~~Conv~~ Types of conjugacy classes:

Conjugacy classes in $GL_n(\mathbb{F}_q)$ are in bijective correspondence with functions:

$$f : \mathrm{Irr}(\mathbb{F}_q[t]) \longrightarrow \Lambda$$

such that $f(t) = \phi$ and

$$\sum_{p \in \mathrm{Irr}(\mathbb{F}_q[t])} (\deg p)|\Lambda| = n.$$

This correspondence is realized by taking f to the $\mathbb{F}_q[t]$-module

$$\bigoplus_{p \in \mathrm{Irr}(\mathbb{F}_q[t])} M_{p, f(p)} \quad \text{where,}$$

for $p(t) \in \mathbb{F}_q[t]$, & $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \Lambda$,

$$M_{p,\lambda} = \mathbb{F}_q[t]/p(t)^{\lambda_1} \oplus \dots \oplus \mathbb{F}_q[t]/p(t)^{\lambda_\ell}.$$

Defn: f_1 & f_2 have the same *type* if $\exists$ a degree-preserving bijection $\mathrm{Irr}(\mathbb{F}_q[t]) \xrightarrow{\alpha} \mathrm{Irr}(\mathbb{F}_q[t])$ such that $f_2 = f_1 \circ \alpha$.

This leads to a partition of conjugacy classes into "types".

Example: For $n = 2$ there are four types.

Now suppose we have found $u_j(t)$ such that

$$p(u_j(t)) \equiv 0 \bmod p(t)^j$$

Can we find $u_{j+1}(t) = u_j(t) + p(t)^j \alpha(t)$ $\ni$

$$\underset{\shortparallel}{p(u_{j+1}(t))} \equiv 0 \bmod p(t)^{j+1} \,?$$

$$p(u_j(t) + p(t)^j \alpha(t)) \equiv p(u_j(t)) + p(t)^j \alpha(t)\, p'(u_j(t)) \bmod p(t)^{j+1}$$

$$= p(t)^j \,[\beta(t) + \alpha(t)\, p'(u_j(t))]$$

But $p'(u_j(t)) \equiv p'(t) \bmod p(t)$, hence is a unit modulo $p(t)$.

$\therefore$ we can solve for $\alpha(t)$. (take $v = p(t)$) Q.E.D.

<u>Corollary</u>. If f_1 and f_2 are of the same type then their centralizers are conjugate.

<u>Proof</u>: Enough to show ~~it for~~ that if p_1 and p_2 are irreducible polynomials of the same degree, then $\exists$ lin. iso. $M_{p_1,\lambda} \xrightarrow{T} M_{p_2,\lambda}$ taking $\ni$

$$\operatorname{End}_{\mathbb{F}_q[t]} M_{p_2,\lambda} = T^{-1} \circ \left(\operatorname{End}_{\mathbb{F}_q[t]} M_{p_1,\lambda}\right) \circ T.$$

$$\shortparallel$$

$$\operatorname{End}_{\mathbb{F}_q[t]} \mathbb{F}_q[t]/p_1(t)^{\lambda_1} \oplus \cdots \oplus \mathbb{F}_q[t]/p_1(t)^{\lambda_\ell}$$

$$\shortparallel$$

$$\operatorname{End}_{\left(\mathbb{F}_q[u]/p_1(u)\right)[v]} \left(\mathbb{F}_q[u]/p_1(u)\right)(v)/v^{\lambda_1} \oplus \cdots$$

$$\shortparallel$$

$$\operatorname{End}_{\mathbb{F}_{q^d}[v]} \left(\mathbb{F}_{q^d}[v]/v^{\lambda_1} \oplus \cdots \oplus \mathbb{F}_{q^d}[v]/v^{\lambda_\ell}\right)$$

<u>Note</u>: This will also help you to calculate $|Z_{GL_n(\mathbb{F}_q)}^{\alpha}|$

Note: define $\mathbb{F}_q[u]/p(u)[v]/(v^k) \to \mathbb{F}_q[t]/p(t)^k$, $v \mapsto p(t)$, $u \mapsto u(t)$

Lemma: ~~Let k be a separable field~~ $\forall\ p(t) \in Irr(\mathbb{F}_q[t])$, $k \in \mathbb{N}$

$$\mathbb{F}_q[t] \big/ p(t)^k \cong \left(\mathbb{F}_q[t] \big/ p(t) \right)[v] \big/ (v^k)$$

Proof: We need to find, in $\mathbb{F}_q[t] \big/ p(t)^k$ a polynomial $u(t)$ such that $p(u(t)) \equiv 0 \mod p(t)^k$.

Take $u_1(t) = t$. Then $p(u_1(t)) \equiv 0 \mod p(t)$.

Strategy (Hensel): we will try to find

$$u_2(t) = u_1(t) + \alpha(t) p(t)$$

such that $p(u_2(t)) \equiv 0 \mod p(t)^2$.

$$\begin{aligned} & p(u_2(t)) \\ &= p(u_1(t) + \alpha(t) p(t)) \\ &\equiv p(u_1(t)) + \alpha(t) p(t) p'(u_1(t)) \mod (p(t))^2. \end{aligned}$$

So we have to solve

$$p(u_1(t)) + \alpha(t) p(t) (p'(u_1(t)) \equiv 0 \mod p(t)^2$$

$$p(t) \left[1 + \alpha(t) p'(t) \right] \equiv 0 \mod p(t)^2$$

$$\text{or} \quad \alpha(t) \equiv -p'(t)^{-1} \mod p(t).$$

We can solve this if we know that $p'(t)$ is a unit modulo $p(t)$. Since $\deg p'(t) < \deg p(t)$ & $p(t)$ is irreducible, the only problem may be that $p'(t) \equiv 0$.

$$\Rightarrow p(t) = a_0 + a_1 x^p + a_2 x^{2p} + \dots + a_n x^{np}$$

$$= \left(a_0^{1/p} + a_1^{1/p} x + \dots + a_n^{1/p} x^n \right)^p \text{ is not irreducible!}$$

Block Jordan canonical form:

If $p(t)$ is any monic polynomial $\mathbb{F}_q[t]/p(t)$ has basis $1, t, \ldots, t^{d-1}$, where d is the degree of $p(t)$.

With respect to this basis, the matrix of t is

$$\begin{pmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & \vdots & -a_1 \\ 0 & 1 & & \vdots & \vdots \\ & 0 & \ddots & 0 & \\ 0 & 0 & & 1 & -a_{d-1} \end{pmatrix} =: C_p \text{ companion matrix of } p.$$

$t^{d-1} \mapsto t^d = -a_0 - a_1 t - \cdots - a_{d-1} t^{d-1}$

Suppose we take, as basis of $\mathbb{F}_p[t]/p(t)^k \cong \left(\mathbb{F}_p[u]/p(u)\right)[v]/v^k$

$u = t - \alpha p \qquad t = u + \alpha p = u + \alpha v$

$v = p$

Take as basis

$1, \cancel{u, u^2, \ldots, u^{d-1}, \alpha^v, \alpha u}, \ldots, \alpha^v u^{d-1}, \alpha^{v^2}, \alpha^2 u, \ldots, \alpha^2 u^{d-1},$

$\ldots\ \alpha^{k-1}, \cancel{\alpha^{k-1} u, \ldots, \alpha^{k-1} u^{d-1}}$

$1, u, u^2, \ldots, u^{d-1}$

$\alpha v, \alpha v u, \ldots, \alpha v u^{d-1}$

$(\alpha v)^2, (\alpha v)^2 u, \ldots, (\alpha v)^2 u^{d-1}$

$\vdots$

$(\alpha v)^{k-1}$

$t(1) = u + \alpha v$

$t u = u^2 + \alpha v u$

$\vdots$

$t u^{d-1} = u^d + \alpha v u^{d-1}$

$u^d = -a_0 - a_1 u - \cdots - a_{d-1} u^{d-1}$

$t \alpha v = \alpha v u + (\alpha v)^2$

$\vdots$

$t(\alpha v)^{k-1} = u(\alpha v)^{k-1} + \cancel{(\alpha v)^k} = 0$

$t(\alpha v)^{k-1} u = u^2 (\alpha v)^{k-1} + 0$

$\vdots$

$$\begin{pmatrix} C_p & & & \\ I & C_p & & 0 \\ & I & \ddots & \\ 0 & & \ddots & \\ & & I & C_p \end{pmatrix}$$

Block J.C.F. !!

<u>Note</u>: Enumerating classes using RCF:

Similarity classes $\longleftrightarrow$ $p_1 \mid p_2 \mid \cdots \mid p_\ell$ $\quad \sum \deg p_i = n$

(no irreducibility condition)

So $\lambda = (\deg p_1, \ldots, \deg p_\ell)$ is a partition of n.

(written in increasing order)

$\therefore$