Λ : poset. (finite)

$\forall$ $\lambda \in \Lambda$ we have a rep. U_λ such that the following condition holds:

$\forall$ $\lambda \in \Lambda$ $\exists$ irrep. V_λ such that

$$U_\lambda = \bigoplus_{\mu \leq \lambda} m_{\mu\lambda} V_\lambda \qquad (*)$$

with $m_{\lambda\lambda} = 1$ for all $\lambda \in \Lambda$.

Then it is possible to determine the $m_{\mu\lambda}$'s ~~and define the~~ from the numbers

$$d_{\mu\lambda} = \dim \mathrm{Hom}(U_\mu, U_\lambda)$$

$$= \sum_{\nu \leq \mu, \lambda} m_{\nu\mu} m_{\nu\lambda} \qquad (+)$$

as follows:

For each minimal element $\lambda \in \Lambda$, $(*)$ says that $U_\lambda = V_\lambda$, so $d_{\lambda\lambda} = 1$.

Now suppose we already know ~~$d_{\mu\lambda}$~~ ~~$d_{\mu\nu}$~~ $m_{\mu\nu}$ for all $\nu < \lambda$ and all $\mu \leq \nu$, then for $\mu < \lambda$, we can determine $m_{\mu\lambda}$ inductively in μ by

$$d_{\mu\lambda} = \cancel{d} \sum_{\nu \leq \mu} m_{\nu\mu} m_{\nu\lambda} = \underbrace{\sum_{\nu < \mu} m_{\nu\mu} m_{\nu\lambda}}_{\text{known, by induction}} + m_{\mu\lambda}$$

Also, we would have

$$d_{\lambda\lambda} = \sum_{\nu \leq \lambda} m_{\nu\lambda}^2 = 1 + \sum_{\nu < \lambda} m_{\nu\lambda}^2 .$$

So in fact, we could prove that $(*)$ holds in this way.

reps. $U_\lambda \ni \dim \mathrm{Hom}(U_\lambda, U_\mu) = d_{\lambda\mu}$ &

<u>Theorem</u>: Suppose $\exists$ non-neg. integers $m_{\nu\mu}$ for all $\nu \leq \mu \ni m_{\mu\mu} = 1$, ~~then~~ and $(\dagger)$ holds, then $(*)$ holds.

<u>Proof</u>: For minimal λ, $(\dagger) \Rightarrow d_{\lambda\lambda} = 1$, so ~~if we define~~ $V_\lambda := U_\lambda$ is forced by $(*)$.

Suppose we have defined $V_\mu \ \forall \mu < \lambda$ such that $(*)$ holds. Then for $\mu < \lambda$, if we already ~~know~~ knew $\dim \mathrm{Hom}(V_\nu, U_\lambda) = m_{\nu\lambda} \ \forall \nu < \mu$

Then $d_{\mu\lambda} = \dim \mathrm{Hom}\left(\sum_{\nu \leq \mu} m_{\nu\mu} V_\mu, U_\lambda\right)$

$$= \sum_{\nu < m} m_{\nu\mu} m_{\nu\lambda} + m_{\mu\lambda}$$

allowing us to ~~calculate~~ show that

$$\dim \mathrm{Hom}(V_\mu, U_\lambda) = m_{\mu\lambda} .$$

If we have done this for all $\mu < \lambda$, we know

$$U_\lambda = \sum_{\mu < \lambda} m_{\mu\lambda} V_\mu \oplus \sum m_i V_i$$

But $d_{\lambda\lambda} = \sum_{\mu < \lambda} m_{\mu\lambda}^2 + 1 \Rightarrow \sum m_i^2 = 1$

So $\exists!$ irrep. $V_\lambda \ni U_\lambda = V_\lambda \oplus \bigoplus_{\mu < \lambda} m_{\mu\lambda} V_\mu .$

From now on, we will write partitions in non-increasing order $\lambda = (\lambda_1 \geq \dots \geq \lambda_l)$, $\mu = (\mu_1 \geq \dots \geq \mu_m)$

__Defn:__ A semistandard tableau of shape μ and type λ is a left-justified array of natural numbers $\{1, \dots, l\}$ with m rows and μ_i numbers in each row where j occurs λ_j times.

__Example__ $n = 4$

__Type__ (4): $\begin{smallmatrix}1&1&1&1\end{smallmatrix}$

__Type__ $(3+1)$: $\begin{smallmatrix}1&1&1&2\end{smallmatrix}$, $\begin{smallmatrix}1&1&1\\2&&\end{smallmatrix}$

__Type__ $(2+2)$: $\begin{smallmatrix}1&1&2&2\end{smallmatrix}$, $\begin{smallmatrix}1&1&2\\2&&\end{smallmatrix}$, $\begin{smallmatrix}1&1\\2&2\end{smallmatrix}$.

__Type__ $(2+1+1)$: $\begin{smallmatrix}1&1&2&3\end{smallmatrix}$, $\begin{smallmatrix}1&1&2\\3&&\end{smallmatrix}$, $\begin{smallmatrix}1&1&3\\2&&\end{smallmatrix}$, $\begin{smallmatrix}1&1\\2&3\end{smallmatrix}$, $\begin{smallmatrix}1&1\\2&\\3&\end{smallmatrix}$

__Type__ $(1+1+1+1)$: $\begin{smallmatrix}1&2&3&4\end{smallmatrix}$, $\begin{smallmatrix}1&2&3\\4&&\end{smallmatrix}$, $\begin{smallmatrix}1&2&4\\3&&\end{smallmatrix}$, $\begin{smallmatrix}1&3&4\\2&&\end{smallmatrix}$, $\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$, $\begin{smallmatrix}1&3\\2&4\end{smallmatrix}$, $\begin{smallmatrix}1&2\\3&\\4&\end{smallmatrix}$, $\begin{smallmatrix}1&3\\2&\\4&\end{smallmatrix}$,

$\begin{smallmatrix}1&4\\2&\\3&\end{smallmatrix}$, $\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}$

__Theorem:__ (Robinson-~~Schensted~~-Knuth): There is a bijective correspondence between ~~mxn~~ $l \times m$ non-negative integer matrices A and pairs (P, Q) of semistand Young tableaux of the same shape, where i occurs exactly $a_{i1} + \dots + a_{im}$ times in Q and j occurs exactly $a_{1j} + \dots + a_{lj}$ times in P.

__Lemma:__ Define $\mu \leq \lambda$ if there exists a semistandard Young tableaux of shape μ and type λ. Then "$\leq$" is a partial order on partitions of n.

Let $m_{\nu\lambda}$ be the no. of semistandard Young tableau of shape ν & type λ.

Example: If $\nu = \lambda$, then $m_{\nu\lambda} = 1$.

Knuth's theorem $\Rightarrow$ $$d_{\mu\lambda} = \sum_{\substack{\nu \leq \mu \\ \nu \leq \lambda}} m_{\nu\mu} \times m_{\nu\lambda}$$

Corollary: The irreps. of S_n are parametrized by partitions of n: $\forall \lambda \vdash n$ $\exists$ irrep. V_λ such that
$$\mathbb{C}[X_\lambda] = \bigoplus_{\mu \leq \lambda} m_{\mu\lambda} V_\mu$$
where $m_{\mu\lambda}$ = no. of s.s. Young tableaux of shape μ & type λ.

$\dim V_{\cancel{\mu}\lambda}$ = no. of semistandard Young tableaux of shape λ & $\underbrace{\text{type } (1,\dots,1) = (1^n)}_{\text{Standard tableaux}}$.

Proof of Lemma: Suppose $\mu \leq \lambda$, i.e., $\exists$ semistandard Young tableau of shape μ & type λ.
Then Since the columns are strictly increasing, no. of columns in μ can not exceed l.

- $m \leq l$.

All the 1's go into the first row

- $\mu_1 \geq \lambda_1$

All the 1's & 2's go into the first two rows

- $\cancel{\mu_2 \geq \lambda_2}$ $\mu_1 + \mu_2 \geq \lambda_1 + \lambda_2$

$\vdots$

- $\cancel{\mu_m}$ $\mu_1 + \dots + \mu_m \geq \lambda_1 + \dots + \lambda_m$.

Conversely, if these conditions hold, one can find a semistandard Young tableau ~~o~~ by putting in

λ_1 1's, λ_2 2's, ..., λ_ℓ ℓ's in order going from left to right and top to bottom.

Both transitivity & antisymmetry are clear from these conditions.

Knuth's theorem was proved by giving an explicit algorithm.

Knuth defined a function, which given a semistandard Young tableau and a ~~non neg~~ positive integer x "inserts x into the tableaux"; the new tableau has shape which is obtained by adding a box to the old shape, at position (s,t).

Example: Insert 3 into

$$\begin{matrix} 1 & 3 & 3 & 5 & 8 \\ 2 & 4 & 6 & 6 & \\ 3 & 5 & 8 & & \\ 4 & & & & \end{matrix}.$$

to get

$$\begin{matrix} 1 & 3 & 3 & 3 & 8 \\ 2 & 4 & 5 & 6 & \\ 3 & 5 & 6 & & \\ 4 & 8 & & & \end{matrix} \qquad (s,t) = (4,2).$$

He also defined an inverse fn., given a semistandard Young tableau and the address of a box, it gives a Young tableau of shape obtained by removing that box.

(44)

Given an integer matrix A, construct a "generalized permutation" as follows:

Arrange ~~rows & cols~~ entries in lexicographic order

$(1,1)$, $(1,2)$, $(1,3)$, …, $(1,m)$, $(2,1)$, …, $(2,m)$, $(3,1)$, …

~~Given~~ Start with A, $\underset{X}{(\ )}$

Find the first (i,j) where $a_{ij} > 0$.

$$A \longleftarrow A - E_{ij}$$

$$X \longleftarrow \left(X : \begin{smallmatrix} i \\ j \end{smallmatrix}\right)$$ until $A = 0$. Output X.

Example:
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & \cdots \\ 3 & 6 & 6 & 1 & 2 & 3 & 4 & 6 & \cdots \end{pmatrix}$$
$$\begin{pmatrix} 4 & 4 & 5 & 5 & 5 & 5 & 6 \\ 3 & 5 & 1 & 1 & 2 & 4 & 7 \end{pmatrix}$$

(If A were a permutation matrix the permutation corresponding to A could be obtained).

Given an array

$$\begin{pmatrix} u_1 & u_2 & u_3 & \dots \\ v_1 & v_2 & v_3 & \dots \end{pmatrix}$$

P and Q are constructed as follows:

Sequentially: insert v_i into P, get (s_i, t_i) and insert u_i into the (s_i, t_i)'th position of Q.