Reduced Kronecker coefficients and Deligne categories

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Kronecker coefficients

Problem

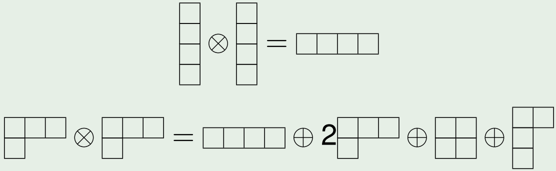
Consider the tensor product of two finite-dimensional complex representations of S_d. How does it decompose into a sum of irreducibles?

Definition (Kronecker coefficient)

Given Young diagrams λ, μ, τ of size d,

$$g^{\lambda}_{\mu,\tau} := [\mu \otimes \tau : \lambda]$$

Example ($d = 4$)



Long top row

Idea: consider Young diagrams with very long top row. What will happen to Kronecker coefficients?

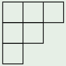
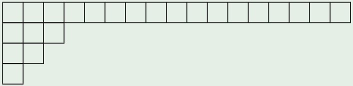
Notation

Given Young diagram λ of **any** *size and $d \geq |\lambda| + \lambda_1$, let*

$$\lambda[d] := \text{add top row of length } d - |\lambda| \text{ to } \lambda.$$

$\lambda[d]$ is a Young diagram with d cells. The corresponding representation of S_d is also denoted $\lambda[d]$ for $d \geq |\lambda| + \lambda_1$ (otherwise we set $\lambda[d] := 0$).

Example

$\lambda =$  $\rightsquigarrow \lambda[23] =$ 

(add a top row so that the total size would be 23).

Murnaghan's result

Proposition (Murnaghan, 1938)

Let μ, τ, λ be Young diagrams of **any** *size. The Kronecker coefficient $g^{\lambda[d]}_{\mu[d],\tau[d]}$ does not depend on d when $d >> 0$. This value is called the* **reduced (or stable) Kronecker coefficient $\bar{g}^{\lambda}_{\mu,\tau}$**.

- Reduced Kronecker coefficients generalize Littlewood-Richardson coefficients (these occur in decomposition of tensor products of GL_n-modules), when $|\lambda| = |\mu| + |\tau|$.
- Briand, Orellana, Rosas (2009): **any** Kronecker coefficient can be expressed through reduced Kronecker coefficients.

Will show that they occur naturally in tensor categories.

$Rep(S_\infty)$

Let $S_\infty = \bigcup_{d \geq 0} S_d$ be the group of permutations $\mathbb{N} \to \mathbb{N}$ with finite support.
Sam, Snowden (2013) defined the abelian symmetric monoidal category $Rep(S_\infty)$ (algebraic representations of S_∞).

- Abelian $\mathbb{C}$-linear category with operation $\otimes$.
- Simple modules $L_\infty(\lambda) \longleftrightarrow$ Young diagrams λ of any size.
  Example: $L_\infty(\emptyset) = \mathbb{C}$, $L_\infty(\square) =$ reflection repr.
- For all $d \geq 0$, have specialization functors

$$Rep(S_\infty) \to Rep(S_d), \; L_\infty(\lambda) \to \lambda[d]$$

  (left-exact, full, essentially surjective, respects $\otimes$).
- Structure constants: reduced Kronecker coefficients!

$$\bar{g}^{\lambda}_{\mu,\tau} := [L_\infty(\mu) \otimes L_\infty(\tau) : L_\infty(\lambda)]$$

Abelian Deligne categories

- Deligne (2007), Comes, Ostrik (2013) constructed $\mathbb{C}$-linear abelian rigid symmetric monoidal categories $Rep(\underline{S}_t)$ for all $t \in \mathbb{C}$.
- Rigidity: can talk about dimensions of objects in $Rep(\underline{S}_t)$.
- $Rep(\underline{S}_t)$ is a highest-weight category.
  - Simple objects $\longleftrightarrow$ Young diagrams of any size.
  - Standard objects ($\sim$ Verma modules) $\longleftrightarrow$ Young diagrams of any size.
    The standard object in $Rep(\underline{S}_t)$ corresponding to a Young diagram λ is denoted by M_λ.
  - Enough projectives/injectives.
- For $t \notin \mathbb{Z}_{\geq 0}$, $Rep(\underline{S}_t)$ is semisimple.

Tilting objects

Definition

A *tilting object* T has a filtration with standard subquotients and so does its dual.
The indecomposable tilting object in $Rep(\underline{S}_t)$ corresponding to a Young diagram λ is denoted by T_λ.
The full subcategory of tilting objects is called $Tilt(\underline{S}_t)$.

Indecomposable object T_λ $\xrightarrow{\text{treat as}}$ S_t irreducible repr. $\lambda[t]$ ("very long" top row of length $t - |\lambda|$)

Example

- $T_\emptyset = \mathbf{1}$ - the unit object.
- Object $\mathfrak{h}_t$ of dim t ("permutation repr.").
- $T_\square$ - the reflection representation (dim. is $t - 1$ for $t \neq 0$).
  If $t \neq 0$, $\mathfrak{h}_t = T_\emptyset \oplus T_\square$.

Properties of $Tilt(\underline{S}_t)$

- It is closed under taking $\otimes$, $\oplus$ and direct summands.
- The categories $Tilt(\underline{S}_t)$, $t \in \mathbb{C}$ form a "polynomial" family interpolating the categories $Rep(S_d)$: $\forall d \in \mathbb{Z}_{\geq 0}$ have a specialization functor,

$$\begin{array}{ccccc} T_\lambda & \in & Tilt(\underline{S}_{t=d}) & \hookrightarrow & Rep(\underline{S}_{t=d}) \\ \big\downarrow & & \big\downarrow \mathcal{S}_d & & \\ \lambda[d] & \in & Rep(S_d) & & \end{array}$$

  It respects $\oplus, \otimes$, is full and essentially surjective.
- Dimensions of objects in $Tilt(\underline{S}_t)$ are polynomials in t.
- For $t \notin \{0, 1, 2, \ldots, |\lambda| + \lambda_1\}$, $T_\lambda = M_\lambda$ and it is simple.
- For $t \notin \mathbb{Z}_{\geq 0}$, $Tilt(\underline{S}_t) = Rep(\underline{S}_t)$.

Question: what is the tensor structure of $Rep(\underline{S}_t)$?
For example, fix $t \in \mathbb{C}$ and consider Young diagrams μ, τ, λ be of **any** size.
- $T_\mu \otimes T_\tau$ is tilting. What is

$$[T_\mu \otimes T_\tau : T_\lambda] = ?$$

- Can we say something about

$$M_\mu \otimes M_\tau = ?$$

Reduced Kronecker coefficients and $Rep(\underline{S}_t)$ - cont.

The following idea can be derived from the polynomiality of the family $Tilt(\underline{S}_t)$:

Theorem (E., 2014)

Let μ, τ, λ be Young diagrams of **any** *size. Then the value of $[T_\mu \otimes T_\tau : T_\lambda]_{Tilt(\underline{S}_t)}$ is constant for almost all t (except finitely many non-negative integer values).*

Moreover, we have:

$$g^{\lambda[d]}_{\mu[d],\tau[d]} = [T_\mu \otimes T_\tau : T_\lambda]_{Tilt(\underline{S}_{t=d})}$$

for $d >> 0$ *(and this is $\bar{g}^{\lambda}_{\mu,\tau}$).*

So, reduced Kronecker coefficients are the structure constants in the tensor categories $Tilt(\underline{S}_t)$ **for** $t \notin \mathbb{Z}_{\geq 0}$**:**

$$[T_\mu \otimes T_\tau : T_\lambda]_{Tilt(\underline{S}_t)} = \bar{g}^{\lambda}_{\mu,\tau} \quad \forall \lambda, \mu, \tau$$

Sketch of proof

Claim:

$$g^{\lambda[d]}_{\mu[d],\tau[d]} = [T_\mu \otimes T_\tau : T_\lambda]_{Tilt(\underline{S}_t)}$$

for $d >> 0$ and generic t.

Sketch of Proof.

Recall the $\otimes$-functor $\mathcal{S}_d : Tilt(\underline{S}_{t=d}) \to Rep(S_d)$. We have $\mathcal{S}_d(T_\lambda) \cong \lambda[d]$, and similarly for T_μ, T_τ.
So $\mathcal{S}_d(T_\mu \otimes T_\tau) \cong \mu[d] \otimes \tau[d]$.

$$\Downarrow$$

$$[T_\mu \otimes T_\tau : T_\lambda]_{Tilt(\underline{S}_{t=d})} = [\mu[d] \otimes \tau[d] : \lambda[d]]_{S_d} =: g^{\lambda[d]}_{\mu[d],\tau[d]}$$

Now, $[T_\mu \otimes T_\tau : T_\lambda]_{Tilt(\underline{S}_t)} \equiv const$ for generic t, so $g^{\lambda[d]}_{\mu[d],\tau[d]}$ doesn't depend on d for $d >> 0$ and the claim is proved. □

More information from the study of the "special cases" of Deligne categories: $Rep(\underline{S}_{t=d})$ when d is an integer.

But what about S_∞?

So what is the connection to $Rep(S_\infty)$?

Theorem (Barter - E. - Heidersdorf, 2017)

For every $t \in \mathbb{C}$ there exists a specialization functor

$$\Gamma_t : Rep(S_\infty) \to Rep(\underline{S}_t), \; L_\infty(\lambda) \to M_\lambda$$

(exact, faithful, respects $\otimes$).

Corollary

The object $M_\mu \otimes M_\tau$ always has a filtration with standard subquotients, and $(M_\mu \otimes M_\tau : M_\lambda) = \overline{g}^{\lambda}_{\mu,\tau}$.

$\otimes$-products of tilting objects

Want to compute structure constants for tilting objects: what is $[T_\mu \otimes T_\tau : T_\lambda]_{Tilt(\underline{S}_{t=d})} = ?$**.**

For this, need to understand the filtration of tilting objects by standard modules.

Equivalence relation $\stackrel{t}{\sim}$

Let $t \in \mathbb{C}$. The following equivalence relation was defined by Comes, Ostrik to describe the blocks of the Deligne categories:

Definition (Equivalence relation $\stackrel{t}{\sim}$ on the set of Young diag.)

$\lambda \stackrel{t}{\sim} \mu$ if the **infinite** sequences $(t - |\lambda|, \lambda_1 - 1, \lambda_2 - 2, ...)$ and $(t - |\mu|, \mu_1 - 1, \mu_2 - 2, ...)$ can be obtained from one another by permuting a finite number of elements.

Here $\forall i$, λ_i is the length of row i of λ (zero if no cells in this row).

Analogy: dot action of the Weyl group on weights of the Lie algebra $\mathfrak{gl}_n$.

Non-trivial $\overset{t}{\sim}$-classes

Non-trivial equivalence classes (more than one element):
Let $d \in \mathbb{Z}_{\geq 0}$.

$$\overset{t=d}{\sim}-\begin{matrix}\text{Non-trivial} \\ \text{equivalence classes}\end{matrix} \overset{\text{bij.}}{\longleftrightarrow} \begin{matrix}\text{Young diagrams} \\ \text{of size } d\end{matrix}$$

To Young diagram λ of size d corresponds a non-trivial class $\{\lambda^{(i)}\}_i$,

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \ldots$$

where $\lambda^{(0)} = \bar{\lambda}$ (λ without its top row), and other $\lambda^{(i)}$ have explicit formulas as well.

Standard filtration of T_λ

Let $t = d \in \mathbb{Z}_{\geq 0}$.

Question: Which standard objects appear as subquotients in the standard filtration of T_λ?

Answer:

- $T_\lambda = M_\lambda$ if λ lies in a trivial $\overset{t=d}{\sim}$-class,
- $T_\lambda = M_\lambda$ if $\lambda = \lambda^{(0)}$ in a non-trivial $\overset{t=d}{\sim}$-class $\lambda^{(0)} \subset \lambda^{(1)} \subset ...$
- $T_{\lambda^{(i)}}$ is an extension of $M_{\lambda^{(i)}}$ and $M_{\lambda^{(i-1)}}$ for $i > 0$ and a non-trivial $\overset{t=d}{\sim}$-class $\lambda^{(0)} \subset \lambda^{(1)} \subset ....$
  Denote: $gr(T_\lambda) = M_\lambda$ and $gr(T_\lambda) := M_{\lambda^{(i)}} \oplus M_{\lambda^{(i-1)}}$/

Briand-Orellana-Rosas result reproved

Let λ, μ, τ be Young diagrams, and $d \geq |\lambda| + \lambda_1$.
Consider the non-trivial $\stackrel{t=d}{\sim}$-class $\lambda = \lambda^{(0)} \subset \lambda^{(1)} \subset ...$

Theorem (E., 2014, reformulated)

In Rep($\underline{S}_{t=d}$), we have:

$$[T_\mu \otimes T_\tau : T_\lambda] = \sum_{i \geq 0} (-1)^i \left(gr(T_\mu) \otimes gr(T_\tau) : M_{\lambda^{(i)}}\right)$$

.

Corollary (Originally due to Briand, Orellana, Rosas, 2009)

Let $d \geq |\lambda| + \lambda_1, |\mu| + \mu_1, |\tau| + \tau_1$

$$g^{\lambda[d]}_{\mu[d],\tau[d]} = \sum_{i \geq 0} (-1)^i \bar{g}^{\lambda^{(i)}}_{\mu,\tau}$$

Corollary

In fact, this shows that **ANY** Kronecker coefficient can be expressed through reduced Kronecker coefficients.
Indeed, let λ, μ, τ be Young diagrams of size d. The corollary says:

$$g^{\lambda}_{\mu,\tau} = \sum_{i \geq 0} (-1)^i \bar{g}^{\lambda^{(i)}}_{\bar{\mu},\bar{\tau}}$$

(here $\bar{\mu} := \mu$ without its top row, similarly for τ).

New identities

What if we take other values of d?

Can obtain new, previously unknown identities involving only reduced Kronecker coefficients.

Idea: Projectives in $Rep(\underline{S}_t)$ are tilting, but not all tiltings are necessarily projective!
In fact, T_λ is not projective in $Rep(\underline{S}_{t=d})$ ($d \in \mathbb{Z}_{\geq 0}$) precisely when $|\lambda| + \lambda_1 \leq d$.

Nice property of projectives in (rigid) tensor categories: projective $\otimes$ any object $=$ projective.

So, if T_μ is projective in $Rep(\underline{S}_{t=d})$ while T_λ is not, then

$$[T_\mu \otimes T_\tau : T_\lambda]_{Rep(\underline{S}_{t=d})} = 0.$$

New identities - cont.

Proposition (E., 2014)

- *Assume* $d \geq |\lambda| + \lambda_1$ *and* $d \in \{|\mu| + \mu_l - l : l = 1, ..., |\mu|\}$ *(i.e.* μ *is in a trivial* $\overset{t=d}{\sim}$*-class). Then*

$$\sum_{i\geq 0} (-1)^i \bar{g}_{\mu,\tau}^{\lambda^{(i)}} = 0$$

- *(Generalization of [BOR]): Consider three non-trivial* $\overset{n}{\sim}$*-classes, denoted by* $\{\lambda^{(i)}\}_{i\geq 0}$, $\{\mu^{(i)}\}_{i\geq 0}$ *and* $\{\tau^{(i)}\}_{i\geq 0}$ *respectively (these classes are not necessarily distinct). Fix any* $k, l \geq 0$*. Then*

$$\sum_{i\geq 0} (-1)^i \bar{g}_{\mu^{(k)},\tau^{(l)}}^{\lambda^{(i)}} = (-1)^{k+l} g_{\mu^{(0)}[n],\tau^{(0)}[n]}^{\lambda^{(0)}[n]}$$

(BOR: $k = 0, l = 0$*).*

Example $d = 0$

Example

Let $d = 0$. The first case holds whenever either μ or τ is *not* a column diagram, and then

$$\bar{g}^{\emptyset}_{\mu,\tau} - \bar{g}^{\square}_{\mu,\tau} + \bar{g}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\mu,\tau} - \bar{g}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}_{\mu,\tau} + \ldots = 0$$

(sum ends with column of length $|\mu| + |\tau|$).

Second case: μ, τ are column partitions, of lengths k, l respectively. Then

$$\bar{g}^{\emptyset}_{\mu,\tau} - \bar{g}^{\square}_{\mu,\tau} + \bar{g}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\mu,\tau} - \bar{g}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}_{\mu,\tau} + \ldots = (-1)^{k+l}\delta_{k,l}$$

Notation remark

In `https://arxiv.org/abs/1407.1506`, the category $Tilt(\underline{S}_t)$ is denoted $Rep(S_t)$ and the abelian category $Rep(\underline{S}_t)$ is denoted $Rep^{ab}(S_t)$.