Chromatic symmetric function of graphs from Borcherds Lie algebra

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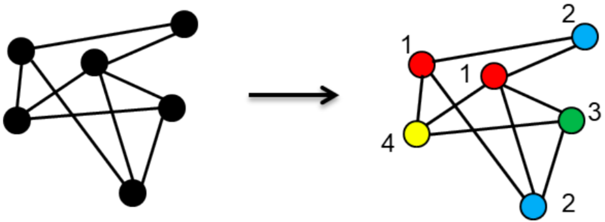
April 29, 2020

G. Arunkumar. Chromatic symmetric function of graphs from Borcherds algebras, https://arxiv.org/abs/1908.08198.

1. What is chromatic symmetric function and why are they interesting?
2. Connection between the Borcherds algebras and Chromatic symmetric functions.
3. Applications.

Graph vertex proper coloring

Let G be a finite simple connected graph with a **totally ordered** vertex set $I = \{\alpha_1, \ldots, \alpha_n\}$. I will be identified with the simple roots of $\mathfrak{g}$.



Chromatic polynomial:

$$\chi_G(q) = \text{The number of ways of coloring } G \text{ 'using' } q \text{ colors.}$$

Results: 1. $\chi_G(q)$ is a polynomial in q. 2. $\chi_G(q) = q(q-1)^{n-1}$ for any tree with n vertices.

Expression for chromatic polynomial

Chromatic polynomial has the following well–known description.

Consider an ordered k–tuples $(P_1, \ldots, P_k)$ such that:

(i) each P_i is a non–empty independent subset of I, i.e. no two vertices have an edge between them; and

(ii) the disjoint union of $P_1, \cdots, P_k$ is equal to I.

We denote by $P_k(G)$ the set of all **stable partitions** with k parts. Then we have

$$\chi_G(q) = \sum_{k \geq 0} |P_k(G)| \binom{q}{k}. \tag{1}$$

Vertex **k**-multicoloring of a graph

$\mathbf{k} = (2,2,2,2,2,2) \in \mathbb{Z}^n_{\geq 0}$ - coloring

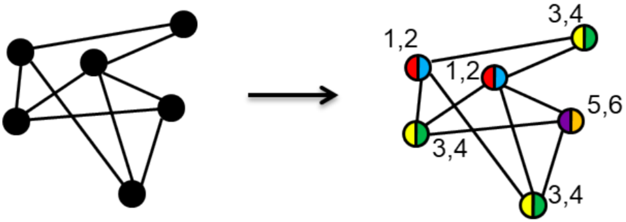


Figure 2: Vertex 6-multicoloring

We let $\pi^G_{\mathbf{k}}(q)$ be the number of such coloring using q colors. This is called the **k**-*generalized chromatic polynomial* of G.

Expression for $\mathbf{k}-$chromatic polynomials

$\mathbf{k}-$chromatic polynomials has the following well–known description.

We denote by $P_k(\mathbf{k}, G)$ the set of all ordered k–tuples $(P_1, \dots, P_k)$ such that:

(i) each P_i is a non–empty independent subset of I, i.e. no two vertices have an edge between them; and

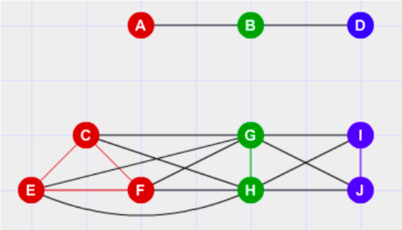
(ii) the disjoint union of $P_1, \cdots, P_k$ is equal to the multiset $\{\underbrace{\alpha_i, \dots, \alpha_i}_{k_i \text{ times}} : i \in I\}$.

Then we have

$$\pi^G_{\mathbf{k}}(q) = \sum_{k \geq 0} |P_k(\mathbf{k}, G)| \binom{q}{k}. \tag{2}$$

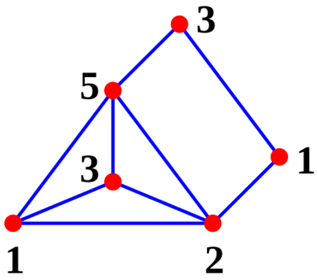
Relation between $\pi_{\mathbf{k}}^G(q)$ and $\chi_G(q)$

The graph $G(\mathbf{k})$ is the join of G with respect to $\mathbf{k}$. Assume $\mathbf{k} = (3,2,2)$



$$\pi_{\mathbf{k}}^G(q) = \frac{1}{\mathbf{k}!}\pi_{\mathbf{1}}^{G(\mathbf{k})}(q), \text{ where } \mathbf{k}! = \prod_{i=1}^{n} k_i!$$

Chromatic symmetric function: R. Stanley, 1995 [2]



$$x^{\kappa} = x_1^2\, x_2\, x_3^2\, x_5$$

$$X_G = X_G(x_1, x_2, \dots) = \sum_{\substack{\kappa \\ \text{proper-coloring}}} x^{\kappa}$$

$$X_G(\underbrace{1, 1, \dots, 1}_{q-\text{times}}, 0, 0, \dots) = \chi_G(q)$$

Expression for Chromatic symmetric function

We have shown that,

$$\chi_G(q) = \sum_{k\geq 0} |P_k(G)| \binom{q}{k}. \tag{3}$$

We have the following expression for chromatic symmetric function.

$$X_G = \sum_{k\geq 1} \sum_{\substack{\mathcal{P}\in P_k(G) \\ \mathcal{P}=(P_1,P_2,\ldots,P_k)}} \sum_{\substack{J\in \mathscr{P}_k(\mathbb{N}) \\ J=\{i_1,i_2,\ldots,i_k\}}} x_{i_1}^{|P_1|} x_{i_2}^{|P_2|} \cdots x_{i_k}^{|P_k|}$$

Stanley: X_G determines the graph G?

Answer: No

These graphs have the same chromatic symmetric function:

These two don't:

$$X_G = X_H = 2\tilde{m}_{221} + 4\tilde{m}_{2111} + \tilde{m}_{11111}$$

$$X_{P4} = 24m_{1111} + 6m_{211} + 2m_{22}, \quad X_{S4} = 24m_{1111} + 6m_{211} + m_{31}$$

S4 is the claw-graph $K_{1,3}$.

Stanley:

Does there exist two non-isomorphic trees with the same chromatic symmetric function?

Positivity results

Monomial symmetric functions

Let a_λ be the number of stable partitions of G of type λ. Then

$$X_G = \sum_{\lambda \vdash d} a_\lambda \widetilde{m_\lambda}.$$

Power sum symmetric functions

For any graph G, the symmetric function $w(X_G)$ is $p-$positive.

Elementary symmetric functions

1.Let G be the claw-graph then $X_G = 3e_4 + 5e_{31} - 2e_{22} + e_{221}$.

2.For any graph G we have $\text{sink}(G, j) = \sum_{\substack{\lambda \vdash d \\ l(\lambda) = j}} c_\lambda$.

3.If the complement of G is bipartite, then $c_\lambda \geq 0$.

Positivity results

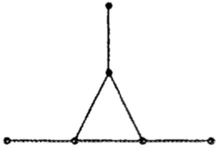
Schur functions

Let G be the claw-graph then $X_G = s_{31} - s_{22} + 5s_{221} + 8s_{1111}$.

Conjecture

Let G be the incomparability graph of a $\mathbf{3+1}$-free poset. Then X_G is $e-$positive.

$\mathrm{Inc}(\mathbf{3+1}) = K_{1,3}$ leads to ask weather any claw-free graph is $e-$positive?



$$X_G = 12e_6 + 18e_{51} + 12e_{42} - 6e_{33} + 6e_{441} + 6e_{321}.$$

Positivity results

Theorem (Gasharov)

Let G be the incomparability graph of a **3+1**-free poset. Then X_G is $s-$positive. Moreover,

$$X_G = \sum_{\lambda \vdash d} f^{\lambda}(P) s_{\lambda}$$

where $f^{\lambda}(P)$ denote the number of $P-$tableaux of shape λ.

Let P be a finite poset with d elements. A $P-$tableau of shape $\lambda \vdash d$ is a map $\tau : P \to \mathbb{N}$ satisfying

1. $\mid \tau^{-1}(i) \mid = \lambda_i$ for all i,
2. τ is a proper coloring of $\text{Inc}(P)$,
3. Suppose $\tau^{-1}(i) = \{u_1 < \cdots < u_{\lambda_i}\}$ and $\tau^{-1}(i+1) = \{v_1 < \cdots < v_{\lambda_{i+1}}\}$ then $v_j \not< u_j$ for all $1 \leq j \leq \lambda_{i+1}$. □

Chromatic polynomial and Kac-Moody Lie algebras

Theorem [R.Venkatesh, Sankaran Viswanath] [4]

Let G be the graph of a Kac-Moody algebra $\mathfrak{g}$. Given a $\pi \in L_G$, define mult $\pi = \prod_{p \in \pi}$ mult (p). Given these notions we have,

$$\chi_G(q) = \sum_{\pi \in L_G} (-1)^{n-|\pi|} \text{ mult } \pi \, q^{|\pi|}$$

Corollary: $|\chi_G(q)[q]| = \text{mult}\,(\alpha_1 + \cdots + \alpha_n)$.
Result:

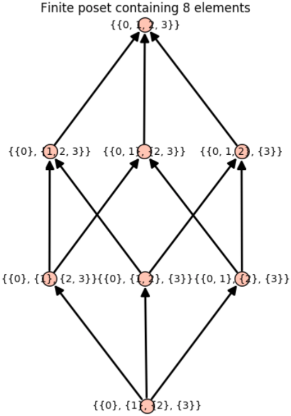
$$\chi_G(q) = \sum_{\pi \in L_G} (-1)^{n-|\pi|} \mu(\pi)\, q^{|\pi|}.$$

Corollary:

$$\text{mult}\,\pi = \mu(\pi) \text{ for } \pi \in L_G.$$

Example

Bond lattice



$$\chi_G(q) = q^4 - 3q^3 + 3q^2 - q$$

$-q$

$3\,q^2$

$-3\,q^3$

q^4

Borcherds-Cartan Matrix

Borcherds–Cartan Matrix

A real matrix $A = (a_{ij})_{i,j \in I}$ is said to be a *Borcherds–Cartan matrix* if the following conditions are satisfied for all $i, j \in I$:

1. A is symmetrizable,
2. $a_{ii} = 2$ or $a_{ii} \leq 0$,
3. $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$,
4. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Recall that a matrix A is called symmetrizable if there exists a diagonal matrix $D = \mathrm{diag}(\epsilon_i, i \in I)$ with positive entries such that DA is symmetric.
Set $I^{\mathrm{re}} = \{i \in I : a_{ii} = 2\}$ (real simple roots) and $I^{\mathrm{im}} = I \backslash I^{\mathrm{re}}$.
If $I = I^{re}$, then A is a generalized Cartan matrix.

Borcherds algebras

The Borcherds algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated to a Borcherds–Cartan matrix A is the Lie algebra generated by $\{e_i, f_i, h_i : i \in I\}$ with the following defining relations:

(R1) $[h_i, h_j] = 0$ for $i, j \in I$,

(R2) $[h_i, e_k] = a_{ik} e_i$, $[h_i, f_k] = -a_{ik} f_i$ for $i, k \in I$,

(R3) $[e_i, f_j] = \delta_{ij} h_i$ for $i, j \in I$.

(R4) $(\text{ad } e_i)^{1-a_{ij}} e_j = 0$, $(\text{ad } f_i)^{1-a_{ij}} f_j = 0$ if $i \in I^{\text{re}}$ and $i \neq j$.

(R5) $[e_i, e_j] = 0$ and $[f_i, f_j] = 0$ if $i, j \in I^{\text{im}}$ and $a_{ij} = 0$.

This definition leads to an interesting combinatorial object, known as free partially commutative Lie algebras, when $I = I^{im}$.

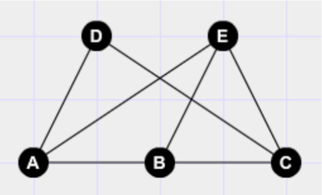
Graph of a Borcherds algebra

The vertex set of G is I, and there is an edge between vertices i, j iff $a_{ij} \neq 0$.

Borcherds-Cartan Matrix

$$\begin{bmatrix} 2 & -3 & 0 & -6 & -1 \\ -3 & -1 & -2 & 0 & -\pi \\ 0 & -2 & 2 & -1 & -4 \\ -6 & 0 & -1 & 2 & 0 \\ -1 & -\pi & -4 & 0 & 0 \end{bmatrix}$$

Associated graph



The graph associated to the matrix A is the graph of the Lie algebra $\mathfrak{g}(A)$.

Generalized chromatic polynomial and Borcherds algebras

Theorem, [-,Deniz Kus,R.Venkatesh], [1]

Let G be the graph of a Borcherds algebra $\mathfrak{g}$, Assume that $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ satisfies: $k_i \in \{0, 1\}$ for all $i \in I^{\mathrm{re}}$. Then

$$\pi^G_{\mathbf{k}}(q) = \varepsilon(\mathbf{k}) \sum_{\mathbf{J} \in L_G(\mathbf{k})} (-1)^{|\mathbf{J}|} \prod_{J \in \mathbf{J}} \binom{q \operatorname{mult}(J)}{D(J, \mathbf{J})}.$$

For $\mathbf{J} \in L_G(\mathbf{k})$ we denote by $D(J, \mathbf{J})$ the multiplicity of J_i in $\mathbf{J}$. When $k_i = 1$ for all i, this reduces to the above given expression.

Corollary: Root multiplicity formula

$$\operatorname{mult} \mathbf{k} = \sum_{\ell | \mathbf{k}} \frac{\mu(\ell)}{\ell} \left| \pi^G_{\mathbf{k}/\ell}(q)\ [q] \right|,$$

$\left|\pi^G_{\mathbf{k}}(q)\ [q]\right|$ denotes the absolute value of the coefficient of q in $\pi^G_{\mathbf{k}}(q)$.

Idea of the proof

denominator identity

$$U := \sum_{w\in W} \epsilon(w) \sum_{\gamma\in\Omega} \epsilon(\gamma) e^{w(\rho-\gamma)-\rho} = \prod_{\alpha\in\Delta_+} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$$

where Ω is the set of all $\gamma \in Q_+$ such that γ is a finite sum of mutually orthogonal distinct imaginary simple roots.

For a Weyl group element $w \in W$, we fix a reduced expression $w = \mathbf{s}_{i_1} \cdots \mathbf{s}_{i_k}$ and let $I(w) = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$. For $\gamma \in \Omega$ we set $I(\gamma) = \{\alpha \in \Pi^{im} : \ \alpha \text{ is a summand of } \gamma\}$ and

$$\mathcal{J}(\gamma) = \{w \in W \backslash \{e\} : I(w) \cup I(\gamma) \text{ is an independent set}\}.$$

Every stable set in G can be thought of as $\mathcal{J}(\gamma)$ for some w and γ.

Idea of the proof

Main lemma

Fix $w \in W$ and $\gamma \in \Omega$. We write $-(w(\rho-\gamma)-\rho) = \sum_{\alpha\in\Pi} b_\alpha(w,\gamma)\alpha$. Then we have

(i) $b_\alpha(w,\gamma) \in \mathbb{Z}_+$ for all $\alpha \in \Pi$ and $b_\alpha(w,\gamma) = 0$ if $\alpha \notin I(w) \cup I(\gamma)$,

(ii) $I(w) = \{\alpha \in \Pi^{\mathrm{re}} : b_\alpha(w,\gamma) \geq 1\}$ and $b_\alpha(w,\gamma) = 1$ if $\alpha \in I(\gamma)$,

(iii) If $w \in \mathcal{J}(\gamma)$, then $b_\alpha(w,\gamma) = 1$ for all $\alpha \in I(w) \cup I(\gamma)$ and $b_\alpha(w,\gamma) = 0$ else,

(iv) If $w \notin \mathcal{J}(\gamma) \cup \{e\}$, then there exists $\alpha \in \Pi^{\mathrm{re}}$ such that $b_\alpha(w,\gamma) > 1$.

We set $\eta(\mathbf{k}) = \sum_{i\in I} k_i\alpha_i \in Q_+$. Calculate the co-efficient of $e^{-\eta(\mathbf{k})}$ in U^q.

We observe that,
$e^{w(\rho-\gamma)-\rho} = \prod_{\alpha\in I(w)\cup I(\gamma)} e^{(-b_\alpha(w,\gamma))\alpha} = \prod_{\alpha\in I(w)\cup I(\gamma)} (X_\alpha)^{b_\alpha(w,\gamma)}$ which is an element of $\mathbb{C}[X_{\alpha_1},\dots,X_{\alpha_n}]$

How to recover X_G from the denominator identity

denominator identity

$$U := \sum_{w \in W} \epsilon(w) \sum_{\gamma \in \Omega} \epsilon(\gamma) e^{w(\rho-\gamma)-\rho} = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$$

Modified denominator identity

Let X be an indeterminate. Then we have $U(X) :=$

$$\sum_{w \in W} \epsilon(w) \sum_{\gamma \in \Omega} \epsilon(\gamma) X^{\mathrm{ht}((\rho-\gamma)-\rho)} e^{w(\rho-\gamma)-\rho} = \prod_{\alpha \in \Delta_+} (1 - X^{\mathrm{ht}(\alpha)} e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$$

Let $X_1, X_2, \ldots$ be a collection of commuting indeterminates. We study the properties of the formal product of the Weyl denominators $\prod_{i=1}^{\infty} U(X_i)$.

Weyl denominators and the G-elementary symmetric functions (Generating function of trivial heaps over G)

Stanley [3] defined the G-analogue of the ith elementary symmetry function as follows.

$$e_i^G = \sum_S \left(\prod_{\alpha \in S} X_\alpha \right),$$

where $X_\alpha = e^{-\alpha}$ and S ranges over all i-element stable subsets of G. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we define $e_\lambda^G = \prod_{i=1}^k e_{\lambda_i}^G$.

$$\begin{aligned} U(X) &= \sum_{(w,\gamma)\in W\times\Omega} \epsilon(w,\gamma) X^{\mathrm{ht}((\rho-\gamma)-\rho)} e^{w(\rho-\gamma)-\rho}, && (4) \\ &= \sum_{\substack{(w,\gamma)\in W\times\Omega \\ \text{stable}}} \epsilon(w,\gamma) e^{w(\rho-\gamma)-\rho} + \sum_{\substack{(w,\gamma)\in W\times\Omega \\ \text{not stable}}} \epsilon(w,\gamma) e^{w(\rho-\gamma)-\rho}, && (5) \\ &= U_1(X) + U_2(X) \ \text{(say)}, && (6) \end{aligned}$$

We define $U_1(X)$ to the stable part of the Weyl denominator $U(X)$

Weyl denominators and the G-elementary symmetric functions

Now, from the main lemma, it is easy to see that

$$U_1(X) = \sum_{\substack{(w,\gamma)\in W\times\Omega \\ \text{stable}}} \epsilon(w,\gamma) X^{\mathrm{ht}((\rho-\gamma)-\rho)} e^{w(\rho-\gamma)-\rho}$$

$$= \sum_{k\geq 0}^{\alpha(G)-\text{independence number of } G} \sum_{\substack{(w,\gamma)\in W\times\Omega \\ \text{stable} \\ |I(w)\cup I(\gamma)|=k}} (-1)^k X^k e^{w(\rho-\gamma)-\rho}$$

$$= \sum_{k\geq 0}^{\alpha(G)} \sum_{\substack{S\text{-stable} \\ |S|=k}} (-X)^k \left(\prod_{\alpha\in S} e^{-\alpha}\right) = \sum_{k\geq 0}^{\alpha(G)} (-X)^k e_k^G,$$

This shows that e_k^G can be recovered from the Weyl denominator.

Weyl denominators and the G-elementary symmetric functions

A number partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is said to be a **stable number partition** of G if $1 \leq \lambda_i \leq \alpha(G)$ for all $1 \leq i \leq k$. The following proposition gives the connection between the modified Weyl denominators, monomial symmetric functions and $G-$elementary symmetric functions.

Proposition

With the notations as above, we have

$$\prod_{i=1}^{\infty} U_1(X_i) = \sum_{\substack{\lambda \\ \text{stable}}} \epsilon(\lambda) M_\lambda(x) e_\lambda^G$$

The proof follows from the following equation

$$\prod_{i=1}^{\infty} U_1(X_i) = \prod_{i=1}^{\infty} \Big(\sum_{k\geq 1}^{\alpha(G)} (-X_i)^k e_k^G \Big).$$

Chromatic symmetric function from the Weyl denominators

Theorem

Fix a tuple of non-negative integers $\mathbf{k} = (k_i : i \in I)$ *such that* $k_i \le 1$ *for* $i \in I^{re}$. *We set* $\eta(\mathbf{k}) = \sum_{i \in I} k_i \alpha_i \in Q_+$. *Then*

$$\Big(\sum_{\substack{\lambda \\ stable}} \epsilon(\lambda) M_\lambda(x) e_\lambda^G \Big)[e^{-\eta(\mathbf{k})}] = \Big(\prod_{i=1}^{\infty} U_1(X_i) \Big)[e^{-\eta(\mathbf{k})}] = \epsilon(\mathbf{k})\ X_{\mathbf{k}}^G.$$

proof

The required coefficient is equal to

$$\sum_{k=1}^{\infty} \sum_{\substack{J\in\mathbb{N}^k \\ J=(i_1,i_2,\dots,i_k)}} \sum_{((w_1,\gamma_1),\dots,(w_k,\gamma_k))\in(W\times\Omega)^k} \epsilon(\gamma)\epsilon(w) \prod_{j=1}^{k} \left(X_{i_j}^{\ell(w)+\mathrm{ht}(\gamma)}\right)$$

where the sum ranges over all k–tuples $((w_1,\gamma_1),(w_2,\gamma_2),\dots,(w_k,\gamma_k)) \in (W\times\Omega)^k$ such that

- (w_i,γ_i) is stable for all $1 \le i \le k$,
- $I(w_1) \,\dot\cup \cdots \dot\cup\, I(w_k) = \{\alpha_i : i \in I^{\mathrm{re}}, k_i = 1\}$,
- $I(\gamma_1) \,\dot\cup \cdots \dot\cup\, I(\gamma_k) = \{\underbrace{\alpha_i,\alpha_i,\dots,\alpha_i}_{k_i-\text{times}} : i \in I^{\mathrm{im}}\}$,
- $I(w_i) \cup I(\gamma_i) \neq \emptyset$ for each $1 \le i \le k$,
- $\gamma_1 + \cdots + \gamma_k = \sum_{i\in I^{\mathrm{im}}} k_i\alpha_i$.

proof

It follows that $\big(I(w_1) \cup I(\gamma_1), \dots, I(w_k) \cup I(\gamma_k)\big) \in P_k\big(\mathbf{k}, G\big)$ and each element is obtained in this way. So the sum ranges over all elements in $P_k(\mathbf{k}, G)$. Hence $\left(\prod_{i=1}^{\infty} U(X_i)\right)[e^{-\eta(\mathbf{k})}]$ is equal to

$$\sum_{k\geq 1} \sum_{\substack{\mathcal{P}\in P_k(\mathbf{k},G) \\ \mathcal{P}=(P_1,P_2,\dots,P_k)}} \sum_{\substack{J\in\mathbb{N}^k \\ J=(i_1,i_2,\dots,i_k)}} x_{i_1}^{|P_1|} x_{i_2}^{|P_2|} \cdots x_{i_k}^{|P_k|}$$

This completes the proof.

Chromatic symmetric function and the root multiplicities

We have the following expression for the chromatic symmetric function in terms of root multiplicities of the Borcherds algebra $\mathfrak{g}$.

Theorem

Let G be the graph of a Borcherds algebra $\mathfrak{g}$. For a fixed tuple of non-negative integers $\mathbf{k} = (k_i : i \in I)$ such that $k_i \leq 1$ for $i \in I^{re}$. Then we have

$$X_{\mathbf{k}}^G = \sum_{\substack{\mathbf{J} \in L_G(\mathbf{k}) \\ \bar{\mathbf{J}} = \{J_1, \dots, J_k\}}} (-1)^{\mathrm{ht}(\eta(\mathbf{k})) + |\bar{\mathbf{J}}|} \left(\prod_{J \in \bar{\mathbf{J}}} \binom{\mathrm{mult}(\beta(J))}{D(J, \mathbf{J})} \right) p_{\mathrm{type}(\mathbf{J})}, \qquad (7)$$

where $\bar{\mathbf{J}}$ is the underlying set of the multiset $\mathbf{J}$.

Chromatic symmetric function and the root multiplicities

Corollary

$$X_G = \sum_{\mathbf{J} \in L_G} (-1)^{l-|\mathbf{J}|} (\mathrm{mult}(\mathbf{J}))\, p_{\mathrm{type}(\mathbf{J})}, \tag{8}$$

where L_G is the bond lattice of G.

We get a Lie theoretic proof of the following theorem of Stanley [2].

Theorem

$$X_G = \sum_{\mathbf{J} \in L_G} \mu(\hat{0}, \mathbf{J})\, p_{\mathrm{type}(\mathbf{J})},$$

where L_G is the bond lattice of G.

G-power sum symmetric functions

The following relation is proved in [3].

$$-\log(1 - e_1^G X + e_2^G X^2 - e_3^G X^3 + \cdots) = p_1^G X + p_2^G \frac{X^2}{2} + p_3^G \frac{X^3}{3} + \cdots \quad (9)$$

The G analogues of power sum symmetric functions are defined using the above equation. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we define $p_\lambda^G = \prod_{i=1}^k p_{\lambda_i}^G$.

Theorem

The G-power sum symmetric function p_λ^G is a polynomial with non-negative integral coefficients.

To prove Theorem 4, it is enough to prove it for p_n^G $(n \in \mathbb{N})$. We assume that all the simple roots of $\mathfrak{g}$ are imaginary, then the modified denominator identity of $\mathfrak{g}$

$$\sum_{w \in W} \epsilon(w) \sum_{\gamma \in \Omega} \epsilon(\gamma) X^{\mathrm{ht}((\rho-\gamma)-\rho)} e^{w(\rho-\gamma)-\rho} = \prod_{\alpha \in \Delta_+} (1 - X^{\mathrm{ht}(\alpha)} e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$$

becomes

$$U(X) := \sum_{\gamma \in \Omega} (-1)^{\mathrm{ht}(-\gamma)} X^{\mathrm{ht}\,\gamma} e^{\gamma} = \prod_{\alpha \in \Delta_+} (1 - X^{\mathrm{ht}\,\alpha} e^{-\alpha})^{\dim \mathfrak{g}_\alpha} \tag{10}$$

We observe that, since all the simple roots are imaginary, the stable part U_1 of U is itself. We have proved that

$$U_1(X) = \sum_{i \geq 0}^{\alpha(G)} (-X)^i e_i^G = U(X).$$

Hence

$$-\log(1 - e_1^G X + e_2^G X^2 - e_3^G X^3 + \cdots) = -\log(U(X))$$

.

This shows that the coefficient of $\frac{X^n}{n}$ in $-\log(U(X))$ is equal to p_n^G. Now, we calculate the same coefficient using the product side of Equation (10).

$$
\begin{aligned}
-\log\Big(\prod_{\alpha\in\Delta_+}(1-X^{\operatorname{ht}\alpha}e^{-\alpha})^{\dim\mathfrak{g}_\alpha}\Big) &= \sum_{\alpha\in\Delta_+}\dim\mathfrak{g}_\alpha\Big(-\log(1-X^{\operatorname{ht}\alpha}e^{-\alpha})\Big),\\
&= \sum_{\alpha\in\Delta_+}\dim\mathfrak{g}_\alpha\Big(\sum_{k\geq 1}\frac{(X^{\operatorname{ht}\alpha}e^{-\alpha})^k}{k})\Big),\\
&= \sum_{k\geq 1}\sum_{m\geq 1}\sum_{\substack{\alpha\in\Delta_+\\ \operatorname{ht}\alpha=m}}((m)(\dim\mathfrak{g}_\alpha)(e^{-k\alpha}))\frac{X^{mk}}{mk}
\end{aligned}
$$

Hence, the coefficient of $\frac{X^n}{n}$ in $-\log(U(X))$ is equal to

$$\sum_{k|n}\Big(\sum_{\substack{\alpha\in\Delta_+\\ \mathrm{ht}\,\alpha=\frac{n}{k}}}(\frac{n}{k})(\dim\mathfrak{g}_\alpha)(e^{-k\alpha})\Big)=\sum_{k|n}\Big(\sum_{\substack{\frac{\alpha}{k}\in\Delta_+\\ \mathrm{ht}\,\frac{\alpha}{k}=\frac{n}{k}}}(\frac{n}{k})(\dim\mathfrak{g}_{\frac{\alpha}{k}})(e^{-\alpha})\Big),$$

$$\sum_{\substack{\alpha\in\Delta_+\\ \mathrm{ht}\,\alpha=n}}\Big(\sum_{k|\alpha}(\frac{n}{k})(\dim\mathfrak{g}_{\frac{\alpha}{k}})\Big)e^{-\alpha},$$

$$=\sum_{k\geq 1}\sum_{m\geq 1}\sum_{\substack{\alpha\in\Delta_+\\ \mathrm{ht}\,\alpha=m}}((m)(\dim\mathfrak{g}_\alpha)(e^{-k\alpha}))\frac{X^{mk}}{mk}.$$

This shows that

$$p_n^G=\sum_{\substack{\alpha\in\Delta_+\\ \mathrm{ht}\,\alpha=n}}\Big(\sum_{k|\alpha}(\frac{n}{k})(\dim\mathfrak{g}_{\frac{\alpha}{k}})\Big)e^{-\alpha}$$

and the theorem follows.

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Thank you