Plethystic inversion and representations of the symmetric group

Sheila Sundaram

Pierrepont School, Westport, CT

2 July 2020

Algebraic Combinatorics Seminar
Institute for Mathematical Sciences, Chennai

Background and notation

Symmetric functions in m variables are characters of polynomial representations of GL_m.

Symmetric functions of degree n, via the Frobenius characteristic map, are in correspondence with the representation ring of the symmetric group S_n.

Plethysm of symmetric functions $f[g]$ corresponds to composition of GL_n characters.

For the symmetric group it corresponds to forming representations of wreath products $S_m[S_n]$, and inducing up to S_{mn}.

Partitions of n and symmetric functions

- $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \geq 1)$ such that $\sum_i \lambda_i = n$ is an integer partition of n; $\ell(\lambda)$ is the number of parts k of λ.
- $p_r = \sum_i {x_i}^r$ is the rth power-sum symmetric function;
- $p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots$ is the power-sum symmetric function indexed by the partition λ;
- s_λ denotes the Schur function indexed by the partition λ;
- Up to a scalar multiple, p_λ is the Frobenius characteristic of the class function that is 1 on the conjugacy class indexed by λ and zero elsewhere;
- s_λ is the Frobenius characteristic of the S_n-irreducible indexed by λ.
- A symmetric function f of homogeneous degree n is *Schur positive* if it is a nonnegative integer combination of Schur functions, i.e. if it corresponds to a true S_n-module.

Symmetric and Exterior Powers

- The homogeneous symmetric function h_n of degree n is the Frobenius characteristic of the trivial representation of S_n.
  It is also the character of $GL(V)$ acting on the nth symmetric power $Sym^n(V)$.
- The elementary symmetric function e_n of degree n is the Frobenius characteristic of the sign representation of S_n.
  It is also the character of $GL(V)$ acting on the nth exterior power $\bigwedge^n(V)$.
  The involution ω in the ring of symmetric functions is defined by $\omega(h_n) = e_n$.
- p_1^n is the Frobenius characteristic of the regular representation of S_n.
  It is also the character of $GL(V)$ acting on the nth tensor power $V^{\otimes n}$.

Define $H = \sum_{n \geq 0} h_n$ and $E = \sum_{n \geq 0} e_n$.

If $F = \sum_{i \geq 1} f_i$ is the $Gl(V)$-character on $W = \oplus_i W_i$, then:
$H[F]$ is the $Gl(V)$-character of the symmetric algebra $Sym(W)$.

$E[F]$ is the $Gl(V)$-character of the exterior algebra $\bigwedge W$.

If λ is the partition with m_i parts equal to i, then:

$H_\lambda[F]$ is the character of the piece $\otimes_i Sym^{m_i}(W_i)$ of the symmetric algebra $Sym(W)$.

$E_\lambda[F]$ is the character of the piece $\otimes_i \bigwedge^{m_i}(W_i)$ of the exterior algebra $\bigwedge W$.

$(1 - p_1)^{-1} = \sum_{n \geq 0} p_1^n$ is the Gl(V)-character on the full tensor algebra $T(V)$.

If $F = \sum_{j\geq 1} f_j$ where each f_j is the Frobenius characteristic ch of an S_j-module W_j, and λ is the partition of n with m_i parts equal to i, then:

$$H_\lambda[F] = \mathrm{ch}\,(\bigotimes_i \mathbf{1}_{S_{m_i}}[W_i])\uparrow^{S_n} = \prod_i h_{m_i}[f_i]$$

$$E_\lambda[F] = \mathrm{ch}\,(\bigotimes_i \mathbf{sgn}_{S_{m_i}}[W_i])\uparrow^{S_n} = \prod_i e_{m_i}[f_i].$$

The modules are induced from the subgroup $\prod_i S_{m_i}[S_i]$, the normaliser in S_n of the direct product of m_i copies of $S_i, i \geq 1$.

The regular representation of a finite group G

$$Reg_G := 1\uparrow_e^G = \sum_{\chi \text{ irreducible repn of } G} (\dim \chi)\chi.$$

For the symmetric group S_n :

Theorem (Reg0)

$$Reg_{S_n} = \sum_{\lambda \vdash n} f^\lambda \chi^\lambda,$$

with Frobenius characteristic

$$\mathrm{ch}\, Reg_{S_n} := \sum_{\lambda \vdash n} f^\lambda s_\lambda,$$

where λ is an integer partition of n,
$f^\lambda = |\{\text{standard Young tableaux of shape } \lambda\}|$, and
s_λ is the Schur function indexed by λ, so $s_\lambda = \mathrm{ch}\, \chi^\lambda$.

The regular representation of S_n — (I)

Let C_n be the cyclic subroup of S_n generated by the long cycle $\sigma = (1\,2\,\ldots\,n)$, let ω_n be a primitive nth root of unity.
For $1 \leq k \leq n$, $\sigma \mapsto \omega_n^k$ yields a representation of C_n, and these are all the distinct irreducibles, so

$$Reg_{C_n} = \sum_{k=1}^{n} (\omega_n^k).$$

This gives a second decomposition of the regular representation of S_n :

Theorem (Reg1)

$$Reg_{S_n} = \sum_{k=1}^{n} (\omega_n^k) \uparrow_{C_n}^{S_n}$$

Proof: Induce the decomposition of the regular representation of C_n up to S_n. (Or take tensor products over $\mathbb{C}C_n$ with $\mathbb{C}S_n$.)

The summands in Reg1

Definition

$Lie_n := \omega_n \uparrow_{C_n}^{S_n}$.

Definition

$Conj_n := \omega_n^n \uparrow_{C_n}^{S_n} = 1 \uparrow_{C_n}^{S_n}$.

$Conj_n$ is the permutation representation of S_n by conjugation on the class of n-cycles, since the stabiliser of an n-cycle is the cyclic group C_n.

Ramanujan sums

$\mu(n)$, $\phi(n)$ are respectively the number-theoretic Möbius and totient functions.

Let $\ell_n^{(k)} := \mathrm{ch}\,\omega_n^k \uparrow_{C_n}^{S_n}, 1 \leq k \leq n$.

Theorem (Foulkes, 1972)

$$\ell_n^{(k)} = \frac{1}{n} \sum_{d|n} \phi(d) \, \frac{\mu(\frac{d}{(d,k)})}{\phi(\frac{d}{(d,k)})} \, p_d^{\frac{n}{d}}.$$

The quantity

$$\phi(d) \, \frac{\mu(\frac{d}{(d,k)})}{\phi(\frac{d}{(d,k)})}$$

is a *Ramanujan sum*; it equals the sum of the kth powers of all primitive dth roots of unity.

Note $\ell_n^{(1)} = \mathrm{ch}\, Lie_n$, $\ell_n^{(n)} = \mathrm{ch}\, Conj_n$. Hence

$$\mathrm{ch}\, Lie_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}}.$$

$$\mathrm{ch}\, Conj_n = \frac{1}{n} \sum_{d|n} \phi(d) p_d^{\frac{n}{d}}.$$

Ring of coinvariants I

Let $\mathcal{R} := \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}_n$ where $\mathcal{I}_n$ is the ideal generated by the non-constant symmetric polynomials.
$\mathcal{R}$ is the ring of coinvariants for S_n, carrying a representation of S_n:

Theorem (Chevalley 1955)

The ring of coinvariants $\mathcal{R}$ affords the regular representation of S_n.

Ring of coinvariants II

The ring $\mathcal{R}$ is graded (by degree):

$$\mathcal{R} = \bigoplus_{i \geq 0} \mathcal{R}_i.$$

Theorem (Kráskiewicz & Weyman (1987/2001))

For each $k = 1, \ldots, n$,
$\bigoplus_{i \equiv k \bmod n} \mathcal{R}_i$ *is isomorphic to* $\omega_n^k \uparrow_{C_n}^{S_n}$.

Decomposition into irreducibles?

A famous theorem in Algebraic Combinatorics

Theorem (Stanley, see EC2 Ex. 7.88, Kráskiewicz & Weyman (1987/2001))

The multiplicity of the irreducible χ^λ *in* $\omega_n^k \uparrow_{C_n}^{S_n}$ *is the number of standard Young tableaux* $\mathbf{t}$ *of shape* λ *whose* **major index** *is congruent to* $k \bmod n$.

Plethysm with the *Lie* characteristic (I)

Let $Lie = \sum_{n \geq 1} Lie_n = \sum_{n \geq 1} \mathrm{ch}\, \omega_n \uparrow_{C_n}^{S_n}$. Then

Theorem (Thrall 1942)

$$H[Lie] = (1 - p_1)^{-1}.$$

Proof.

Use the identity $H = \exp(\sum_{i \geq 1} \frac{p_i}{i})$ and the following facts:

1. $p_i[f] = f[p_i]$; $p_1[f] = f[p_1] = f$; $c[f] = c$ for constants c.
2. $p_i[p_j] = p_{ij} = p_j[p_i]$.
3. $(f + g)[r] = f[r] + g[r]$;
4. $(fg)[r] = f[r]\, g[r]$; $(f/g)[r] = f[r]/g[r]$.

□

The regular representation of S_n — (II)

The plethystic identity

$$H[Lie] = (1 - p_1)^{-1}$$

is equivalent to

Theorem (Reg2 : Thrall 1942)

$$\mathrm{ch}\, Reg_{S_n} = \sum_{\lambda \vdash n} H_\lambda[Lie] = \sum_{\lambda \vdash n} h_{m_1}[Lie_1] h_{m_2}[Lie_2] \ldots,$$

where λ has m_i parts equal to i.

Plethystic Inverses

Definition

We say symmetric functions f and g are plethystic inverses if

$$f[g] = g[f] = p_1.$$

Associativity of plethysm implies that $f[g] = p_1 \iff g[f] = p_1$.

Theorem

If f is a symmetric function with no constant term and nonzero coefficient for p_1, then f has a plethystic inverse.

Proof: Let $f = p_1 + \sum_{n\geq 2} f_n$ where f_n is of homogeneous degree n. Then

$$p_1 = f[g] = g + (\sum_{n\geq 2} f_n[g]).$$

Isolate by degree and recursively compute the terms of g.

Examples

1. $(p_1 - 1)$ and $(p_1 + 1)$ are inverses.
2. $\frac{p_1}{1+p_1}$ and $\frac{p_1}{1-p_1}$ are inverses.
3. $\sum_{n\geq 1} p_n$ and $\sum_{n\geq 1} \mu(n)p_n$ are inverses.

Theorem (INV1: Cadogan 1971)

$$\sum_{n\geq 1} h_n[\sum_{n\geq 1}(-1)^{n-1}\omega(Lie_n)] = p_1.$$

Plethystic Identities: A meta theorem

Fix $\psi : \mathbb{N}_+ \to \mathbb{R}$.

Let $f_n := \frac{1}{n} \sum_{d|n} \psi(d) p_d^{\frac{n}{d}}$, and $f_n(t) := \frac{1}{n} \sum_{d|n} \psi(d) t^{\frac{n}{d}}$.

Let $F := \sum_{n \geq 1} f_n$, $\quad F^{alt} := \sum_{n \geq 1} (-1)^{n-1} f_n$ (symmetric functions).

Theorem (S 2017)

$$H[F] = \prod_{m \geq 1} (1 - p_m)^{-f_m(1)} \quad (1)$$

$$\iff E[F] = \prod_{m \geq 1} (1 - p_m)^{f_m(-1)} \quad (2)$$

$$\iff H[\omega(F)^{alt}] = \prod_{m \geq 1} (1 + p_m)^{f_m(1)} \quad (3)$$

$$\iff E[\omega(F)^{alt}] = \prod_{m \geq 1} (1 + p_m)^{-f_m(-1)} \quad (4)$$

Specialisations of $\psi(d)$ — I

Take $\psi(d) = \mu(d)$ to obtain, for $F = Lie$, since $f_m(1) = \delta_{m,1}$,

$$H[Lie] = (1 - p_1)^{-1} \iff H[\omega(Lie)^{alt}] = 1 + p_1.$$

So Thrall's theorem is equivalent to Cadogan's computation of the plethystic inverse of $\sum_{n\geq 1} h_n$, and also to:

Theorem (INV2 : Plethystic inverse of *Lie*)

(Orlik-Solomon, Lehrer-Solomon 1986)

$$\sum_{n\geq 1} (-1)^{n-1} e_n[Lie] = p_1$$

Recall $E = \sum_{n\geq 0} e_n$.

Definition (S 2018)

Let k_n be the 2-valuation of the positive integer n. Define

$$Lie_n^{(2)} := \omega_n^{k_n}\uparrow_{C_n}^{S_n}.$$

Notice: $Lie_n^{(2)} = \begin{cases} Lie_n & n \text{ odd,} \\ Conj_n & n \text{ a power of 2.} \end{cases}$.

Meta theorem + the corresponding specialisation of $\psi(d)$ gives

Theorem (Reg3: S 2018)

$$E[\sum_{n\geq 1} Lie_n^{(2)}] = (1-p_1)^{-1}.$$

$$\mathrm{ch}\, Reg_{S_n} = \sum_{\lambda\vdash n} E_\lambda[Lie_n^{(2)}] = \sum_{\lambda\vdash n} e_{m_1}[Lie_1^{(2)}]e_{m_2}[Lie_2^{(2)}]\ldots$$

The inverses of $E - 1 = \sum_{n\geq 1} e_n$ and $\sum_{n\geq 1} Lie_n^{(2)}$

The meta theorem implies

Corollary (S 2018)

$$(INV3) \qquad \sum_{n\geq 1} e_n[\sum_{n\geq 1}(-1)^{n-1}\omega(Lie_n^{(2)})] = p_1.$$

and

$$(INV4) \qquad \sum_{n\geq 1}(-1)^{n-1}h_n[\sum_{n\geq 1} Lie_n^{(2)}] = p_1.$$

Question: Is there a more conceptual explanation for (Reg3), (Inv3), (Inv4)? An acyclic complex?

Theorem (S 2018)

$$Lie_n = Lie_n^{(2)} - Lie_{\frac{n}{2}}^{(2)}[p_2].$$

Proof: First note that $H[p_1 - p_2] = E$.
Also, $(p_1 - p_2)$ and $\sum_{k\geq 0} p_{2^k}$ are plethystic inverses.

$$\begin{aligned}(1-p_1)^{-1} &= H[Lie] = H[(p_1 - p_2)[\sum_{k\geq 0} p_{2^k}]]\,[Lie] \\ &= (H[p_1 - p_2])[\sum_{k\geq 0} p_{2^k}[Lie]] = E[\sum_{k\geq 0} p_{2^k}[Lie]] \\ &\implies \sum_{n\geq 1} Lie_n^{(2)} = \sum_{k\geq 0} p_{2^k}[Lie].\end{aligned}$$

Specialisations of $\psi(d)$ — III

Recall that $\ell_n^{(k)} = \mathrm{ch}\, \omega_n^k \uparrow_{C_n}^{S_n}$.

Theorem (S 2017)

Fix $k \geq 1$. Then

$$H[\sum_{m \geq 1} \ell_n^{(k)}] = 1 + \sum_{\substack{\lambda \\ \lambda_i | k}} p_\lambda.$$

This generalises Thrall's Theorem.

From this one can deduce

Corollary (S 2018)

$$\ell_n^{(k)} = \sum_{m \mid \gcd(n,k)} Lie_{\frac{n}{m}}[p_m],$$

confirming that $\ell_n^{(k)}$ and the representation $\omega_n^k \uparrow_{C_n}^{S_n}$ depend only on $\gcd(n, k)$.

The plethystic inverse of the odd Lie representations

Theorem (INV5: S 2020; conjectured by Richard Stanley)

The plethystic inverse of $\sum_{n\geq 1} Lie_{2n-1}$ *is*

$$\frac{e_1 + e_3 + \dots}{1 + e_2 + e_4 + \dots}.$$

Theorem (S 2020)

Let $\delta_n = (n-1, n-2, \dots, 1), n \geq 2$. *(Set* $\delta_1 = \emptyset$.*) Then*

$$\frac{e_1 + e_3 + \dots}{1 + e_2 + e_4 + \dots} = s_{(1)} + \sum_{n\geq 3} (-1)^n s_{\delta_n/\delta_{n-2}}$$

$$= \tanh\Big(\sum_{i\geq 1} \operatorname{arctanh} x_i\Big).$$

The plethystic inverse of the *alternating* odd Lie's

Theorem (INV6: S 2020)

The plethystic inverse of $\sum_{n\geq 0}(-1)^n Lie_{2n+1}$ *is*

$$\frac{e_1 - e_3 + e_5 - \dots}{1 - e_2 + e_4 - \dots}.$$

Theorem (Carlitz 1973)

Let $\delta_n = (n-1, n-2, \dots, 1), n \geq 2$. *(Set* $\delta_1 = \emptyset$.*) Then*

$$\frac{e_1 - e_3 + e_5 - \dots}{1 - e_2 + e_4 - \dots} = s_{(1)} + \sum_{n\geq 3} s_{\delta_n/\delta_{n-2}}$$
$$= \tan\Big(\sum_{i\geq 1} \arctan x_i\Big).$$

Define Hk_n to be the Frobenius characteristic of the (multiplicity-free) sum of all irreducibles indexed by hooks $(n-r, 1^r), r = 0, 1, \ldots, n-1$; $Hk_1 = h_1$. Then one has (yet another) decomposition of the regular representation of S_n :

Theorem (Reg4: S 2020)

$$\mathrm{ch}\, Reg_{S_n} = \sum_{n\geq 1} Hk_n[\sum_{n\geq 1} Lie_{2n-1}].$$

Question: Is there a more conceptual explanation?

From sums to alternating sums: another meta theorem

Fix $k \geq 2$. Let $F_{1,k} := \sum_{m\geq 0} f_{mk+1}$, $G_{1,k} := \sum_{m\geq 0} g_{mk+1}$ be two series of symmetric functions where the f_i, g_i are of homogeneous degree i, and $f_1 = g_1 = p_1$. Define $F^{alt}_{1,k} := \sum_{m\geq 0}(-1)^m f_{mk+1}$ and similarly $G^{alt}_{1,k}$.

Theorem (S 2020)

$$F_{1,k}[G_{1,k}] = p_1 \iff F^{alt}_{1,k}[G^{alt}_{1,k}] = p_1.$$

As a vector space, Lie_n is the degree n multilinear component of the free Lie algebra on m generators. Its Gl_m-character was computed by Brandt (1944) to be

$$\mathrm{ch}\, Lie_n = Lie_n(x_1, \ldots, x_m) = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}}.$$

Recall (Reg2):

$$\mathrm{ch}\, Reg_{S_n} = \sum_{\lambda \vdash n} H_\lambda[Lie] = \sum_{\lambda \vdash n} h_{m_1}[Lie_1] h_{m_2}[Lie_2] \ldots,$$

In the Gl_m-context, the Poincaré-Birkhoff-Witt theorem for the free Lie algebra says its universal enveloping algebra is the full tensor algebra. Hence by Schur-Weyl duality, Thrall's theorem gives the decomposition of the full tensor algebra as a sum of symmetrised Lie modules.

Other series that have *nice* plethystic inverses: 1 mod k

Let $\beta_n^{(j)}, 0 \leq j \leq k-1$, be the Frobenius characteristic of the unique nonvanishing homology S_n-module of the subposet of partitions of n with block sizes congruent to j mod k.

Theorem (Calderbank-Hanlon-Robinson 1986)

Fix $k \geq 2$. The symmetric functions

$$(INV7) \qquad \sum_{m\geq 0} h_{1+mk} \quad \text{and} \quad \sum_{m\geq 0} (-1)^m \beta_{1+mk}^{(1)}$$

are plethystic inverses.

From the meta theorem we obtain:

Theorem (S 2020)

Fix $k \geq 2$. The following pairs of symmetric functions are plethystic inverses:

$$(INV8) \qquad \sum_{m\geq 0} (-1)^m h_{1+mk} \quad \text{and} \quad \sum_{m\geq 0} \beta^{(1)}_{1+mk}.$$

$$(INV9) \qquad \sum_{m\geq 0} e_{1+mk} \quad \text{and} \quad \sum_{m\geq 0} (-1)^m \omega(\beta^{(1)}_{1+mk}),\ k \text{ even}.$$

$$(INV10) \qquad \sum_{m\geq 0} (-1)^m e_{1+mk} \quad \text{and} \quad \sum_{m\geq 0} \omega(\beta^{(1)}_{1+mk}),\ k \text{ even}.$$

More on Thrall's decomposition, Reg2

Recall

$$\mathrm{ch}\, Reg_{S_n} = \sum_{\lambda \vdash n} H_\lambda[Lie] = \sum_{\lambda \vdash n} h_{m_1}[Lie_1] h_{m_2}[Lie_2] \ldots .$$

The modules $H_\lambda[Lie]$ are the *higher Lie* modules. Their irreducible decomposition is known only for the cases $\lambda = (n), (1^n)$ and (2^a) or $(2^a, 1)$.

Let J_n be the degree n multilinear component of the free Jordan algebra on m generators. Instead of the Lie bracket, we have the bracket

$$[x, y] = x \otimes y + y \otimes x.$$

Schur-Weyl duality:
View J_n as an S_n-module, with Frobenius characteristic η_n. Set $\eta_0 = 1$.

Theorem (Calderbank-Hanlon-S 1994)

The Frobenius characteristic of the S_n-module on the free Jordan algebra satisfies

$$H[\sum_{n\geq 1} Lie_{2n-1}] = \sum_{n\geq 0} \eta_n.$$

Fix $p \geq 2$. Let α_p be a primitive pth root of unity. Consider the degree n multilinear component of the free algebra with bracket $[x, y] := x \otimes y - \alpha_p\, y \otimes x$. The representation $\eta_n(p)$ of S_n on this component is a sum of higher Lie modules:

Theorem (Calderbank-Hanlon-S 1994)

$$\mathrm{ch}\,\eta_n(p) = \sum_{\substack{\lambda \vdash n \\ \lambda_i \not\equiv 0 \bmod p}} H_\lambda[Lie].$$

The higher conjugacy modules 1

The meta theorem can be used to prove:

Theorem (Solomon 1961)

$$H[\sum_{n\geq 1} Conj_n] = \sum_{\lambda} H_\lambda[\sum_{n\geq 1} Conj_n] = \sum_{\mu\vdash n} p_\mu.$$

$H_\lambda[\sum_{n\geq 1} Conj_n]$ is the action by conjugation on the class indexed by λ. The character value on σ is the order of the centraliser of σ.

Theorem (S 2017)

$$E\Big[\sum_{n\geq 1} Conj_n\Big] = \sum_{\lambda} E_\lambda\Big[\sum_{n\geq 1} Conj_n\Big] = \sum_{\substack{\mu\vdash n \\ \text{all parts odd}}} p_\mu .$$

Open problems

(♣) Why does the number of standard Young tableaux $\mathbf{t}$ of shape λ with major index $\equiv k$ mod n depend only on (k, n)?

What is the irreducible decomposition of

1. $H_\lambda[Lie]$? (Thrall's problem – 1942)
2. $H_\lambda[Conj]$?
3. More generally, for $H_\lambda[\ell_n^{(k)}]$?

Known only for

1. $\lambda = (n)$: (Stanley, Kráskiewicz-Weyman)
2. $\lambda = (1^n)$ (the trivial module)
3. λ has all parts equal to 1 or 2.
4. $\lambda = (n-k, 1^k)$ (recent work of Hegedus-Roichman).

Character tables of S_n

	$\underline{2}$	$\underline{1^2}$	Row Sum	Odd Row Sum
(2)	1	1	**2**	**1**
(1^2)	-1	1	**0**	**1**

	$\underline{3}$	$\underline{2,1}$	$\underline{1^3}$	Row sum	Odd row sum
(3)	1	1	1	**3**	**2**
$(2,1)$	-1	0	2	**1**	**1**
(1^3)	1	-1	1	**1**	**2**

Character tables for S_4

	$\underline{4}$	$\underline{3,1}$	$\underline{2^2}$	$\underline{2,1^2}$	$\underline{1^4}$	Row sum	Odd row sum
(4)	1	1	1	1	1	**5**	**2**
$(3,1)$	-1	0	-1	1	3	**2**	**3**
(2^2)	0	-1	2	0	2	**3**	**1**
$(2,1^2)$	1	0	-1	-1	3	**2**	**3**
(1^4)	-1	1	1	-1	1	**1**	**2**

Character tables: S_5

	5	$4,1$	$3,2$	$3,1^2$	$2^2,1$	$2,1^3$	1^5	RowΣ	OddΣ
(5)	1	1	1	1	1	1	1	**7**	**3**
$(4,1)$	-1	0	-1	1	0	2	4	**5**	**4**
$(3,2)$	0	-1	1	-1	1	1	5	**6**	**4**
$(3,1^2)$	1	0	0	0	-2	0	6	**5**	**7**
$(2^2,1)$	0	1	-1	-1	1	-1	5	**4**	**4**
$(2,1^3)$	-1	0	1	1	0	-2	4	**3**	**4**
(1^5)	1	-1	-1	1	1	-1	1	**1**	**3**

	$\underline{C^1}$	$\underline{\mathbf{C^2}}$	$\underline{C^3}$	$\underline{C^4}$	$\underline{\mathbf{C^5}}$	$\underline{C^6}$	$\underline{\mathbf{C^7}}$	$\underline{C^8}$	$\underline{C^9}$	$\underline{C^{10}}$	$\underline{\mathbf{C^{11}}}$	R	**O**
(6)	1	1	1	1	1	1	1	1	1	1	1	**11**	**4**
$(5,1)$	-1	0	-1	1	-1	0	2	-1	1	3	5	**8**	**6**
$(4,2)$	0	-1	1	-1	0	0	0	3	1	3	9	**15**	**8**
$(4,1^2)$	1	0	0	0	1	-1	1	-2	-2	2	10	**10**	**12**
(3^2)	0	0	-1	-1	2	1	-1	-3	1	1	5	**4**	**6**
$(3,2,1)$	0	1	0	0	-2	0	-2	0	0	0	16	**13**	**13**
$(3,1^3)$	-1	0	0	0	1	1	1	2	-2	-2	10	**10**	**12**
(2^3)	0	0	-1	1	2	-1	-1	3	1	-1	5	**8**	**6**
$(2^2,1^2)$	0	-1	1	1	0	0	0	-3	1	-3	9	**5**	**8**
$(2,1^4)$	1	0	-1	-1	-1	0	2	1	1	-3	5	**4**	**6**
(1^6)	-1	1	1	-1	1	-1	1	-1	1	-1	1	**1**	**4**

References

1. Christophe Reutenauer, Free Lie Algebras.
2. Richard Stanley, Enumerative Combinatorics Vol. 2, Ex. 7.71, 7.88, 7.89.
3. Sheila Sundaram, *The conjugacy action of S_n and modules induced from centralisers*, J. Algebraic Combin. 48 (2018), no. 2, 179225. DOI 10.1007/s10801-017-0796-9
4. ——, *Variations on the S_n-module Lie_n*, arXiv.math.RT/1803.09368.
5. ——, *On a curious variant of the S_n-module Lie_n*, Algebraic Combinatorics, to appear.
6. ——, *The plethystic inverse of the odd Lie representations Lie_{2n+1}*, arXiv:2003.10700v1 [math.CO].

Notes 1

Notes 2

Notes 3

Notes 4

THANK YOU!