Random t-cores and hook lengths in random partitions

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Preliminaries

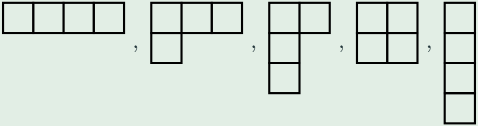
(Integer) Partitions

Definition

A **partition** λ *is a non-increasing tuple of non-negetive integers. We denote* $\mathcal{P}(n)$ *to be the set of partitions* λ *of size* n *and* $p(n) := |\mathcal{P}(n)|$.

Example

For $n = 4$, we have $p(4) = 5$; the following are Young diagrams of these five partitions in $\mathcal{P}(4)$.



Rim-hooks

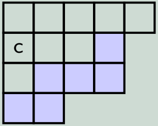
Definition

Let c be a cell in the Young diagram of a partition λ,

- **Hook** *of c = cells to the right and to the bottom of c (in the same row/column).*
- **Rim-hook** *of c = boundary joining the two ends of the hook.*
- **Hook-length** *h_c is the size of Rim-hooks of c.*

Example

Let $\lambda = (5, 4, 4, 2)$ and c is the cell at position $(1, -2)$. The shaded set is the rim-hook of c.



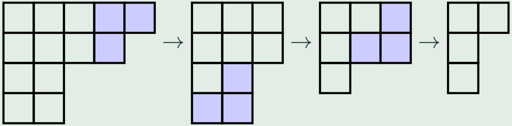
t-core of a partition

Definition

*The t-**core of a partition** λ, denoted $\mathrm{core}_t(\lambda)$, is the partition obtained by removing as many rim hooks of size t as possible.*

Example

Let $t = 3$ and $\lambda = (5, 4, 2, 2)$, then $\mathrm{core}_3(\lambda) = (2, 1, 1)$.



t-core and t-divisible partition

Definition

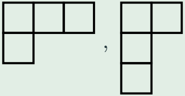
We say a partition λ is a

- t-**core** *if* $\mathrm{core}_t(\lambda) = \lambda$.
- t-**divisible** *if* $\mathrm{core}_t(\lambda) = (0)$. *(Note that t must divides $|\lambda|$)*

Let $\mathfrak{C}_t(n)$ and $\mathcal{D}_t(n)$ be the set of t-core and t-divisible partition resp of size n. We also denote $c_t(n)$ and $d_t(n)$ to be the cardinality of the above sets.

Example

For $t = 3$ and $n = 4$, we see $\mathcal{D}_3(4)$ is empty and $\mathfrak{C}_3(4)$ consists of



Partition Division

Let $\mathcal{P}$, $\mathcal{D}_t$ and $\mathfrak{C}_t$ be union of $\mathcal{P}(n)$, $\mathcal{D}_t(n)$ and $\mathfrak{C}_t(n)$ over all n.

Theorem (Partition Division)

There is a natural bijection

$$\Delta_t : \mathcal{P} \to \mathfrak{C}_t \times \mathcal{D}_t$$

given by $\Delta_t(\lambda) = (\rho, \nu)$, *where* $\rho = \mathrm{core}_t(\lambda)$ *and* ν *is the 't-quotient'. Moreover,*

$$|\lambda| = |\rho| + |\nu|$$

Remark

This a generalization of the euclidean division for natural numbers which can be stated as $\Delta_t : \mathbb{N} \to \{0, 1, \ldots, t-1\} \times t\mathbb{N}$ *is a bijection.*

Partition Division

Corollary

We have the following identity of generating functions

$$\sum_{k=0}^{\infty} p(k)q^k = \left(\sum_{k=0}^{\infty} c_t(k)q^k\right)\left(\sum_{k=0}^{\infty} d_t(k)q^k\right)$$

The following identities are known :

$$\sum_{n=0}^{\infty} c_t(n)x^n = \prod_{k=1}^{\infty} \frac{(1-x^{tk})^t}{1-x^k}$$

$$\sum_{n=0}^{\infty} d_t(n)x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^{tk})^t}.$$

Asymptotics

Remark

We know the following asymptotic results (work by Hardy-Ramanujan)

$$p(n) \sim \frac{\exp\left(\pi\sqrt{\frac{2n}{3}}\right)}{4n\sqrt{3}}, \quad d_t(n) \sim \frac{t^{(t+2)/2}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)}{2^{(3t+5)/4}3^{(t+1)/4}n^{(t+3)/4}}. \tag{1}$$

But $c_t(n)$ does not have a similar asymptotic result.

- $c_2(n) = 1$ for n a triangular number and zero otherwise.
- Gravnille and Ono

$$c_3(n) = \sum_{d|(3n+1)} \left(\frac{d}{3}\right)$$

- Gravnille and Ono showed t-core partition conjecture that for $t > 3$,

$$c_t(n) > 0$$

First Theorem

Definition

$$C_t(n) := \sum_{i=0}^{\lfloor \frac{n}{t} \rfloor} c_t(n - i\, t).$$

There is a combinatorial reason for taking this sum. Let $\pi : \mathfrak{C}_t \times \mathcal{D}_t \to \mathfrak{C}_t$ be the projection, then

$$\begin{aligned} C_t(n) =& |\pi(\Delta_t(\mathcal{P}(n))| \\ =& \#\{\mathrm{core}_t(\lambda) \mid \lambda \quad \text{a partition of } n\}. \end{aligned}$$

First Theorem

Remark

When t is a prime number, $C_t(n)$ can be defined to be the number of t-blocks in the t-modular representation theory of the symmetric group S_n.

Theorem (AA,SS)

We have

$$C_t(n) = \frac{(2\pi)^{(t-1)/2}}{t^{(t+2)/2}\,\Gamma(\frac{t+1}{2})}\left(n + \frac{t^2-1}{24}\right)^{(t-1)/2} + O(n^{(t-2)/2}).$$

Distribution of the size of t-core of uniformely random $\lambda \in \mathcal{P}(n)$

Random Variable

Definition

Fix $t \geq 2$. Let λ be a uniformly random partition of n and Y_n be random variable on $\mathbb{N}_{\geq 0}$ given by

$$Y_n = |\operatorname{core}_t(\lambda)|.$$

The probability mass function of Y_n is given by

$$\begin{aligned}\mu_n(k) &= \frac{\#\{\lambda \vdash n : |\operatorname{core}_t(\lambda)| = k\}}{p(n)} \\ &= \frac{c_t(k)d_t(n-k)}{p(n)}\end{aligned}$$

We are interested in the convergence of Y_n.

Continuous Random Variable

Definition

Let X_n be continuous random variables defined on $[0, \infty)$ with the probability density function $f_n \equiv f_{n,t}$ given by

$$f_n(x) = \frac{\sqrt{n}\, c_t(\lfloor x\sqrt{n} \rfloor)\, d_t(n - \lfloor x\sqrt{n} \rfloor)}{p(n)}.$$

Note that

$$\int_0^\infty f_n(x) = \sum_{k=0}^{n} \frac{c_t(k) d_t(n-k)}{p(n)} = 1.$$

Theorem (Main Result (AA,SS))

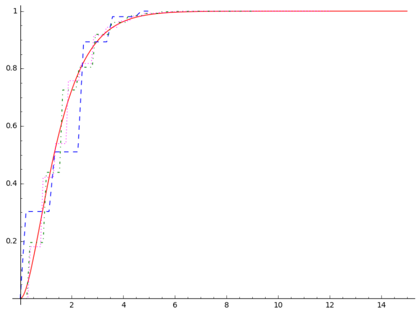
The random variable X_n converges weakly to a gamma-distributed random variable X with shape parameter $\alpha = (t-1)/2$ and rate parameter $\beta = \pi/\sqrt{6}$.

Recall that the *gamma distribution* with *shape parameter* $\alpha > 0$ and *rate parameter* $\beta > 0$ is a continuous random variable on $[0, \infty)$ with density given by

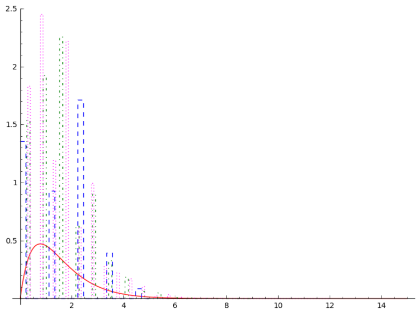
$$\gamma(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x), & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{2}$$

where Γ is the standard gamma function.

Plots



(a)



(b)

Figure 1: Comparison of the limiting CDFs and densities for small values of n with $t = 5$. A red solid line is used for the limiting distribution, dashed blue for $n = 20$, dash-dotted green for $n = 62$ and dotted magenta for $n = 103$. In (a) the CDFs, and in (b) the densities, are plotted for X and these X_n's.

Expected size of the t-cores

Theorem (AA,SS)

The expectation of the size of t-core for a uniformly random partition of size n is asymptotic to $(t-1)\sqrt{6n}/2\pi$.

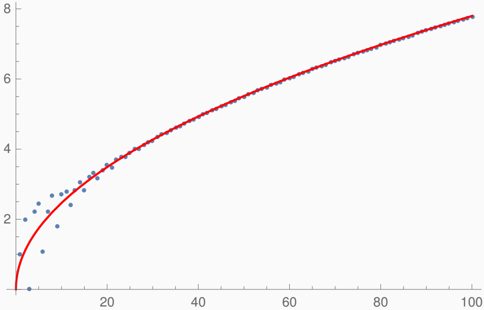


Figure 2: The average size of the 3-core for partitions of size 1 to 100 in blue circles, along with the result from Corollary, $\sqrt{6x}/\pi$, as a red line.

A brief introduction to abacus

Abacus

Definition

An **abacus** *or* **1-runner** *is a function* $w : \mathbb{Z} \to \{0, 1\}$ *such that* $w(m) = 1$ *for* $m << 0$ *and* $w(m) = 0$ *for* $m >> 0$*. We say* w *is* **justified** *at position* p *if* $w_i = 1$ *(resp* $w_i = 0$*) for* $i < p$ *(resp.* $i \geq p$*).*

Any abacus can be transformed to a justified by moving 1*'s to the left. We say* w *is* **balanced** *if after this transformation, we get abacus justified at* 0*.*

Example

$$w = \cdots 11100110\underline{1}1001000 \cdots$$

$$\downarrow$$

$$w' = \cdots 11111111\underline{0}0000000 \cdots$$

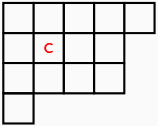
Abaci and Partitions

Theorem

There is a natural bijection

$$\mathcal{P} = \left\{ \textit{partitions } \lambda \right\} \leftrightarrow \left\{ \textit{Balanced Abaci } w \right\}.$$

- *cell c in λ $\leftrightarrow$ pair of positions $i < j$ with $(w_i, w_j) = (0, 1)$.*



$\rightarrow \cdots 11101{\color{red}0}00\underline{1}{\color{red}1}010000 \cdots$

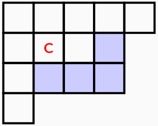
Abaci and Partitions

Theorem

There is a natural bijection

$$\mathcal{P} = \Big\{\textit{partitions } \lambda\Big\} \leftrightarrow \Big\{\textit{Balanced Abaci } w\Big\}.$$

- *cell c in $\lambda \leftrightarrow$ pair of positions $i < j$ with $(w_i, w_j) = (0, 1)$.*
- *Removing rim-hook of $c \leftrightarrow$ Swapping $(w_i, w_j) = (0, 1)$ to $(w'_i, w'_j) = (1, 0)$.*



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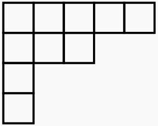


$\rightarrow \cdots 11101\textcolor{red}{1}00\underline{1}\textcolor{red}{0}010000\cdots$

Definition

*Let w be a 1-runner abacus, then the t-**runner** of w is the t-tuple $(\lambda^0, \dots, \lambda^{t-1})$ of 1-runner abaci given by*

$$\lambda^i_n = w_{nt+i}$$

Example

Let $t = 3$ and $\lambda = (5, 4, 2, 2)$ then $w = (\cdots 110011\underline{0}010100 \cdots)$ then

$$\begin{aligned}
\lambda^0 &= (\dots, & 1, && 1, && 0, && \underline{0}, && 0, && 0, && \dots), \\
\lambda^1 &= (\dots, & 1, && 1, && 1, && 0, && 1, && 0, && \dots), \\
\lambda^2 &= (\dots, & 1, && 0, && 1, && 1, && 0, && 0, && \dots).
\end{aligned}$$

What is $\mathrm{core}_t(\lambda)$?

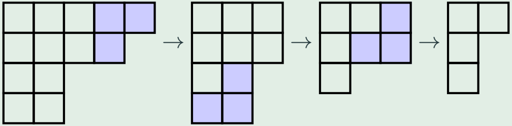
t-core of a partition

Definition

The t-**core of a partition** λ, *denoted* $\mathrm{core}_t(\lambda)$, *is the partition obtained by removing as many rim hooks of size* t *as possible.*

Example

Let $t = 3$ and $\lambda = (5, 4, 2, 2)$, then $\mathrm{core}_3(\lambda) = (2, 1, 1)$.



t-runner abaci

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Example

Let $t = 3$ and $\lambda = (5, 4, 2, 2)$ then $w = (\cdots 110011\underline{0}010100 \cdots)$ then

$$\begin{array}{rcccccccc} \lambda^0 = & (\dots, & 1, & 1, & 0, & \underline{0}, & 0, & 0, & \dots), \\ \lambda^1 = & (\dots, & 1, & 1, & 1, & 0, & 1, & 0, & \dots), \\ \lambda^2 = & (\dots, & 1, & 0, & 1, & 1, & 0, & 0, & \dots). \end{array}$$

Then $\mathrm{core}_3(\lambda) = (2, 1, 1)$ is given by

$$\begin{array}{rcccccccc} \rho^0 = & (\dots, & 1, & 1, & 0, & \underline{0}, & 0, & 0, & \dots), \\ \rho^1 = & (\dots, & 1, & 1, & 1, & 1, & 0, & 0, & \dots), \\ \rho^2 = & (\dots, & 1, & 1, & 1, & 0, & 0, & 0, & \dots). \end{array}$$

Let (p_0, p_1, p_2) be the position of justification which is (-1,1,0).

t-quotient partition

Definition

*Let λ be a partition, $(\nu^0, \dots, \nu^{t-1})$ be corresponding t-runners. Let ν^i be the balanced 1-runner obtained by shifting λ^i appropriately. We denote t-**quotient** partition to be the partition ν corresponding to t-runner abacus $(\nu^0, \dots, \nu^{t-1})$.*

Example

Let $t = 3$ and $\lambda = (5, 4, 2, 2)$ then $w = (\cdots 110011\underline{0}010100 \cdots)$ then

$$\begin{aligned}
\lambda^0 = \quad & (\dots, \quad 1, \quad 1, \quad 0, \quad \underline{0}, \quad 0, \quad 0, \quad \dots), \\
\lambda^1 = \quad & (\dots, \quad 1, \quad 1, \quad 1, \quad 0, \quad 1, \quad 0, \quad \dots), \\
\lambda^2 = \quad & (\dots, \quad 1, \quad 0, \quad 1, \quad 1, \quad 0, \quad 0, \quad \dots).
\end{aligned}$$

Then the $\rho = \mathrm{core}_3(\lambda) = (2, 1, 1)$ and 3-quotient $\nu = (3, 3, 2, 1)$ is given by

$$\begin{aligned}
\nu^0 = \quad & (\dots, \quad 1, \quad 1, \quad 1, \quad \underline{0}, \quad 0, \quad 0, \quad \dots), \\
\nu^1 = \quad & (\dots, \quad 1, \quad 1, \quad 0, \quad 1, \quad 0, \quad 0, \quad \dots), \\
\nu^2 = \quad & (\dots, \quad 1, \quad 0, \quad 1, \quad 1, \quad 0, \quad 0, \quad \dots).
\end{aligned}$$

Note also $|\lambda| = |\rho| + |\nu|$ and ν is by construction a 3-divisible partition.

Random Hook-length

Hook Length

Corollary (of the Main Result)

For a uniformly random cell c of a uniformly random partition λ of n, the probability that the hook length of c in λ is divisible by t is $1/t + O(n^{-1/2})$.

Proof.

Observe that removing a t-rim hook reduces the number of hooks h_c with $t|h_c$ by exactly one. So, for any partition $\lambda \vdash n$

$$\#\{c \in \lambda \mid t|h_c\} = \frac{n - |\operatorname{core}_t(\lambda)|}{t}.$$

Thus the probability that t divides h_c as $n \to \infty$ is

$$\lim_{n\to\infty} \frac{n - E(|\operatorname{core}_t(\lambda)|)}{tn} = \frac{1}{t} - \lim_{n\to\infty} \frac{t-1}{2t\pi}\sqrt{\frac{6}{n}},$$

$\square$

Removing size t rim-hook

Example

Consider the partition $\lambda = (6, 4, 3, 1)$ and let $t = 4$.

$$\begin{array}{|c|c|c|c|c|c|}\hline 9 & 7 & 6 & 4 & 2 & 1 \\ \hline 6 & 4 & 3 & 1 \\ \cline{1-4} 4 & 2 & 1 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|}\hline 7 & 6 & 5 & 4 & 2 & 1 \\ \hline 4 & 3 & 2 & 1 \\ \cline{1-4}\end{array}$$

Removing the shaded size 4 rim-hook we obtain $\lambda' = (6, 4)$.

i	No. of hook lengths in $\lambda \equiv i$ mod 4	No. of hook lengths in $\lambda' \equiv i$ mod 4
0	3	2
1	5	3
2	4	3
3	2	2

Therefore, we have removed two cells congruent to 1 modulo 4, but none congruent to 3 modulo 4.

Partition Division

Let $\mathcal{P}$, $\mathcal{D}_t$ and $\mathfrak{C}_t$ be union of $\mathcal{P}(n)$, $\mathcal{D}_t(n)$ and $\mathfrak{C}_t(n)$ over all n.

Theorem (Partition Division)

There is a natural bijection

$$\Delta_t : \mathcal{P} \to \mathfrak{C}_t \times \mathcal{D}_t$$

given by $\Delta_t(\lambda) = (\rho, \nu)$*, where* $\rho = \mathrm{core}_t(\lambda)$ *and* ν *is the 't-quotient'. Moreover,*

$$|\lambda| = |\rho| + |\nu|$$

A natural action of S_t

Definition

We define the action of S_t on $\mathcal{D}_t(n)$, the set of t-divisible of n, partitions induced by the action S_t on the corresponding t-runners

$$\sigma \cdot (\nu^0, \nu^1, \ldots, \nu^{t-1}) = (\nu^{\sigma_0}, \nu^{\sigma_1}, \ldots, \nu^{\sigma_{t-1}}) \tag{3}$$

Note that the above action preserves the size of the t-divisible partition.

Definition

We define the action of $\sigma \in S_t$ on $\mathcal{P}(n)$ by $\sigma\lambda = \Delta_t^{-1}(\rho, \sigma\nu)$.

b-smoothing of t-divisible partitions

Definition

*The b-**smoothing** of a t-divisible partition ν, denoted C_ν^b, is the union of cells in the Young diagram of ν whose corresponding $(0,1)$ pairs are at least $(b+1)$ columns apart in the t-runner abacus of ν.*

Example

Let $\nu = (7,3,2)$ be a 3-divisible partition, whose 3-runner abacus is given by

$$\begin{array}{rl} \nu^0 = & (\ldots, \quad 1, \quad 1, \quad 0, \quad \underline{0}, \quad 0, \quad 1, \quad 0, \quad \ldots) \\ \nu^1 = & (\ldots, \quad 1, \quad 1, \quad 0, \quad 1, \quad 0, \quad 0, \quad 0, \quad \ldots) \\ \nu^2 = & (\ldots, \quad 1, \quad 1, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0, \quad \ldots) \end{array}$$

Then the following are the b-smoothings C_ν^b :

$\nu \in \mathcal{D}_3$	C_ν^0	C_ν^1	C_ν^2
$(7,3,2)$	$(7,2)$	(4)	(2)

b-smoothing

Remark

C_ν^b gives a sequence of sub partitions of ν,

$$\nu \supset C_\nu^0 \supset C_\nu^1 \supset C_\nu^2 \dots$$

Moreover, the b-smoothings are invariant under action of S_t.

$$C_{\sigma\nu}^b = C_\nu^b$$

Lemma

For a uniformly random cell $c \in C_\nu^b$ of uniformly random ν, the probability that the hook length h_c^ν is congruent to i modulo t, where $i \neq 0$, is independent of i.

Randomizing modulo class of hook-length

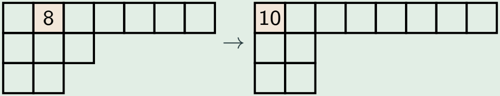
Example

Let $\nu = (7,3,2)$ be a 3-divisible partition, whose 3-runner abacus is given by

$$\begin{array}{rl} \nu^0 = & (\ldots, \quad 1, \quad 1, \quad 0, \quad \underline{0}, \quad 0, \quad 1, \quad 0, \quad \ldots) \\ \nu^1 = & (\ldots, \quad 1, \quad 1, \quad 0, \quad 1, \quad 0, \quad 0, \quad 0, \quad \ldots) \\ \nu^2 = & (\ldots, \quad 1, \quad 1, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0, \quad \ldots) \end{array}$$

Let $\sigma = (213) \in S_3$, then $\sigma \cdot \nu = (8,2,2)$

$$\begin{array}{rl} \nu^{\sigma_0} = & (\ldots, \quad 1, \quad 1, \quad 0, \quad \underline{1}, \quad 0, \quad 0, \quad 0, \quad \ldots) \\ \nu^{\sigma_1} = & (\ldots, \quad 1, \quad 1, \quad 0, \quad 0, \quad 0, \quad 1, \quad 0, \quad \ldots) \\ \nu^{\sigma_2} = & (\ldots, \quad 1, \quad 1, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0, \quad \ldots) \end{array}$$



(4)

Canonical smoothing

Let $\Delta_t(\lambda) = (\rho, \nu)$ and $(\rho^1, \dots, \rho^{t-1})$ be the t-runner of the t-core ρ. Recall ρ^i are justified (say at positions p_i).

Definition

Let

$$b_\lambda = \max_{1 \le i < j \le t-1} |p_i - p_j|$$

then we define canonical smoothing $C_\lambda = C_\nu^{b_\lambda}$ as sub partition of ν.

Example

Let $\rho = (1)$ and $t = 3$

$$\begin{array}{rcccccccccc} \rho^0 = & (\dots, & 1, & 1, & 1, & \underline{1}, & 0, & 0, & 0, & \dots) \\ \rho^1 = & (\dots, & 1, & 1, & 1, & 0, & 0, & 0, & 0, & \dots). \\ \rho^2 = & (\dots, & 1, & 1, & 0, & 0, & 0, & 0, & 0, & \dots) \end{array}$$

So $(p_0, p_1, p_2) = (1, 0, -1)$, hence $b_\lambda = 2$

Example

Let $\lambda = (10, 3)$. Then $\Delta_3(\lambda) = (\rho, \nu)$, where $\nu = (7, 3, 2)$ and $\rho = (1)$. The 3-runner abacus of ρ is given by $(p_0, p_1, p_2) = (1, 0, -1)$ which implies $b_\lambda = 2$.

$$\begin{array}{llllllllll}
\nu^0 = & (\dots, & 1, & 1, & \hat{0}, & \underline{0}, & 0, & \tilde{\hat{1}}, & 0, & \dots) \\
\nu^1 = & (\dots, & 1, & 1, & \tilde{0}, & 1, & 0, & 0, & 0, & \dots). \\
\nu^2 = & (\dots, & 1, & 1, & 1, & 0, & 0, & 0, & 0, & \dots)
\end{array}$$

and in the t-runner of λ

$$\begin{array}{llllllllll}
\lambda^0 = & (\dots, & 1, & 1, & 1, & \underline{0}, & 0, & 0, & 1, & \dots) \\
\lambda^1 = & (\dots, & 1, & 1, & 0, & 1, & 0, & 0, & 0, & \dots), \\
\lambda^2 = & (\dots, & 1, & 1, & 0, & 0, & 0, & 0, & 0, & \dots)
\end{array}$$

Canonical smoothing inside λ

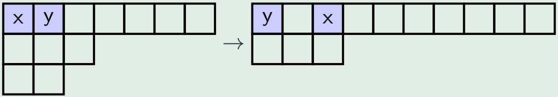
Lemma

Let λ be a partition with $\Delta_t(\lambda) = (\rho, \nu)$. Then there exists an injective map ϕ that takes the cells in C_λ to the cells in λ such that for any cell $c \in C_\lambda$, the hook lengths

$$h^{\lambda}_{\phi(c)} \equiv h^{\nu}_{c} \mod t$$

.

Example



$\nu = (7, 3, 2)$ $\qquad$ $\lambda = (10, 3)$

$h^{\nu}_{x} = 9, h^{\nu}_{y} = 8$ $\qquad$ $h^{\lambda}_{x} = 9, h^{\lambda}_{y} = 11$

Benefit of Canonical smoothing

Lemma

For a uniformly random cell $c \in \phi(C_\lambda) \subset \lambda$ of uniformly random λ, the probability that the hook length h_c^λ is congruent to i modulo t, where $i \neq 0$, is independent of i.

So it will be enough to show that for random λ,

$\phi(C_\lambda)$ is 'majority' of cells in λ.

$\phi(C_\lambda)$ is 'most' of λ

Lemma

For any partition $\lambda \vdash n$ with $\Delta_t(\lambda) = (\rho, \nu)$,

$$|\nu| - |C_\lambda| < \#\{c \in \nu \mid h_c < t(b_\lambda + 1)\} = O(t(b_\lambda + 1)\sqrt{n}). \tag{5}$$

A simple counting also gives us $b_\lambda \leq 2\sqrt{|\operatorname{core}_t(\lambda)|}$.

Since expected size of $\operatorname{core}_t(\lambda)$ is $O(\sqrt{n})$,

$$\begin{aligned} |\lambda| - |\phi(C_\lambda)| &= |\operatorname{core}_t(\lambda)| + |\nu| - |C_\lambda| \\ &= O(n^{3/4}) \end{aligned}$$

Second main result

So far we have

- For $i = 0$, probability that $h_c \equiv i \mod t$ is $1/t$.
- For $i \neq 0$ and $c \in \phi(C_\lambda)$, probability that $h_c \equiv i \mod t$ is independent of i.

Using the size estimate and the above two result we obtain :

Theorem

For a uniformly random cell c of a uniformly random partition λ of n, the probability that the hook length of c in λ is congruent to i modulo t is asymptotic to $1/t$ for any $i \in \{0, 1, \ldots, t-1\}$.

Summary

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- Hook-lengths are randomly distributed over modulo classes of t.

Thank you!

Questions