G. Arunkumar, Chromatic symmetric function of graphs from Borcherds Lie algebra

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1. What is chromatic symmetric function and why are they interesting?
2. Connection between the Borcherds algebras and Chromatic symmetric functions.
3. Applications.

Graph vertex proper coloring

Let $G$ be a finite simple connected graph with a totally ordered vertex set $I = \{\alpha_1, \ldots, \alpha_n\}$. $I$ will be identified with the simple roots of $g$.

Chromatic polynomial:

$$\chi_G(q) = \text{The number of ways of coloring } G \text{ 'using' } q \text{ colors.}$$

Results: 1. $\chi_G(q)$ is a polynomial in $q$. 2. $\chi_G(q) = q(q - 1)^{n-1}$ for any tree with $n$ vertices.
Chromatic polynomial has the following well–known description. Consider an ordered $k$–tuples $(P_1, \ldots, P_k)$ such that:

(i) each $P_i$ is a non–empty independent subset of $I$, i.e. no two vertices have an edge between them; and

(ii) the disjoint union of $P_1, \cdots, P_k$ is equal to $I$.

We denote by $P_k(G)$ the set of all stable partitions with $k$ parts. Then we have

$$\chi_G(q) = \sum_{k \geq 0} |P_k(G)| \binom{q}{k}. \quad (1)$$
Vertex $k$-multicoloring of a graph

$k = (2, 2, 2, 2, 2, 2) \in \mathbb{Z}_{\geq 0}^n$ - coloring

We let $\pi_k^G(q)$ be the number of such coloring using $q$ colors. This is called the $k$-generalized chromatic polynomial of $G$.
Expression for \(k\)–chromatic polynomials

\(k\)–chromatic polynomials has the following well–known description.

We denote by \(P_k(k, G)\) the set of all ordered \(k\)–tuples \((P_1, \ldots, P_k)\) such that:

(i) each \(P_i\) is a non–empty independent subset of \(I\), i.e. no two vertices have an edge between them; and

(ii) the disjoint union of \(P_1, \ldots, P_k\) is equal to the multiset \(\{\alpha_i, \ldots, \alpha_i : i \in I\}\). \(k_i\) times

Then we have

\[
\pi^G_k(q) = \sum_{k \geq 0} |P_k(k, G)| \binom{q}{k}.
\] (2)
Relation between $\pi^G_k(q)$ and $\chi_G(q)$

The graph $G(k)$ is the join of $G$ with respect to $k$. Assume $k = (3, 2, 2)$

$$\pi^G_k(q) = \frac{1}{k!} \pi^G_1(q), \text{ where } k! = \prod_{i=1}^{n} k_i!$$

\[ x^\kappa = x_1^2 x_2 x_3^2 x_5 \]

\[ X_G = X_G(x_1, x_2, \ldots) = \sum_{\kappa \text{ proper-coloring}} x^\kappa \]

\[ X_G(1, 1, \ldots, 1, 0, 0, \ldots) = \chi_G(q) \]
We have shown that,

$$\chi_G(q) = \sum_{k \geq 0} |P_k(G)| \binom{q}{k}. \quad (3)$$

We have the following expression for chromatic symmetric function.

$$X_G = \sum_{k \geq 1} \sum_{P \in P_k(G)} \sum_{J \in \mathcal{P}_k(\mathbb{N})} \sum_{P = (P_1, P_2, \ldots, P_k)} x_{i_1} |P_1| x_{i_2} |P_2| \cdots x_{i_k} |P_k|$$

Stanley: $X_G$ determines the graph $G$?
Answer: No

These graphs have the same chromatic symmetric function:

\[ X_G = X_H = 2\tilde{m}_{221} + 4\tilde{m}_{2111} + \tilde{m}_{11111} \]

\[ X_{P4} = 24m_{1111} + 6m_{211} + 2m_{22}, \quad X_{S4} = 24m_{1111} + 6m_{211} + m_{31} \]

S4 is the claw-graph \( K_{1,3} \).

Stanley:

Does there exist two non-isomorphic trees with the same chromatic symmetric function?
Positivity results

Monomial symmetric functions
Let $a_{\lambda}$ be the number of stable partitions of $G$ of type $\lambda$. Then

$$X_G = \sum_{\lambda \vdash d} a_{\lambda} \tilde{m}_{\lambda}.$$ 

Power sum symmetric functions
For any graph $G$, the symmetric function $w(X_G)$ is $p-$positive.

Elementary symmetric functions
1. Let $G$ be the claw-graph then $X_G = 3e_4 + 5e_{31} - 2e_{22} + e_{221}.$
2. For any graph $G$ we have $\text{sink}(G,j) = \sum_{\lambda \vdash d} c_{\lambda}.$
3. If the complement of $G$ is bipartite, then $c_{\lambda} \geq 0.$
Positivity results

**Schur functions**

Let $G$ be the claw-graph then $X_G = s_{31} - s_{22} + 5s_{221} + 8s_{1111}$.

**Conjecture**

Let $G$ be the incomparability graph of a $3+1$-free poset. Then $X_G$ is $e-$positive.

Inc($3+1$) = $K_{1,3}$ leads to ask whether any claw-free graph is $e-$positive?

\[
X_G = 12e_6 + 18e_{51} + 12e_{42} - 6e_{33} + 6e_{441} + 6e_{321}.
\]
Positivity results

Theorem (Gasharov)

Let $G$ be the incomparability graph of a $3+1$-free poset. Then $X_G$ is $s$–positive. Moreover,

$$X_G = \sum_{\lambda | d} f^\lambda(P) s_\lambda$$

where $f^\lambda(P)$ denote the number of $P$–tableaux of shape $\lambda$.

Let $P$ be a finite poset with $d$ elements. A $P$–tableau of shape $\lambda | d$ is a map $\tau : P \to \mathbb{N}$ satisfying

1. $|\tau^{-1}(i)| = \lambda_i$ for all $i$,
2. $\tau$ is a proper coloring of $\text{Inc}(P)$,
3. Suppose $\tau^{-1}(i) = \{u_1 < \cdots < u_{\lambda_i}\}$ and $\tau^{-1}(i + 1) = \{v_1 < \cdots < v_{\lambda_i+1}\}$ then $v_j \not< u_j$ for all $1 \leq j \leq \lambda_i+1$. \qed
Theorem [R.Venkatesh, Sankaran Viswanath] [4]

Let $G$ be the graph of a Kac-Moody algebra $g$. Given a $\pi \in L_G$, define $\text{mult } \pi = \prod_{p \in \pi} \text{mult } (p)$. Given these notions we have,

$$\chi_G(q) = \sum_{\pi \in L_G} (-1)^{n-|\pi|} \text{mult } \pi \ q^{|\pi|}$$

Corollary: $|\chi_G(q)[q]| = \text{mult } (\alpha_1 + \cdots + \alpha_n)$.

Result:

$$\chi_G(q) = \sum_{\pi \in L_G} (-1)^{n-|\pi|} \mu(\pi) \ q^{|\pi|}.$$ 

Corollary:

$$\text{mult } \pi = \mu(\pi) \text{ for } \pi \in L_G.$$
Example

Bond lattice

\( \chi_G(q) = q^4 - 3q^3 + 3q^2 - q - q \)

\( 3q^2 \)

\( -3q^3 \)

\( q^4 \)
A real matrix \( A = (a_{ij})_{i,j \in I} \) is said to be a \textit{Borcherds–Cartan matrix} if the following conditions are satisfied for all \( i, j \in I \):

1. \( A \) is symmetrizable,
2. \( a_{ii} = 2 \) or \( a_{ii} \leq 0 \),
3. \( a_{ij} \leq 0 \) if \( i \neq j \) and \( a_{ij} \in \mathbb{Z} \) if \( a_{ii} = 2 \),
4. \( a_{ij} = 0 \) if and only if \( a_{ji} = 0 \).

Recall that a matrix \( A \) is called symmetrizable if there exists a diagonal matrix \( D = \text{diag}(\epsilon_i, i \in I) \) with positive entries such that \( DA \) is symmetric.

Set \( I^{re} = \{ i \in I : a_{ii} = 2 \} \) (real simple roots) and \( I^{im} = I \setminus I^{re} \).

If \( I = I^{re} \), then \( A \) is a generalized Cartan matrix.
Borcherds algebras

The Borcherds algebra \( g = g(A) \) associated to a Borcherds–Cartan matrix \( A \) is the Lie algebra generated by \( \{ e_i, f_i, h_i : i \in I \} \) with the following defining relations:

1. \( [h_i, h_j] = 0 \) for \( i, j \in I \),
2. \( [h_i, e_k] = a_{ik} e_i, \ [h_i, f_k] = -a_{ik} f_i \) for \( i, k \in I \),
3. \( [e_i, f_j] = \delta_{ij} h_i \) for \( i, j \in I \).
4. \( (\text{ad } e_i)^{1-a_{ij}} e_j = 0, \ (\text{ad } f_i)^{1-a_{ij}} f_j = 0 \) if \( i \in I^{\text{re}} \) and \( i \neq j \).
5. \( [e_i, e_j] = 0 \) and \( [f_i, f_j] = 0 \) if \( i, j \in I^{\text{im}} \) and \( a_{ij} = 0 \).

This definition leads to an interesting combinatorial object, known as free partially commutative Lie algebras, when \( I = I^{\text{im}} \).
The vertex set of $G$ is $I$, and there is an edge between vertices $i, j$ iff $a_{ij} \neq 0$.

Borcherds-Cartan Matrix

$$
\begin{bmatrix}
2 & -3 & 0 & -6 & -1 \\
-3 & -1 & -2 & 0 & -\pi \\
0 & -2 & 2 & -1 & -4 \\
-6 & 0 & -1 & 2 & 0 \\
-1 & -\pi & -4 & 0 & 0
\end{bmatrix}
$$

Associated graph

The graph associated to the matrix $A$ is the graph of the Lie algebra $g(A)$. 
Theorem, [-,Deniz Kus,R.Venkatesh], [1]

Let $G$ be the graph of a Borcherds algebra $\mathfrak{g}$, Assume that $k = (k_1, k_2, \ldots, k_n)$ satisfies: $k_i \in \{0, 1\}$ for all $i \in I^\text{re}$. Then

$$\pi_k^G(q) = \varepsilon(k) \sum_{J \in L_G(k)} (-1)^{|J|} \prod_{J \in J} \left( q^{\text{mult}(J)} D(J, J) \right).$$

For $J \in L_G(k)$ we denote by $D(J, J)$ the multiplicity of $J_i$ in $J$. When $k_i = 1$ for all $i$, this reduces to the above given expression.

Corollary: Root multiplicity formula

$$\text{mult } k = \sum_{\ell \mid k} \frac{\mu(\ell)}{\ell} \left| \pi_{k/\ell}^G(q) \right|,$$

$|\pi_k^G(q)|$ denotes the absolute value of the coefficient of $q$ in $\pi_k^G(q)$. 
Idea of the proof

denominator identity

\[ U := \sum_{w \in W} \epsilon(w) \sum_{\gamma \in \Omega} \epsilon(\gamma) e^{w(\rho - \gamma) - \rho} = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim g_\alpha} \]

where \( \Omega \) is the set of all \( \gamma \in Q_+ \) such that \( \gamma \) is a finite sum of mutually orthogonal distinct imaginary simple roots.

For a Weyl group element \( w \in W \), we fix a reduced expression \( w = s_{i_1} \cdots s_{i_k} \) and let \( I(w) = \{ \alpha_{i_1}, \ldots, \alpha_{i_k} \} \). For \( \gamma \in \Omega \) we set \( I(\gamma) = \{ \alpha \in \Pi^m : \alpha \text{ is a summand of } \gamma \} \) and

\[ \mathcal{J}(\gamma) = \{ w \in W \setminus \{ e \} : I(w) \cup I(\gamma) \text{ is an independent set} \} \].

Every stable set in \( G \) can be thought of as \( \mathcal{J}(\gamma) \) for some \( w \) and \( \gamma \).
Idea of the proof

Main lemma

Fix \( w \in W \) and \( \gamma \in \Omega \). We write 
\[-(w(\rho - \gamma) - \rho) = \sum_{\alpha \in \Pi} b_\alpha(w, \gamma)\alpha.\]
Then we have

(i) \( b_\alpha(w, \gamma) \in \mathbb{Z}_+ \) for all \( \alpha \in \Pi \) and \( b_\alpha(w, \gamma) = 0 \) if \( \alpha \notin I(w) \cup I(\gamma) \),
(ii) \( I(w) = \{ \alpha \in \Pi^{re} : b_\alpha(w, \gamma) \geq 1 \} \) and \( b_\alpha(w, \gamma) = 1 \) if \( \alpha \in I(\gamma) \),
(iii) If \( w \in J(\gamma) \), then \( b_\alpha(w, \gamma) = 1 \) for all \( \alpha \in I(w) \cup I(\gamma) \) and \( b_\alpha(w, \gamma) = 0 \) else,
(iv) If \( w \notin J(\gamma) \cup \{ e \} \), then there exists \( \alpha \in \Pi^{re} \) such that \( b_\alpha(w, \gamma) > 1 \).

We set \( \eta(k) = \sum_{i \in I} k_i \alpha_i \in Q_+ \). Calculate the co-efficient of \( e^{-\eta(k)} \) in \( U^q \).

We observe that,
\[e^{w(\rho - \gamma) - \rho} = \prod_{\alpha \in I(w) \cup I(\gamma)} e^{-b_\alpha(w, \gamma)}\alpha = \prod_{\alpha \in I(w) \cup I(\gamma)} (X_\alpha)^{b_\alpha(w, \gamma)} \text{ which is an element of } \mathbb{C}[X_{\alpha_1}, \ldots, X_{\alpha_n}]\]
How to recover \( X_G \) from the denominator identity

denominator identity

\[
U := \sum_{w \in W} \epsilon(w) \sum_{\gamma \in \Omega} \epsilon(\gamma) e^{w(\rho-\gamma)-\rho} = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim g_\alpha}
\]

Modified denominator identity

Let \( X \) be an indeterminate. Then we have \( U(X) := \)

\[
\sum_{w \in W} \epsilon(w) \sum_{\gamma \in \Omega} \epsilon(\gamma) X^{\text{ht}((\rho-\gamma)-\rho)} e^{w(\rho-\gamma)-\rho} = \prod_{\alpha \in \Delta_+} (1 - X^{\text{ht}(\alpha)} e^{-\alpha})^{\dim g_\alpha}
\]

Let \( X_1, X_2, \ldots \) be a collection of commuting indeterminates. We study the properties of the formal product of the Weyl denominators \( \prod_{i=1}^{\infty} U(X_i) \).
Weyl denominators and the G-elementary symmetric functions (Generating function of trivial heaps over $G$)

Stanley [3] defined the $G$-analogue of the $i$th elementary symmetry function as follows.

$$e_i^G = \sum_S \left( \prod_{\alpha \in S} X_\alpha \right),$$

where $X_\alpha = e^{-\alpha}$ and $S$ ranges over all $i$-element stable subsets of $G$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we define $e^G_\lambda = \prod_{i=1}^k e^G_{\lambda_i}$.

\[ U(X) = \sum_{(w, \gamma) \in W \times \Omega} \epsilon(w, \gamma) X^{ht((\rho-\gamma)-\rho)} e^{w(\rho-\gamma)-\rho}, \]

\[ = \sum_{(w, \gamma) \in W \times \Omega \text{ stable}} \epsilon(w, \gamma) e^{w(\rho-\gamma)-\rho} + \sum_{(w, \gamma) \in W \times \Omega \text{ not stable}} \epsilon(w, \gamma) e^{w(\rho-\gamma)-\rho}, \]

\[ = U_1(X) + U_2(X) \text{ (say)}, \]
Weyl denominators and the G-elementary symmetric functions

Now, from the main lemma, it is easy to see that

$$U_1(X) = \sum_{(w,\gamma) \in W \times \Omega \text{ stable}} \epsilon(w,\gamma) X^{\operatorname{ht}((\rho-\gamma)-\rho)} e^{w(\rho-\gamma)-\rho}$$

$$\alpha(G)$$—independence number of $G$

$$= \sum_{k \geq 0} \sum_{(w,\gamma) \in W \times \Omega \text{ stable} \atop |I(w) \cup I(\gamma)| = k} (\epsilon(-1)^{k} X^{k} e^{w(\rho-\gamma)-\rho})$$

$$= \sum_{k \geq 0} \sum_{S-\text{stable} \atop |S| = k} (-X)^{k} \left( \prod_{\alpha \in S} e^{-\alpha} \right) = \sum_{k \geq 0} (-X)^{k} e_{G}^{k},$$

This shows that $e_{G}^{k}$ can be recovered from the Weyl denominator.
Weyl denominators and the G-elementary symmetric functions

A number partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is said to be a **stable number partition** of \( G \) if \( 1 \leq \lambda_i \leq \alpha(G) \) for all \( 1 \leq i \leq k \). The following proposition gives the connection between the modified Weyl denominators, monomial symmetric functions and \( G \)-elementary symmetric functions.

**Proposition**

With the notations as above, we have

\[
\prod_{i=1}^{\infty} U_1(X_i) = \sum_{\lambda \text{ stable}} \epsilon(\lambda) M_{\lambda}(x) e^G_{\lambda}
\]

The proof follows from the following equation

\[
\prod_{i=1}^{\infty} U_1(X_i) = \prod_{i=1}^{\infty} \left( \sum_{k \geq 1} \left( -X_i \right)^k e^G_k \right).
\]
Theorem

Fix a tuple of non-negative integers $k = (k_i : i \in I)$ such that $k_i \leq 1$ for $i \in I^{re}$. We set $\eta(k) = \sum_{i \in I} k_i \alpha_i \in \mathbb{Q}_+$. Then

$$\left( \sum_{\lambda \text{ stable}} \epsilon(\lambda) M_{\lambda}(x) e^{G}_{\lambda} \right) [e^{-\eta(k)}] = \left( \prod_{i=1}^{\infty} U_1(X_i) \right) [e^{-\eta(k)}] = \epsilon(k) \ X_k^G.$$
The required coefficient is equal to
\[
\sum_{k=1}^{\infty} \sum_{J \in \mathbb{N}^k} \sum_{J = (i_1, i_2, \ldots, i_k)} \epsilon(\gamma) \epsilon(w) \prod_{j=1}^{k} \left( X_{i_j}^{\ell(w) + \text{ht}(\gamma)} \right)
\]
where the sum ranges over all \( k \)-tuples
\(((w_1, \gamma_1), (w_2, \gamma_2), \ldots, (w_k, \gamma_k)) \in (W \times \Omega)^k \) such that

- \((w_i, \gamma_i)\) is stable for all \( 1 \leq i \leq k \),
- \( I(w_1) \cup \cdots \cup I(w_k) = \{ \alpha_i : i \in I^{\text{re}}, k_i = 1 \} \),
- \( I(\gamma_1) \cup \cdots \cup I(\gamma_k) = \underbrace{\{ \alpha_i, \alpha_i, \ldots, \alpha_i \}}_{k_i \text{-times}} : i \in I^{\text{im}} \),
- \( I(w_i) \cup I(\gamma_i) \neq \emptyset \) for each \( 1 \leq i \leq k \),
- \( \gamma_1 + \cdots + \gamma_k = \sum_{i \in I^{\text{im}}} k_i \alpha_i \).
It follows that \((I(w_1) \cup I(\gamma_1), \ldots, I(w_k) \cup I(\gamma_k)) \in P_k(k, G)\) and each element is obtained in this way. So the sum ranges over all elements in \(P_k(k, G)\). Hence \(\left(\prod_{i=1}^{\infty} U(X_i)\right)[e^{-\eta(k)}]\) is equal to

\[
\sum_{k \geq 1} \sum_{\mathcal{P} \in P_k(k, G)} \sum_{\mathcal{J} \in \mathbb{N}^k} x_{i_1}^{P_1} x_{i_2}^{P_2} \cdots x_{i_k}^{P_k}
\]

This completes the proof.
Chromatic symmetric function and the root multiplicities

We have the following expression for the chromatic symmetric function in terms of root multiplicities of the Borcherds algebra $\mathfrak{g}$.

**Theorem**

Let $G$ be the graph of a Borcherds algebra $\mathfrak{g}$. For a fixed tuple of non-negative integers $k = (k_i : i \in I)$ such that $k_i \leq 1$ for $i \in I^{\text{re}}$. Then we have

$$X_k^G = \sum_{\mathbf{J} \in L_G(k)} (-1)^{\text{ht}(\eta(k)) + |\bar{\mathbf{J}}|} \left( \prod_{J \in \bar{\mathbf{J}}} \left( \frac{\text{mult}(\beta(J))}{D(J, J)} \right) \right) p_{\text{type}(\mathbf{J})},$$

where $\bar{\mathbf{J}}$ is the underlying set of the multiset $\mathbf{J}$.
Corollary

\[ X_G = \sum_{J \in L_G} (-1)^{|J|} (\text{mult}(J)) \, p_{\text{type}}(J), \]  

where \( L_G \) is the bond lattice of \( G \).

We get a Lie theoretic proof of the following theorem of Stanley [2].

Theorem

\[ X_G = \sum_{J \in L_G} \mu(\hat{0}, J) \, p_{\text{type}}(J), \]  

where \( L_G \) is the bond lattice of \( G \).
The following relation is proved in [3].

\[- \log(1 - e_1^G X + e_2^G X^2 - e_3^G X^3 + \cdots) = p_1^G X + p_2^G \frac{X^2}{2} + p_3^G \frac{X^3}{3} + \cdots \] (9)

The \( G \) analogues of power sum symmetric functions are defined using the above equation. For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), we define

\[ p_\lambda^G = \prod_{i=1}^{k} p_{\lambda_i}^G. \]

**Theorem**

*The \( G \)-power sum symmetric function \( p_\lambda^G \) is a polynomial with non-negative integral coefficients.*
To prove Theorem 4, it is enough to prove it for $p^G_n \ (n \in \mathbb{N})$. We assume that all the simple roots of $g$ are imaginary, then the modified denominator identity of $g$

$$
\sum_{w \in W} \epsilon(w) \sum_{\gamma \in \Omega} \epsilon(\gamma) X^{ht((\rho-\gamma)-\rho)} e^{w(\rho-\gamma)-\rho} = \prod_{\alpha \in \Delta_+} (1 - X^{ht(\alpha)} e^{-\alpha})^{\dim g_{\alpha}}
$$

becomes

$$
U(X) := \sum_{\gamma \in \Omega} (-1)^{ht(-\gamma)} X^{ht\gamma} e^\gamma = \prod_{\alpha \in \Delta_+} (1 - X^{ht\alpha} e^{-\alpha})^{\dim g_{\alpha}} \quad (10)
$$

We observe that, since all the simple roots are imaginary, the stable part $U_1$ of $U$ is itself. We have proved that

$$
U_1(X) = \sum_{i \geq 0} (-X)^i e_i^G = U(X).
$$

Hence

$$
- \log(1 - e_1^G X + e_2^G X^2 - e_3^G X^3 + \cdots) = - \log(U(X))
$$
This shows that the coefficient of $\frac{X^n}{n}$ in $-\log(U(X))$ is equal to $p_n^G$. Now, we calculate the same coefficient using the product side of Equation (10).

\[
-\log \left( \prod_{\alpha \in \Delta_+} (1 - X^{ht\alpha} e^{-\alpha})^{\dim g_{\alpha}} \right) = \sum_{\alpha \in \Delta_+} \dim g_{\alpha} \left( -\log(1 - X^{ht\alpha} e^{-\alpha}) \right),
\]

\[
= \sum_{\alpha \in \Delta_+} \dim g_{\alpha} \left( \sum_{k \geq 1} \frac{(X^{ht\alpha} e^{-\alpha})^k}{k} \right),
\]

\[
= \sum_{k \geq 1} \sum_{m \geq 1} \sum_{\alpha \in \Delta_+} \frac{(m)(\dim g_{\alpha})(e^{-k\alpha})}{mk} X^{mk}.
\]
Hence, the coefficient of $\frac{X^n}{n}$ in $-\log(U(X))$ is equal to

$$\sum_{k|n} \left( \sum_{\alpha \in \Delta_+ \atop \text{ht } \alpha = \frac{n}{k}} \left( \sum_{\frac{\alpha}{k} \in \Delta_+ \atop \text{ht } \frac{\alpha}{k} = \frac{n}{k}} \left( \frac{n}{k} \right) \left( \dim g_{\alpha k} \right) (e^{-k \alpha}) \right) \right) = \sum_{k|n} \left( \sum_{\frac{\alpha}{k} \in \Delta_+ \atop \text{ht } \frac{\alpha}{k} = \frac{n}{k}} \left( \sum_{\alpha \in \Delta_+ \atop \text{ht } \alpha = n} \left( \frac{n}{k} \right) \left( \dim g_{\alpha k} \right) \right) e^{-\alpha} \right),$$

$$= \sum_{k \geq 1} \sum_{m \geq 1} \sum_{\alpha \in \Delta_+ \atop \text{ht } \alpha = m} ((m) \left( \dim g_{\alpha} \right) (e^{-k \alpha})) \frac{X^{mk}}{mk}.$$

This shows that

$$p_n^G = \sum_{\alpha \in \Delta_+ \atop \text{ht } \alpha = n} \left( \sum_{k \mid \alpha} \left( \sum_{\frac{\alpha}{k} \in \Delta_+ \atop \text{ht } \frac{\alpha}{k} = \frac{n}{k}} \left( \frac{n}{k} \right) \left( \dim g_{\alpha k} \right) \right) \right) e^{-\alpha}$$

and the theorem follows.
G. Arunkumar, Deniz Kus, and R. Venkatesh.  
Root multiplicities for Borcherds algebras and graph coloring.  

Richard P. Stanley.  
A symmetric function generalization of the chromatic polynomial of a graph.  

Richard P. Stanley.  
Graph colorings and related symmetric functions: ideas and applications: a description of results, interesting applications, & notable open problems.  
Selected papers in honor of Adriano Garsia (Taormina, 1994).
Thank you