Twisting

1. Conjugacy classes: G : gp, ZG: center.

The map $ZG \times G \longrightarrow G$ descends to a group
$(z, g) \longmapsto zg$

action of ZG on the set of conjugacy classes.

Example: For $GL_n(\mathbb{F}_q)$, $ZG = \mathbb{F}_q^* \, Id$.

$\therefore$ the conjugacy classes in $GL_n(\mathbb{F}_q)$ come with an action of $\mathbb{F}_q^*$.

2. ~~Class~~ ~~Eo~~ Representations: $\hat{G}$: iso. classes of irreps of G,

$X'(G) := Hom(G, \mathbb{C}^*)$.

For $\rho \in \hat{G}$, $\chi \in X'(G)$ define

$$\rho \otimes \chi(g) = \chi(g)\rho(g).$$

$(\chi, \rho) \longmapsto \rho \otimes \chi$ defines a gp. action

$$X'(G) \times \hat{G} \longrightarrow \hat{G}.$$

Let $[G, G]$ = subgp. of G gen. by $\{xyx^{-1}y^{-1} \mid x, y \in G\}$.

Then $X'(G) = \widehat{G/[G, G]}$.

Example: For $GL_n(\mathbb{F}_q)$, $[G, G] = SL_n(\mathbb{F}_q)$ if q is odd.

$\boxed{n=2}$

- $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} a+1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]$
- so $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in [G, G]$ for $a \neq -1$.
- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in [G, G] \Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in [G, G]$
- so $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in [G, G] \quad \forall\, a \in \mathbb{F}_q \quad \ldots\ (1)$

- similarly $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \in [G,G] \quad \forall\ a \in \mathbb{F}_q$
- $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a^{-1} & 1 \end{pmatrix}$
- $\begin{pmatrix} 0 & a \\ -a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}$
- in ptclr., $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in [G,G]$. ... (2)
- $\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$
- so $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in [G,G] \quad \forall\ a \in \mathbb{F}_q^*$.

$\boxed{n \geq 2}$ Using these steps we can show that if

$$E_{ij}(a) = \begin{array}{c} \begin{pmatrix} 1 & & & \\ & \ddots & a & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \\ \uparrow \\ j\text{th col.} \end{array} \begin{array}{l} \leftarrow i\text{th row} \\ \\ \\ \\ \\ \end{array}$$

$$e_{ij}(a) = \begin{array}{c} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & a & \cdots & \cdots & \\ & & \vdots & a^{-1} & \cdots & \\ & & \vdots & \vdots & \ddots & \\ & & & & & 1 \end{pmatrix} \\ \uparrow \quad \uparrow \\ i\text{th col} \quad j\text{th col} \end{array} \begin{array}{l} \\ \\ \leftarrow i\text{th row} \\ \leftarrow j\text{th row} \\ \\ \\ \\ \\ \end{array}$$

$$s_{ij} = \begin{array}{c} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & \cdots & 1 & \\ & & \vdots & \ddots & \vdots & \\ & & -1 & \cdots & 0 & \\ & & & & & \ddots \ 1 \end{pmatrix} \\ \uparrow \quad \uparrow \\ i\text{th col.} \quad j\text{th col.} \end{array} \begin{array}{l} \\ \\ \leftarrow i\text{th row} \\ \\ \leftarrow j\text{th row} \\ \\ \\ \\ \end{array}$$

Then $E_{ij}(a) \in [G,G] \quad \forall\ a \in \mathbb{F}_q \quad \forall\ i \neq j$

$e_{ij}(a) \in [G,G] \quad \forall\ a \in \mathbb{F}_q^* \quad \forall\ i \neq j$

$s_{ij} \in [G,G] \quad \forall\ i \neq j$.