}_k$.

Ex: Let $x_1 = (12)$, $x_2 = (123)$, $x_3 = (1)$, $x_4 = (1)$ be 4 elements in S_3 . Let $\sigma = (1234) \in S_4$. Then $(x_1, x_2, x_3, x_4; \sigma)$ is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 6 & 4 & 7 & 8 & 9 & 10 & 11 & 12 & 2 & 1 & 3 \end{pmatrix}.$$

Wreath Product Modules 1

k any field.

Let W be an S_n -module, V an S_m -module. Define the $S_m[S_n]$ -module $V[W]$ to be the following:

- 1 As a vector space over k , $V[W]$ is the tensor product $W^{\otimes m} \otimes V$.
- 2 The action of $S_m[S_n]$ is defined as follows. Let $w_i \in W$, $v \in V$. Let $x_i \in S_n$, $1 \leq i \leq m$, and $\sigma \in S_m$. Then

$$\begin{aligned} (x_1, \dots, x_m; \sigma) \cdot (w_1 \otimes \dots \otimes w_m \otimes v) \\ := x_1 w_{\sigma^{-1}(1)} \otimes \dots \otimes x_m w_{\sigma^{-1}(m)} \otimes (\sigma \cdot v). \end{aligned}$$

- 3 $\dim V[W] = (\dim W)^m \dim V$.
- 4 $\text{ch } V[W] \uparrow_{S_m[S_n]}^{S_{mn}} = \text{ch } V[\text{ch } W]$.

Wreath Product Modules 2

Let W be an S_n -module, V an S_m -module.

- ① (*Linearity in first variable*) Let V_i be S_m -modules, $i = 1, 2$. Then $V_1[W] \oplus V_2[W] \simeq (V_1 \oplus V_2)[W]$ as $S_m[S_n]$ -modules.
- ② (*Multiplicativity in first variable*) Let U_i be S_{m_i} -modules, $i = 1, 2$. Then $(U_1[W] \otimes U_2[W]) \uparrow_{S_{m_1}[S_n] \times S_{m_2}[S_n]}^{S_{m_1+m_2}[S_n]} \simeq (U_1 \otimes U_2)[W]$ as $S_{m_1+m_2}[S_n]$ -modules, via the natural embedding of $S_{m_1} \times S_{m_2}$ as a subgroup of $S_{m_1+m_2}$;
- ③ (*Inner transitivity of induction*) Let $H < K$ be subgroups. Let W_H be an H -module. Then $V[W_H] \uparrow_{S_m[H]}^{S_m[K]} \simeq V[W_H \uparrow_H^K]$ as $S_m[K]$ -modules.

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THANK YOU!