# Plethystic inversion and representations of the symmetric group

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Symmetric functions in m variables are characters of polynomial representations of  $GL_m$ .

Symmetric functions of degree n, via the Frobenius characteristic map, are in correspondence with the representation ring of the symmetric group  $S_n$ .

Plethysm of symmetric functions f[g] corresponds to composition of  $GL_n$  characters.

For the symmetric group it corresponds to forming representations of wreath products  $S_m[S_n]$ , and inducing up to  $S_{mn}$ .

### Partitions of *n* and symmetric functions

- $\lambda = (\lambda_1 \ge \lambda_2 \ge \dots \lambda_k \ge 1)$  such that  $\sum_i \lambda_i = n$  is an integer partition of n;  $\ell(\lambda)$  is the number of parts k of  $\lambda$ .
- $p_r = \sum_i x_i^r$  is the *r*th power-sum symmetric function;
- *p*<sub>λ</sub> = *p*<sub>λ1</sub>*p*<sub>λ2</sub>... is the power-sum symmetric function indexed by the partition λ;
- $s_{\lambda}$  denotes the Schur function indexed by the partition  $\lambda$ ;
- Up to a scalar multiple, p<sub>λ</sub> is the Frobenius characteristic of the class function that is 1 on the conjugacy class indexed by λ and zero elsewhere;
- s<sub>λ</sub> is the Frobenius characteristic of the S<sub>n</sub>-irreducible indexed by λ.
- A symmetric function *f* of homogeneous degree *n* is *Schur positive* if it is a nonnegative integer combination of Schur functions, i.e. if it corresponds to a true *S*<sub>n</sub>-module.

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- The homogeneous symmetric function  $h_n$  of degree n is the Frobenius characteristic of the trivial representation of  $S_n$ . It is also the character of GL(V) acting on the nth symmetric power  $Sym^n(V)$ .
- The elementary symmetric function e<sub>n</sub> of degree n is the Frobenius characteristic of the sign representation of S<sub>n</sub>. It is also the character of GL(V) acting on the nth exterior power ∧<sup>n</sup>(V). The involution ω in the ring of symmetric functions is defined

by  $\omega(h_n) = e_n$ .

•  $p_1^n$  is the Frobenius characteristic of the regular representation of  $S_n$ .

It is also the character of GL(V) acting on the *n*th tensor power  $V^{\otimes n}$ .

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### Plethysm and Schur-Weyl duality

Define  $H = \sum_{n \ge 0} h_n$  and  $E = \sum_{n \ge 0} e_n$ .

If  $F = \sum_{i \ge 1} f_i$  is the GI(V)-character on  $W = \bigoplus_i W_i$ , then: H[F] is the GI(V)-character of the symmetric algebra Sym(W). E[F] is the GI(V)-character of the exterior algebra  $\bigwedge W$ .

If  $\lambda$  is the partition with  $m_i$  parts equal to i, then:

 $H_{\lambda}[F]$  is the character of the piece  $\otimes_i Sym^{m_i}(W_i)$  of the symmetric algebra Sym(W).

 $E_{\lambda}[F]$  is the character of the piece  $\otimes_i \bigwedge^{m_i}(W_i)$  of the exterior algebra  $\bigwedge W$ .

 $(1 - p_1)^{-1} = \sum_{n \ge 0} p_1^n$  is the GI(V)-character on the full tensor algebra T(V).

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If  $F = \sum_{j\geq 1} f_j$  where each  $f_j$  is the Frobenius characteristic ch of an  $S_j$ -module  $W_j$ , and  $\lambda$  is the partition of n with  $m_j$  parts equal to i, then:

$$H_{\lambda}[F] = \operatorname{ch}\left(\bigotimes_{i} \mathbf{1}_{S_{m_{i}}}[W_{i}]\right) \uparrow^{S_{n}} = \prod_{i} h_{m_{i}}[f_{i}]$$
$$E_{\lambda}[F] = \operatorname{ch}\left(\bigotimes_{i} \operatorname{sgn}_{S_{m_{i}}}[W_{i}]\right) \uparrow^{S_{n}} = \prod_{i} e_{m_{i}}[f_{i}].$$

The modules are induced from the subgroup  $\prod_i S_{m_i}[S_i]$ , the normaliser in  $S_n$  of the direct product of  $m_i$  copies of  $S_i$ ,  $i \ge 1$ .

### The regular representation of a finite group G

$$Reg_G := 1 \uparrow_e^G = \sum_{\chi \text{ irreducible repn of } G} (\dim \chi) \chi.$$

For the symmetric group  $S_n$ :

Theorem (Reg0)

$$\operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} f^{\lambda} \chi^{\lambda},$$

with Frobenius characteristic

$$\operatorname{ch} \operatorname{Reg}_{S_n} := \sum_{\lambda \vdash n} f^{\lambda} s_{\lambda},$$

where  $\lambda$  is an integer partition of n,  $f^{\lambda} = |\{\text{standard Young tableaux of shape }\lambda\}|$ , and  $s_{\lambda}$  is the Schur function indexed by  $\lambda$ , so  $s_{\lambda} = \operatorname{ch} \chi^{\lambda}$ .

Let  $C_n$  be the cyclic subroup of  $S_n$  generated by the long cycle  $\sigma = (12 \dots n)$ , let  $\omega_n$  be a primitive *n*th root of unity. For  $1 \le k \le n$ ,  $\sigma \mapsto \omega_n^k$  yields a representation of  $C_n$ , and these are all the distinct irreducibles, so

$$Reg_{C_n} = \sum_{k=1}^n (\omega_n^k).$$

This gives a second decomposition of the regular representation of  $S_n$ :

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### The regular representation of $S_n$ — (I) (continued)

#### Theorem (Reg1)

$$Reg_{S_n} = \sum_{k=1}^n (\omega_n^k) \uparrow_{C_n}^{S_n}$$

**Proof:** Induce the decomposition of the regular representation of  $C_n$  up to  $S_n$ . (Or take tensor products over  $\mathbb{C}C_n$  with  $\mathbb{C}S_n$ .)

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#### Definition

$$Lie_n := \omega_n \uparrow_{C_n}^{S_n}.$$

#### Definition

$$Conj_n := \omega_n^n \uparrow_{C_n}^{S_n} = 1 \uparrow_{C_n}^{S_n}.$$

 $Conj_n$  is the permutation representation of  $S_n$  by conjugation on the class of *n*-cycles, since the stabiliser of an *n*-cycle is the cyclic group  $C_n$ .

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### Ramanujan sums

 $\mu(n), \ \phi(n)$  are respectively the number-theoretic Möbius and totient functions.

Let 
$$\ell_n^{(k)} := \operatorname{ch} \omega_n^k \uparrow_{C_n}^{S_n}, 1 \le k \le n.$$

Theorem (Foulkes, 1972)

$$\ell_n^{(k)} = \frac{1}{n} \sum_{d|n} \phi(d) \; \frac{\mu(\frac{d}{(d,k)})}{\phi(\frac{d}{(d,k)})} \, p_d^{\frac{n}{d}}.$$

The quantity

$$\phi(d) \; rac{\mu(rac{d}{(d,k)})}{\phi(rac{d}{(d,k)})}$$

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is a *Ramanujan sum*; it equals the sum of the *k*th powers of all primitive *d*th roots of unity.

Note 
$$\ell_n^{(1)} = \operatorname{ch} Lie_n$$
,  $\ell_n^{(n)} = \operatorname{ch} Conj_n$ . Hence  
 $\operatorname{ch} Lie_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}}$ .  
 $\operatorname{ch} Conj_n = \frac{1}{n} \sum_{d|n} \phi(d) p_d^{\frac{n}{d}}$ .

<ロト < 個ト < 目ト < 目ト 目 のので 12 / 50 Let  $\mathcal{R} := \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}_n$  where  $\mathcal{I}_n$  is the ideal generated by the non-constant symmetric polynomials.

 $\mathcal{R}$  is the ring of coinvariants for  $S_n$ , carrying a representation of  $S_n$ :

#### Theorem (Chevalley 1955)

The ring of coinvariants  $\mathcal{R}$  affords the regular representation of  $S_n$ .

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The ring  $\mathcal{R}$  is graded (by degree):

$$\mathcal{R} = \bigoplus_{i \ge 0} \mathcal{R}_i.$$

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Theorem (Kráskiewicz & Weyman (1987/2001))

For each k = 1, ..., n,  $\bigoplus_{i \equiv k \mod n} \mathcal{R}_i$  is isomorphic to  $\omega_n^k \uparrow_{C_n}^{S_n}$ .

Decomposition into irreducibles?

## Theorem (Stanley, see EC2 Ex. 7.88, Kráskiewicz & Weyman (1987/2001))

The multiplicity of the irreducible  $\chi^{\lambda}$  in  $\omega_n^k \uparrow_{C_n}^{S_n}$  is the number of standard Young tableaux **t** of shape  $\lambda$  whose **major index** is congruent to k mod n.

### Plethysm with the *Lie* characteristic (I)

Let 
$$Lie = \sum_{n\geq 1} Lie_n = \sum_{n\geq 1} \operatorname{ch} \omega_n \uparrow_{C_n}^{S_n}$$
. Then

Theorem (Thrall 1942)

$$H[Lie] = (1 - p_1)^{-1}.$$

#### Proof.

Use the identity  $H = \exp(\sum_{i \ge 1} \frac{p_i}{i})$  and the following facts:

• 
$$p_i[f] = f[p_i]; p_1[f] = f[p_1] = f; c[f] = c$$
 for constants c.

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$$p_i[p_j] = p_{ij} = p_j[p_i].$$

$$(f+g)[r] = f[r] + g[r];$$

$$(fg)[r] = f[r]g[r]; (f/g)[r] = f[r]/g[r].$$

The regular representation of  $S_n$  — (II)

The plethystic identity

$$H[Lie] = (1 - p_1)^{-1}$$

is equivalent to

Theorem (Reg2 : Thrall 1942)

$$\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} H_{\lambda}[\operatorname{Lie}] = \sum_{\lambda \vdash n} h_{m_1}[\operatorname{Lie}_1]h_{m_2}[\operatorname{Lie}_2]\dots,$$

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where  $\lambda$  has  $m_i$  parts equal to i.

### Definition

We say symmetric functions f and g are plethystic inverses if

$$f[g]=g[f]=p_1.$$

Associativity of plethysm implies that  $f[g] = p_1 \iff g[f] = p_1$ .

#### Theorem

If f is a symmetric function with no constant term and nonzero coefficient for  $p_1$ , then f has a plethystic inverse.

**Proof:** Let  $f = p_1 + \sum_{n \ge 2} f_n$  where  $f_n$  is of homogeneous degree *n*. Then

$$p_1 = f[g] = g + (\sum_{n \ge 2} f_n[g]).$$

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Isolate by degree and recursively compute the terms of g.

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### The plethystic inverse of $H-1 = \sum_{n\geq 1} h_n$

### Theorem (INV1: Cadogan 1971)

$$\sum_{n\geq 1}h_n[\sum_{n\geq 1}(-1)^{n-1}\omega(Lie_n)]=p_1.$$

### Plethystic Identities: A meta theorem

Fix 
$$\psi : \mathbb{N}_+ \to \mathbb{R}$$
.  
Let  $f_n := \frac{1}{n} \sum_{d|n} \psi(d) p_d^{\frac{n}{d}}$ , and  $f_n(t) := \frac{1}{n} \sum_{d|n} \psi(d) t^{\frac{n}{d}}$ .  
Let  $F := \sum_{n \ge 1} f_n$ ,  $F^{alt} := \sum_{n \ge 1} (-1)^{n-1} f_n$  (symmetric functions).

### Theorem (S 2017)

$$H[F] = \prod_{m \ge 1} (1 - p_m)^{-f_m(1)} \quad (1)$$
  
$$\iff E[F] = \prod_{m \ge 1} (1 - p_m)^{f_m(-1)} \quad (2)$$
  
$$\iff H[\omega(F)^{alt}] = \prod_{m \ge 1} (1 + p_m)^{f_m(1)} \quad (3)$$
  
$$\iff E[\omega(F)^{alt}] = \prod_{m \ge 1} (1 + p_m)^{-f_m(-1)} \quad (4)$$

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Take  $\psi(d) = \mu(d)$  to obtain, for F = Lie, since  $f_m(1) = \delta_{m,1}$ ,

$$H[Lie] = (1 - p_1)^{-1} \iff H[\omega(Lie)^{alt}] = 1 + p_1.$$

So Thrall's theorem is equivalent to Cadogan's computation of the plethystic inverse of  $\sum_{n>1} h_n$ , and also to:

Theorem (INV2 : Plethystic inverse of *Lie*)

(Orlik-Solomon, Lehrer-Solomon 1986)

$$\sum_{n\geq 1} (-1)^{n-1} e_n[Lie] = p_1$$

### Specialisations of $\psi(d) - \mathsf{II}$

Recall 
$$E = \sum_{n \ge 0} e_n$$
.

### Definition (S 2018)

Let  $k_n$  be the 2-valuation of the positive integer n. Define

$$Lie_n^{(2)} := \omega_n^{k_n} \uparrow_{C_n}^{S_n}$$

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Notice: 
$$Lie_n^{(2)} = \begin{cases} Lie_n & n \text{ odd}, \\ Conj_n & n \text{ a power of } 2. \end{cases}$$

Meta theorem + the corresponding specialisation of  $\psi(d)$  gives

#### Theorem (Reg3: S 2018)

$$E[\sum_{n\geq 1}Lie_n^{(2)}]=(1-p_1)^{-1}.$$

$$\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} E_{\lambda}[\operatorname{Lie}_n^{(2)}] = \sum_{\lambda \vdash n} e_{m_1}[\operatorname{Lie}_1^{(2)}]e_{m_2}[\operatorname{Lie}_2^{(2)}]\dots$$

The inverses of 
$$E-1=\sum_{n\geq 1}e_n$$
 and  $\sum_{n\geq 1}Lie_n^{(2)}$ 

#### The meta theorem implies

Corollary (S 2018)		
	( <i>INV</i> 3)	$\sum_{n\geq 1} e_n[\sum_{n\geq 1} (-1)^{n-1} \omega(Lie_n^{(2)})] = p_1.$
and		
	( <i>INV</i> 4)	$\sum_{n\geq 1} (-1)^{n-1} h_n [\sum_{n\geq 1} Lie_n^{(2)}] = p_1.$

**Question:** Is there a more conceptual explanation for (Reg3), (Inv3), (Inv4)? An acyclic complex?

### Lie versus Lie<sup>(2)</sup>

### Theorem (S 2018)

$$Lie_n = Lie_n^{(2)} - Lie_{\frac{n}{2}}^{(2)}[p_2].$$

**Proof:** First note that  $H[p_1 - p_2] = E$ . Also,  $(p_1 - p_2)$  and  $\sum_{k\geq 0} p_{2^k}$  are plethystic inverses.

$$(1 - p_1)^{-1} = H[Lie] = H[(p_1 - p_2)[\sum_{k \ge 0} p_{2^k}]] [Lie]$$
$$= (H[p_1 - p_2])[\sum_{k \ge 0} p_{2^k}[Lie]] = E[\sum_{k \ge 0} p_{2^k}[Lie]]$$
$$\Longrightarrow \sum_{n \ge 1} Lie_n^{(2)} = \sum_{k \ge 0} p_{2^k}[Lie].$$

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Recall that 
$$\ell_n^{(k)} = \operatorname{ch} \omega_n^k \uparrow_{C_n}^{S_n}$$
.

Theorem (S 2017) Fix  $k \ge 1$ . Then  $H[\sum_{m \ge 1} \ell_n^{(k)}] = 1 + \sum_{\substack{\lambda \\ \lambda_i \mid k}} p_{\lambda}.$ 

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This generalises Thrall's Theorem.

#### From this one can deduce

Corollary (S 2018)

$$\ell_n^{(k)} = \sum_{m \mid \gcd(n,k)} Lie_{\frac{n}{m}}[p_m],$$

confirming that  $\ell_n^{(k)}$  and the representation  $\omega_n^k \uparrow_{C_n}^{S_n}$  depend only on gcd(n,k).

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### The plethystic inverse of the odd Lie representations

Theorem (INV5: S 2020; conjectured by Richard Stanley)

The plethystic inverse of  $\sum_{n>1} Lie_{2n-1}$  is

$$\frac{e_1+e_3+\ldots}{1+e_2+e_4+\ldots}$$

Theorem (S 2020)

Let 
$$\delta_n = (n - 1, n - 2, \dots, 1), n \ge 2$$
. (Set  $\delta_1 = \emptyset$ .) Then

$$\frac{e_1 + e_3 + \dots}{1 + e_2 + e_4 + \dots} = s_{(1)} + \sum_{n \ge 3} (-1)^n s_{\delta_n / \delta_{n-2}}$$
$$= \tanh(\sum_{i \ge 1} \operatorname{arctanh} x_i).$$

### The plethystic inverse of the alternating odd Lie's

#### Theorem (INV6: S 2020)

The plethystic inverse of  $\sum_{n>0} (-1)^n Lie_{2n+1}$  is

$$\frac{e_1 - e_3 + e_5 - \dots}{1 - e_2 + e_4 - \dots}$$

### Theorem (Carlitz 1973)

Let 
$$\delta_n = (n - 1, n - 2, ..., 1), n \ge 2$$
. (Set  $\delta_1 = \emptyset$ .) Then

$$rac{e_1-e_3+e_5-\dots}{1-e_2+e_4-\dots}=s_{(1)}+\sum_{n\geq 3}s_{\delta_n/\delta_{n-2}}\ = an(\sum_{i\geq 1}rctan\ x_i).$$

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Define  $Hk_n$  to be the Frobenius characteristic of the (multiplicity-free) sum of all irreducibles indexed by hooks  $(n - r, 1^r), r = 0, 1, ..., n - 1$ ;  $Hk_1 = h_1$ . Then one has (yet another) decomposition of the regular representation of  $S_n$ :

## Theorem (Reg4: S 2020) $\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{n \ge 1} \operatorname{Hk}_n[\sum_{n \ge 1} \operatorname{Lie}_{2n-1}].$

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Question: Is there a more conceptual explanation?

Fix  $k \ge 2$ . Let  $F_{1,k} := \sum_{m \ge 0} f_{mk+1}$ ,  $G_{1,k} := \sum_{m \ge 0} g_{mk+1}$  be two series of symmetric functions where the  $f_i, g_i$  are of homogeneous degree i, and  $f_1 = g_1 = p_1$ . Define  $F_{1,k}^{alt} := \sum_{m \ge 0} (-1)^m f_{mk+1}$  and similarly  $G_{1,k}^{alt}$ .

Theorem (S 2020)

$$F_{1,k}[G_{1,k}] = p_1 \iff F_{1,k}^{alt}[G_{1,k}^{alt}] = p_1.$$

### The free Lie algebra

As a vector space,  $Lie_n$  is the degree *n* multilinear component of the free Lie algebra on *m* generators. Its  $Gl_m$ -character was computed by Brandt (1944) to be

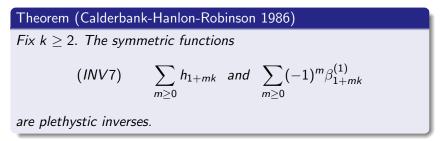
ch Lie<sub>n</sub> = Lie<sub>n</sub>(x<sub>1</sub>,...,x<sub>m</sub>) = 
$$\frac{1}{n}\sum_{d|n}\mu(d)p_d^{\frac{n}{d}}$$
.

Recall (Reg2):

$$\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} H_{\lambda}[\operatorname{Lie}] = \sum_{\lambda \vdash n} h_{m_1}[\operatorname{Lie}_1]h_{m_2}[\operatorname{Lie}_2]\dots,$$

In the  $Gl_m$ -context, the Poincaré-Birkhoff-Witt theorem for the free Lie algebra says its universal enveloping algebra is the full tensor algebra. Hence by Schur-Weyl duality, Thrall's theorem gives the decomposition of the full tensor algebra as a sum of symmetrised Lie modules.

Let  $\beta_n^{(j)}$ ,  $0 \le j \le k - 1$ , be the Frobenius characteristic of the unique nonvanishing homology  $S_n$ -module of the subposet of partitions of n with block sizes congruent to  $j \mod k$ .



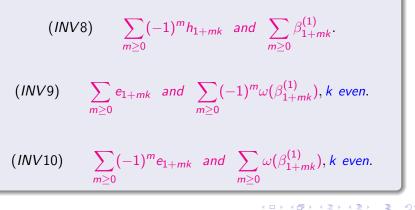
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### $1 \mod k$ continued

From the meta theorem we obtain:

Theorem (S 2020)

Fix  $k \ge 2$ . The following pairs of symmetric functions are plethystic inverses:



#### Recall

$$\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} H_{\lambda}[\operatorname{Lie}] = \sum_{\lambda \vdash n} h_{m_1}[\operatorname{Lie}_1]h_{m_2}[\operatorname{Lie}_2]\dots$$

The modules  $H_{\lambda}[Lie]$  are the *higher Lie* modules. Their irreducible decomposition is known only for the cases  $\lambda = (n), (1^n)$  and  $(2^a)$  or  $(2^a, 1)$ .

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Let  $J_n$  be the degree *n* multilinear component of the free Jordan algebra on *m* generators. Instead of the Lie bracket, we have the bracket

$$[x,y] = x \otimes y + y \otimes x.$$

Schur-Weyl duality:

View  $J_n$  as an  $S_n$ -module, with Frobenius characteristic  $\eta_n$ . Set  $\eta_0 = 1$ .

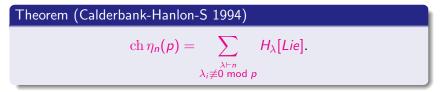
#### Theorem (Calderbank-Hanlon-S 1994)

The Frobenius characteristic of the  $S_n$ -module on the free Jordan algebra satisfies

$$H[\sum_{n\geq 1} Lie_{2n-1}] = \sum_{n\geq 0} \eta_n.$$

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Fix  $p \ge 2$ . Let  $\alpha_p$  be a primitive *p*th root of unity. Consider the degree *n* multilinear component of the free algebra with bracket  $[x, y] := x \otimes y - \alpha_p y \otimes x$ . The representation  $\eta_n(p)$  of  $S_n$  on this component is a sum of higher Lie modules:



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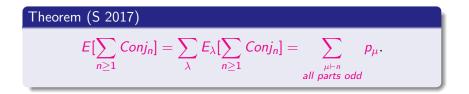
The meta theorem can be used to prove:

Theorem (Solomon 1961)  
$$H[\sum_{n\geq 1} Conj_n] = \sum_{\lambda} H_{\lambda}[\sum_{n\geq 1} Conj_n] = \sum_{\mu\vdash n} p_{\mu}.$$

 $H_{\lambda}[\sum_{n\geq 1} Conj_n]$  is the action by conjugation on the class indexed by  $\lambda$ . The character value on  $\sigma$  is the order of the centraliser of  $\sigma$ .

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(\*) Why does the number of standard Young tableaux t of shape  $\lambda$  with major index  $\equiv k \mod n$  depend only on (k, n)?

What is the irreducible decomposition of

- $H_{\lambda}[Lie]$ ? (Thrall's problem 1942)
- 2  $H_{\lambda}[Conj]?$
- **3** More generally, for  $H_{\lambda}[\ell_n^{(k)}]$ ?

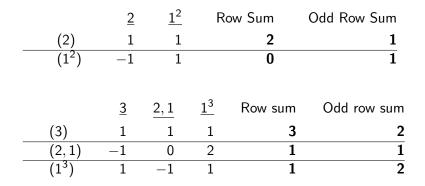
Known only for

$$old \lambda=({\it n})$$
 : (Stanley, Kráskiewicz-Weyman)

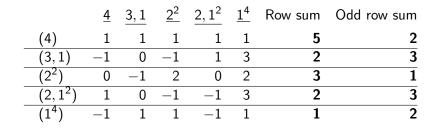
2 
$$\lambda = (1^n)$$
 (the trivial module)

- $\textcircled{0} \lambda \text{ has all parts equal to 1 or 2.}$
- $\lambda = (n k, 1^k)$  (recent work of Hegedus-Roichman).

#### Character tables of $S_n$

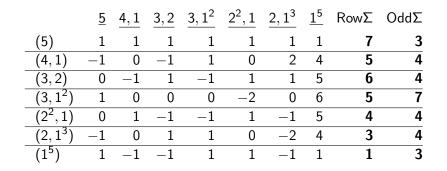


#### Character tables for $S_4$



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#### Character tables: $S_5$



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### Character tables: $S_6$

	$\underline{C}^1$	<u>C</u> <sup>2</sup>	<u>C<sup>3</sup></u>	<u>C</u> <sup>4</sup>	<u>C</u> <sup>5</sup>	<u>C</u> <sup>6</sup>	<u>C</u> <sup>7</sup>	<u>C</u> <sup>8</sup>	<u>C</u> <sup>9</sup>	<u>C</u> <sup>10</sup>	<u>C<sup>11</sup></u>	R	0
(6)	1	1	1	1	1	1	1	1	1	1	1	11	4
(5,1)	-1	0	-1	1	-1	0	2	-1	1	3	5	8	6
(4,2)	0	-1	1	-1	0	0	0	3	1	3	9	15	8
$(4, 1^2)$	1	0	0	0	1	-1	1	-2	-2	2	10	10	12
$(3^2)$	0	0	-1	-1	2	1	-1	-3	1	1	5	4	6
(3, 2, 1)	0	1	0	0	-2	0	-2	0	0	0	16	13	13
$(3,1^3)$	-1	0	0	0	1	1	1	2	-2	-2	10	10	12
$(2^3)$	0	0	-1	1	2	-1	-1	3	1	-1	5	8	6
$(2^2, 1^2)$	0	-1	1	1	0	0	0	-3	1	-3	9	5	8
$(2,1^4)$	1	0	-1	$^{-1}$	-1	0	2	1	1	-3	5	4	6
(16)	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	4

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#### THANK YOU!

