Plethystic inversion and representations of the symmetric group

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Symmetric functions in m variables are characters of polynomial representations of GL_m .

Symmetric functions of degree n, via the Frobenius characteristic map, are in correspondence with the representation ring of the symmetric group S_n .

Plethysm of symmetric functions f[g] corresponds to composition of GL_n characters.

For the symmetric group it corresponds to forming representations of wreath products $S_m[S_n]$, and inducing up to S_{mn} .

Partitions of *n* and symmetric functions

- $\lambda = (\lambda_1 \ge \lambda_2 \ge \dots \lambda_k \ge 1)$ such that $\sum_i \lambda_i = n$ is an integer partition of n; $\ell(\lambda)$ is the number of parts k of λ .
- $p_r = \sum_i x_i^r$ is the *r*th power-sum symmetric function;
- *p*_λ = *p*_{λ1}*p*_{λ2}... is the power-sum symmetric function indexed by the partition λ;
- s_{λ} denotes the Schur function indexed by the partition λ ;
- Up to a scalar multiple, p_λ is the Frobenius characteristic of the class function that is 1 on the conjugacy class indexed by λ and zero elsewhere;
- s_λ is the Frobenius characteristic of the S_n-irreducible indexed by λ.
- A symmetric function *f* of homogeneous degree *n* is *Schur positive* if it is a nonnegative integer combination of Schur functions, i.e. if it corresponds to a true *S*_n-module.

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- The homogeneous symmetric function h_n of degree n is the Frobenius characteristic of the trivial representation of S_n . It is also the character of GL(V) acting on the nth symmetric power $Sym^n(V)$.
- The elementary symmetric function e_n of degree n is the Frobenius characteristic of the sign representation of S_n. It is also the character of GL(V) acting on the nth exterior power ∧ⁿ(V). The involution ω in the ring of symmetric functions is defined

by $\omega(h_n) = e_n$.

• p_1^n is the Frobenius characteristic of the regular representation of S_n .

It is also the character of GL(V) acting on the *n*th tensor power $V^{\otimes n}$.

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Plethysm and Schur-Weyl duality

Define $H = \sum_{n \ge 0} h_n$ and $E = \sum_{n \ge 0} e_n$.

If $F = \sum_{i \ge 1} f_i$ is the GI(V)-character on $W = \bigoplus_i W_i$, then: H[F] is the GI(V)-character of the symmetric algebra Sym(W). E[F] is the GI(V)-character of the exterior algebra $\bigwedge W$.

If λ is the partition with m_i parts equal to i, then:

 $H_{\lambda}[F]$ is the character of the piece $\otimes_i Sym^{m_i}(W_i)$ of the symmetric algebra Sym(W).

 $E_{\lambda}[F]$ is the character of the piece $\otimes_i \bigwedge^{m_i}(W_i)$ of the exterior algebra $\bigwedge W$.

 $(1 - p_1)^{-1} = \sum_{n \ge 0} p_1^n$ is the GI(V)-character on the full tensor algebra T(V).

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If $F = \sum_{j\geq 1} f_j$ where each f_j is the Frobenius characteristic ch of an S_j -module W_j , and λ is the partition of n with m_j parts equal to i, then:

$$H_{\lambda}[F] = \operatorname{ch}\left(\bigotimes_{i} \mathbf{1}_{S_{m_{i}}}[W_{i}]\right) \uparrow^{S_{n}} = \prod_{i} h_{m_{i}}[f_{i}]$$
$$E_{\lambda}[F] = \operatorname{ch}\left(\bigotimes_{i} \operatorname{sgn}_{S_{m_{i}}}[W_{i}]\right) \uparrow^{S_{n}} = \prod_{i} e_{m_{i}}[f_{i}].$$

The modules are induced from the subgroup $\prod_i S_{m_i}[S_i]$, the normaliser in S_n of the direct product of m_i copies of S_i , $i \ge 1$.

The regular representation of a finite group G

$$Reg_G := 1 \uparrow_e^G = \sum_{\chi \text{ irreducible repn of } G} (\dim \chi) \chi.$$

For the symmetric group S_n :

Theorem (Reg0)

$$\operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} f^{\lambda} \chi^{\lambda},$$

with Frobenius characteristic

$$\operatorname{ch} \operatorname{Reg}_{S_n} := \sum_{\lambda \vdash n} f^{\lambda} s_{\lambda},$$

where λ is an integer partition of n, $f^{\lambda} = |\{\text{standard Young tableaux of shape }\lambda\}|$, and s_{λ} is the Schur function indexed by λ , so $s_{\lambda} = \operatorname{ch} \chi^{\lambda}$.

Let C_n be the cyclic subroup of S_n generated by the long cycle $\sigma = (12 \dots n)$, let ω_n be a primitive *n*th root of unity. For $1 \le k \le n$, $\sigma \mapsto \omega_n^k$ yields a representation of C_n , and these are all the distinct irreducibles, so

$$Reg_{C_n} = \sum_{k=1}^n (\omega_n^k).$$

This gives a second decomposition of the regular representation of S_n :

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The regular representation of S_n — (I) (continued)

Theorem (Reg1)

$$Reg_{S_n} = \sum_{k=1}^n (\omega_n^k) \uparrow_{C_n}^{S_n}$$

Proof: Induce the decomposition of the regular representation of C_n up to S_n . (Or take tensor products over $\mathbb{C}C_n$ with $\mathbb{C}S_n$.)

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Definition

$$Lie_n := \omega_n \uparrow_{C_n}^{S_n}.$$

Definition

$$Conj_n := \omega_n^n \uparrow_{C_n}^{S_n} = 1 \uparrow_{C_n}^{S_n}.$$

 $Conj_n$ is the permutation representation of S_n by conjugation on the class of *n*-cycles, since the stabiliser of an *n*-cycle is the cyclic group C_n .

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Ramanujan sums

 $\mu(n), \ \phi(n)$ are respectively the number-theoretic Möbius and totient functions.

Let
$$\ell_n^{(k)} := \operatorname{ch} \omega_n^k \uparrow_{C_n}^{S_n}, 1 \le k \le n.$$

Theorem (Foulkes, 1972)

$$\ell_n^{(k)} = \frac{1}{n} \sum_{d|n} \phi(d) \; \frac{\mu(\frac{d}{(d,k)})}{\phi(\frac{d}{(d,k)})} \, p_d^{\frac{n}{d}}.$$

The quantity

$$\phi(d) \; rac{\mu(rac{d}{(d,k)})}{\phi(rac{d}{(d,k)})}$$

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is a *Ramanujan sum*; it equals the sum of the *k*th powers of all primitive *d*th roots of unity.

Note
$$\ell_n^{(1)} = \operatorname{ch} Lie_n$$
, $\ell_n^{(n)} = \operatorname{ch} Conj_n$. Hence
 $\operatorname{ch} Lie_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}}$.
 $\operatorname{ch} Conj_n = \frac{1}{n} \sum_{d|n} \phi(d) p_d^{\frac{n}{d}}$.

<ロト < 個ト < 目ト < 目ト 目 のので 12 / 50 Let $\mathcal{R} := \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}_n$ where \mathcal{I}_n is the ideal generated by the non-constant symmetric polynomials.

 \mathcal{R} is the ring of coinvariants for S_n , carrying a representation of S_n :

Theorem (Chevalley 1955)

The ring of coinvariants \mathcal{R} affords the regular representation of S_n .

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The ring \mathcal{R} is graded (by degree):

$$\mathcal{R} = \bigoplus_{i \ge 0} \mathcal{R}_i.$$

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Theorem (Kráskiewicz & Weyman (1987/2001))

For each k = 1, ..., n, $\bigoplus_{i \equiv k \mod n} \mathcal{R}_i$ is isomorphic to $\omega_n^k \uparrow_{C_n}^{S_n}$.

Decomposition into irreducibles?

Theorem (Stanley, see EC2 Ex. 7.88, Kráskiewicz & Weyman (1987/2001))

The multiplicity of the irreducible χ^{λ} in $\omega_n^k \uparrow_{C_n}^{S_n}$ is the number of standard Young tableaux **t** of shape λ whose **major index** is congruent to k mod n.

Plethysm with the *Lie* characteristic (I)

Let
$$Lie = \sum_{n\geq 1} Lie_n = \sum_{n\geq 1} \operatorname{ch} \omega_n \uparrow_{C_n}^{S_n}$$
. Then

Theorem (Thrall 1942)

$$H[Lie] = (1 - p_1)^{-1}.$$

Proof.

Use the identity $H = \exp(\sum_{i \ge 1} \frac{p_i}{i})$ and the following facts:

•
$$p_i[f] = f[p_i]; p_1[f] = f[p_1] = f; c[f] = c$$
 for constants c.

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$$p_i[p_j] = p_{ij} = p_j[p_i].$$

$$(f+g)[r] = f[r] + g[r];$$

$$(fg)[r] = f[r]g[r]; (f/g)[r] = f[r]/g[r].$$

The regular representation of S_n — (II)

The plethystic identity

$$H[Lie] = (1 - p_1)^{-1}$$

is equivalent to

Theorem (Reg2 : Thrall 1942)

$$\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} H_{\lambda}[\operatorname{Lie}] = \sum_{\lambda \vdash n} h_{m_1}[\operatorname{Lie}_1]h_{m_2}[\operatorname{Lie}_2]\dots,$$

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where λ has m_i parts equal to i.

Definition

We say symmetric functions f and g are plethystic inverses if

$$f[g]=g[f]=p_1.$$

Associativity of plethysm implies that $f[g] = p_1 \iff g[f] = p_1$.

Theorem

If f is a symmetric function with no constant term and nonzero coefficient for p_1 , then f has a plethystic inverse.

Proof: Let $f = p_1 + \sum_{n \ge 2} f_n$ where f_n is of homogeneous degree *n*. Then

$$p_1 = f[g] = g + (\sum_{n \ge 2} f_n[g]).$$

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Isolate by degree and recursively compute the terms of g.

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The plethystic inverse of $H-1 = \sum_{n\geq 1} h_n$

Theorem (INV1: Cadogan 1971)

$$\sum_{n\geq 1}h_n[\sum_{n\geq 1}(-1)^{n-1}\omega(Lie_n)]=p_1.$$

Plethystic Identities: A meta theorem

Fix
$$\psi : \mathbb{N}_+ \to \mathbb{R}$$
.
Let $f_n := \frac{1}{n} \sum_{d|n} \psi(d) p_d^{\frac{n}{d}}$, and $f_n(t) := \frac{1}{n} \sum_{d|n} \psi(d) t^{\frac{n}{d}}$.
Let $F := \sum_{n \ge 1} f_n$, $F^{alt} := \sum_{n \ge 1} (-1)^{n-1} f_n$ (symmetric functions).

Theorem (S 2017)

$$H[F] = \prod_{m \ge 1} (1 - p_m)^{-f_m(1)} \quad (1)$$

$$\iff E[F] = \prod_{m \ge 1} (1 - p_m)^{f_m(-1)} \quad (2)$$

$$\iff H[\omega(F)^{alt}] = \prod_{m \ge 1} (1 + p_m)^{f_m(1)} \quad (3)$$

$$\iff E[\omega(F)^{alt}] = \prod_{m \ge 1} (1 + p_m)^{-f_m(-1)} \quad (4)$$

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Take $\psi(d) = \mu(d)$ to obtain, for F = Lie, since $f_m(1) = \delta_{m,1}$,

$$H[Lie] = (1 - p_1)^{-1} \iff H[\omega(Lie)^{alt}] = 1 + p_1.$$

So Thrall's theorem is equivalent to Cadogan's computation of the plethystic inverse of $\sum_{n>1} h_n$, and also to:

Theorem (INV2 : Plethystic inverse of *Lie*)

(Orlik-Solomon, Lehrer-Solomon 1986)

$$\sum_{n\geq 1} (-1)^{n-1} e_n[Lie] = p_1$$

Specialisations of $\psi(d) - \mathsf{II}$

Recall
$$E = \sum_{n \ge 0} e_n$$
.

Definition (S 2018)

Let k_n be the 2-valuation of the positive integer n. Define

$$Lie_n^{(2)} := \omega_n^{k_n} \uparrow_{C_n}^{S_n}$$

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Notice:
$$Lie_n^{(2)} = \begin{cases} Lie_n & n \text{ odd}, \\ Conj_n & n \text{ a power of } 2. \end{cases}$$

Meta theorem + the corresponding specialisation of $\psi(d)$ gives

Theorem (Reg3: S 2018)

$$E[\sum_{n\geq 1}Lie_n^{(2)}]=(1-p_1)^{-1}.$$

$$\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} E_{\lambda}[\operatorname{Lie}_n^{(2)}] = \sum_{\lambda \vdash n} e_{m_1}[\operatorname{Lie}_1^{(2)}]e_{m_2}[\operatorname{Lie}_2^{(2)}]\dots$$

The inverses of
$$E-1=\sum_{n\geq 1}e_n$$
 and $\sum_{n\geq 1}Lie_n^{(2)}$

The meta theorem implies

| Corollary (S 2018) | | |
|--------------------|-----------------|--|
| | (<i>INV</i> 3) | $\sum_{n\geq 1} e_n[\sum_{n\geq 1} (-1)^{n-1} \omega(Lie_n^{(2)})] = p_1.$ |
| and | | |
| | (<i>INV</i> 4) | $\sum_{n\geq 1} (-1)^{n-1} h_n [\sum_{n\geq 1} Lie_n^{(2)}] = p_1.$ |

Question: Is there a more conceptual explanation for (Reg3), (Inv3), (Inv4)? An acyclic complex?

Lie versus Lie⁽²⁾

Theorem (S 2018)

$$Lie_n = Lie_n^{(2)} - Lie_{\frac{n}{2}}^{(2)}[p_2].$$

Proof: First note that $H[p_1 - p_2] = E$. Also, $(p_1 - p_2)$ and $\sum_{k\geq 0} p_{2^k}$ are plethystic inverses.

$$(1 - p_1)^{-1} = H[Lie] = H[(p_1 - p_2)[\sum_{k \ge 0} p_{2^k}]] [Lie]$$
$$= (H[p_1 - p_2])[\sum_{k \ge 0} p_{2^k}[Lie]] = E[\sum_{k \ge 0} p_{2^k}[Lie]]$$
$$\Longrightarrow \sum_{n \ge 1} Lie_n^{(2)} = \sum_{k \ge 0} p_{2^k}[Lie].$$

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Recall that
$$\ell_n^{(k)} = \operatorname{ch} \omega_n^k \uparrow_{C_n}^{S_n}$$
.

Theorem (S 2017) Fix $k \ge 1$. Then $H[\sum_{m \ge 1} \ell_n^{(k)}] = 1 + \sum_{\substack{\lambda \\ \lambda_i \mid k}} p_{\lambda}.$

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This generalises Thrall's Theorem.

From this one can deduce

Corollary (S 2018)

$$\ell_n^{(k)} = \sum_{m \mid \gcd(n,k)} Lie_{\frac{n}{m}}[p_m],$$

confirming that $\ell_n^{(k)}$ and the representation $\omega_n^k \uparrow_{C_n}^{S_n}$ depend only on gcd(n,k).

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The plethystic inverse of the odd Lie representations

Theorem (INV5: S 2020; conjectured by Richard Stanley)

The plethystic inverse of $\sum_{n>1} Lie_{2n-1}$ is

$$\frac{e_1+e_3+\ldots}{1+e_2+e_4+\ldots}$$

Theorem (S 2020)

Let
$$\delta_n = (n - 1, n - 2, \dots, 1), n \ge 2$$
. (Set $\delta_1 = \emptyset$.) Then

$$\frac{e_1 + e_3 + \dots}{1 + e_2 + e_4 + \dots} = s_{(1)} + \sum_{n \ge 3} (-1)^n s_{\delta_n / \delta_{n-2}}$$
$$= \tanh(\sum_{i \ge 1} \operatorname{arctanh} x_i).$$

The plethystic inverse of the alternating odd Lie's

Theorem (INV6: S 2020)

The plethystic inverse of $\sum_{n>0} (-1)^n Lie_{2n+1}$ is

$$\frac{e_1 - e_3 + e_5 - \dots}{1 - e_2 + e_4 - \dots}$$

Theorem (Carlitz 1973)

Let
$$\delta_n = (n - 1, n - 2, ..., 1), n \ge 2$$
. (Set $\delta_1 = \emptyset$.) Then

$$rac{e_1-e_3+e_5-\dots}{1-e_2+e_4-\dots}=s_{(1)}+\sum_{n\geq 3}s_{\delta_n/\delta_{n-2}}\ = an(\sum_{i\geq 1}rctan\ x_i).$$

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Define Hk_n to be the Frobenius characteristic of the (multiplicity-free) sum of all irreducibles indexed by hooks $(n - r, 1^r), r = 0, 1, ..., n - 1$; $Hk_1 = h_1$. Then one has (yet another) decomposition of the regular representation of S_n :

Theorem (Reg4: S 2020) $\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{n \ge 1} \operatorname{Hk}_n[\sum_{n \ge 1} \operatorname{Lie}_{2n-1}].$

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Question: Is there a more conceptual explanation?

Fix $k \ge 2$. Let $F_{1,k} := \sum_{m \ge 0} f_{mk+1}$, $G_{1,k} := \sum_{m \ge 0} g_{mk+1}$ be two series of symmetric functions where the f_i, g_i are of homogeneous degree i, and $f_1 = g_1 = p_1$. Define $F_{1,k}^{alt} := \sum_{m \ge 0} (-1)^m f_{mk+1}$ and similarly $G_{1,k}^{alt}$.

Theorem (S 2020)

$$F_{1,k}[G_{1,k}] = p_1 \iff F_{1,k}^{alt}[G_{1,k}^{alt}] = p_1.$$

The free Lie algebra

As a vector space, Lie_n is the degree *n* multilinear component of the free Lie algebra on *m* generators. Its Gl_m -character was computed by Brandt (1944) to be

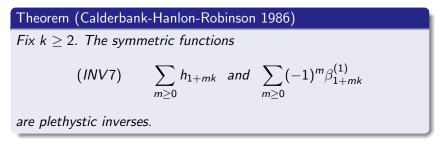
ch Lie_n = Lie_n(x₁,...,x_m) =
$$\frac{1}{n}\sum_{d|n}\mu(d)p_d^{\frac{n}{d}}$$
.

Recall (Reg2):

$$\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} H_{\lambda}[\operatorname{Lie}] = \sum_{\lambda \vdash n} h_{m_1}[\operatorname{Lie}_1]h_{m_2}[\operatorname{Lie}_2]\dots,$$

In the Gl_m -context, the Poincaré-Birkhoff-Witt theorem for the free Lie algebra says its universal enveloping algebra is the full tensor algebra. Hence by Schur-Weyl duality, Thrall's theorem gives the decomposition of the full tensor algebra as a sum of symmetrised Lie modules.

Let $\beta_n^{(j)}$, $0 \le j \le k - 1$, be the Frobenius characteristic of the unique nonvanishing homology S_n -module of the subposet of partitions of n with block sizes congruent to $j \mod k$.



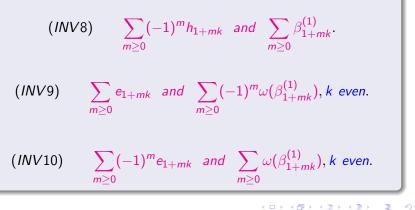
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$1 \mod k$ continued

From the meta theorem we obtain:

Theorem (S 2020)

Fix $k \ge 2$. The following pairs of symmetric functions are plethystic inverses:



Recall

$$\operatorname{ch} \operatorname{Reg}_{S_n} = \sum_{\lambda \vdash n} H_{\lambda}[\operatorname{Lie}] = \sum_{\lambda \vdash n} h_{m_1}[\operatorname{Lie}_1]h_{m_2}[\operatorname{Lie}_2]\dots$$

The modules $H_{\lambda}[Lie]$ are the *higher Lie* modules. Their irreducible decomposition is known only for the cases $\lambda = (n), (1^n)$ and (2^a) or $(2^a, 1)$.

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Let J_n be the degree *n* multilinear component of the free Jordan algebra on *m* generators. Instead of the Lie bracket, we have the bracket

$$[x,y] = x \otimes y + y \otimes x.$$

Schur-Weyl duality:

View J_n as an S_n -module, with Frobenius characteristic η_n . Set $\eta_0 = 1$.

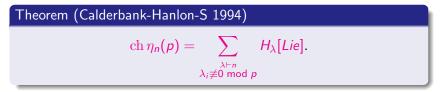
Theorem (Calderbank-Hanlon-S 1994)

The Frobenius characteristic of the S_n -module on the free Jordan algebra satisfies

$$H[\sum_{n\geq 1} Lie_{2n-1}] = \sum_{n\geq 0} \eta_n.$$

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Fix $p \ge 2$. Let α_p be a primitive *p*th root of unity. Consider the degree *n* multilinear component of the free algebra with bracket $[x, y] := x \otimes y - \alpha_p y \otimes x$. The representation $\eta_n(p)$ of S_n on this component is a sum of higher Lie modules:



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The meta theorem can be used to prove:

Theorem (Solomon 1961)
$$H[\sum_{n\geq 1} Conj_n] = \sum_{\lambda} H_{\lambda}[\sum_{n\geq 1} Conj_n] = \sum_{\mu\vdash n} p_{\mu}.$$

 $H_{\lambda}[\sum_{n\geq 1} Conj_n]$ is the action by conjugation on the class indexed by λ . The character value on σ is the order of the centraliser of σ .

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(*) Why does the number of standard Young tableaux t of shape λ with major index $\equiv k \mod n$ depend only on (k, n)?

What is the irreducible decomposition of

- $H_{\lambda}[Lie]$? (Thrall's problem 1942)
- 2 $H_{\lambda}[Conj]?$
- **3** More generally, for $H_{\lambda}[\ell_n^{(k)}]$?

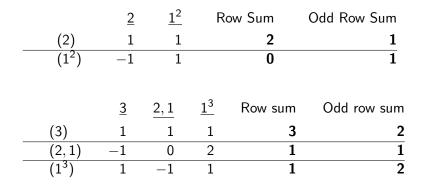
Known only for

$$old \lambda=({\it n})$$
 : (Stanley, Kráskiewicz-Weyman)

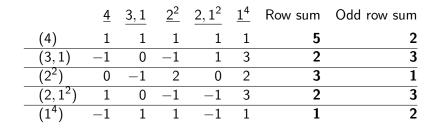
2
$$\lambda = (1^n)$$
 (the trivial module)

- $\textcircled{0} \lambda \text{ has all parts equal to 1 or 2.}$
- $\lambda = (n k, 1^k)$ (recent work of Hegedus-Roichman).

Character tables of S_n

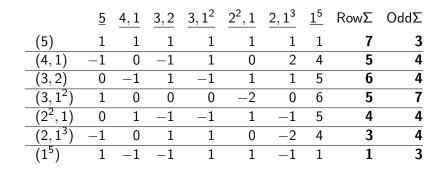


Character tables for S_4



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Character tables: S_5



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Character tables: S_6

| | \underline{C}^1 | <u>C</u> ² | <u>C³</u> | <u>C</u> ⁴ | <u>C</u> ⁵ | <u>C</u> ⁶ | <u>C</u> ⁷ | <u>C</u> ⁸ | <u>C</u> ⁹ | <u>C</u> ¹⁰ | <u>C¹¹</u> | R | 0 |
|--------------|-------------------|-----------------------|----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|------------------------|-----------------------|----|----|
| (6) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 11 | 4 |
| (5,1) | -1 | 0 | -1 | 1 | -1 | 0 | 2 | -1 | 1 | 3 | 5 | 8 | 6 |
| (4,2) | 0 | -1 | 1 | -1 | 0 | 0 | 0 | 3 | 1 | 3 | 9 | 15 | 8 |
| $(4, 1^2)$ | 1 | 0 | 0 | 0 | 1 | -1 | 1 | -2 | -2 | 2 | 10 | 10 | 12 |
| (3^2) | 0 | 0 | -1 | -1 | 2 | 1 | -1 | -3 | 1 | 1 | 5 | 4 | 6 |
| (3, 2, 1) | 0 | 1 | 0 | 0 | -2 | 0 | -2 | 0 | 0 | 0 | 16 | 13 | 13 |
| $(3,1^3)$ | -1 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | -2 | -2 | 10 | 10 | 12 |
| (2^3) | 0 | 0 | -1 | 1 | 2 | -1 | -1 | 3 | 1 | -1 | 5 | 8 | 6 |
| $(2^2, 1^2)$ | 0 | -1 | 1 | 1 | 0 | 0 | 0 | -3 | 1 | -3 | 9 | 5 | 8 |
| $(2,1^4)$ | 1 | 0 | -1 | $^{-1}$ | -1 | 0 | 2 | 1 | 1 | -3 | 5 | 4 | 6 |
| (16) | -1 | 1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | 1 | 4 |

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