Flag representations

For each partition λ of n $\quad (\lambda \vdash n)$

$$\lambda = (\lambda_1 \leq \dots \leq \lambda_\ell)$$

$$\lambda_1 + \dots + \lambda_\ell = n$$

Let $X_\lambda = \begin{cases} \{x_1 \subset \dots \subset x_\ell \subset \underline{n} \mid |x_i| = \lambda_1 + \dots + \lambda_i\} \\ \{x_1 \subset \dots \subset x_\ell \subset \mathbb{F}_q^n \mid \dim x_i = \lambda_1 + \dots + \lambda_i\} \end{cases}$

$$G = \begin{cases} S_n \\ GL_n(\mathbb{F}_q) \end{cases}$$

$G \circlearrowright \mathbb{C}[X_\lambda]$. $\quad (X_k = X_{(k, n-k)})$

<u>Question</u>: Given $\lambda, \mu \vdash n$, what does

$$G \backslash X_\lambda \times X_\mu \text{ look like?}$$

$x = (x_1 \subset \dots \subset x_\ell) \in X_\lambda$

$y = (y_1 \subset \dots \subset y_m) \in X_\mu \quad (\mu = (\mu_1 \leq \dots \leq \mu_m))$

Then look at

$$d_{ij} = \cancel{\dim\left(\frac{x_i \cap y_j}{\quad}\right)} \dim \left(\frac{x_i \cap y_j}{x_{i-1} \cap y_j + x_i \cap y_{j-1}} \right)$$

<u>Claim</u>: d_{ij} is an $\ell \times m$ matrix, whose rows and columns are non-negative integers which add up to λ & μ respectively

<u>Pf</u>: $\dim x_i / x_{i-1} = \lambda_i$

This subspace is filtered by y:

$$\frac{y_1 \cap x_i}{y_1 \cap x_{i-1}} \subset \frac{y_2 \cap x_i}{y_2 \cap x_{i-1}} \subset \dots \subset \frac{y_m \cap x_i}{y_m \cap x_{i-1}}$$

these dimensions increase from 0 to λ_i

~~So if~~ The dimensions of the successive quotients

~~y~~ $$\frac{x_i \cap y_j}{x_{i-1} \cap y_j} \Big/ \frac{x_i \cap y_{j-1}}{x_{i-1} \cap y_{j-1}} \cong \frac{x_i \cap y_j}{x_{i-1} \cap y_j + x_i \cap y_{j-1}}.$$

add up to λ_i: $\sum_j d_{ij} = \lambda_i$

By ~~the symmetric~~ interchanging the roles of x & y in this argument,

$$\sum_i d_{ij} = \mu_j.$$

Examples: $\lambda = (1, \dots, 1)$. (n times)

$=: (1^n)$

d is a permutation matrix.

Theorem: If $d(x, y) = d(x', y')$ then $\exists$ $g \in G$ such that ~~x', g~~ $x' = gx$ and $y' = gy$.

Proof: In the case of sets — ~~lets start with an example~~.

Take any fn. $x_1 \cap y_1 \longrightarrow x_1' \cap y_1'$

Extend to

$$\begin{tikzcd}[column sep=small]
 & x_2 \cap y_2 & & x_1 \cap y_3 & & & x_2' \cap y_2' & & x_1' \cap y_3' \\
x_2 \cap y_1 \arrow[ur, dash] & & x_1 \cap y_2 \arrow[ul, dash] \arrow[ur, dash] & & \longrightarrow & x_2' \cap y_1' \arrow[ur, dash] & & x_1' \cap y_2' \arrow[ul, dash] \arrow[ur, dash] \\
 & x_1 \cap y_1 \arrow[ul, dash] \arrow[ur, dash] & & & & & x_1' \cap y_1' \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Same can be done with subspaces!

Question: Can every such matrix appear?

Let $w \in S_n$, $w \in GL_n(\mathbb{F}_q)$ be a permutation / corresponding permutation matrix. $\therefore$ $we_i = e_{w(i)}$ for the ith coordinate vector. $\lambda = \mu = (1^n)$

Take $x_i = \text{span}\{e_1, \ldots, e_i\}$.

Qn: What is $d(x, wx)$? $y = wx$

$$x_i \cap y_j = \text{span}\{e_\alpha \mid \alpha \le i,\ w^{-1}\alpha \le j\}$$

$$x_i \cap y_{j-1} = \text{span}\{e_\alpha \mid \alpha \le i,\ w^{-1}\alpha \le j-1\}$$

$$x_{i-1} \cap y_j = \text{span}\{e_\alpha \mid \alpha \le i-1,\ w^{-1}\alpha \le j\}.$$

So $\dim \dfrac{x_i \cap y_j}{x_i \cap y_{j-1} + x_{i-1} \cap y_j}$

= no. of α such that $\alpha = i$ and $w^{-1}\alpha = j$.

$$= \begin{cases} 1 & \text{if } w(j) = i \\ 0 & \text{otherwise} \end{cases}$$

Consequence: every permutation matrix does arise as a relative position of two full flags.

$\lambda = \mu = (1^n)$

<u>What about general λ & μ?</u>

<u>Lemma:</u> Every (λ,μ)-matrix (d_{ij}) (i.e., $l \times m$ matrix whose ith row sums to λ_i & jth columns to μ_j) can be refined to a permutation matrix, i.e., the (i,j)th element can be expanded into a $\lambda_i \times \mu_j$ matrix $\forall\, i,j$ ~~in such a way that~~ which has ~~only 0 or 1~~ sum d_{ij} and the resulting block matrix is a permutation matrix.

<u>Pf:</u> Assume ~~$\lambda_l > 1$ and~~ $\lambda_i > 1$, $\lambda_{i'} = 1 \;\forall\, i' < i$

~~Write $\lambda' = (\lambda_1, \dots, \lambda_{l-1}, 1, \lambda_l - 1)$~~

$\lambda' = (\underbrace{1, 1, \dots 1}_{i\text{ times}}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_l)$

$$d = \begin{pmatrix} \\ d_{i1} & d_{i2} & \cdots & d_{im} \\ \\ d_{l1} & d_{l2} & \cdots & d_{lm} \end{pmatrix} \qquad d_{l1} + \cdots + d_{lm} = \lambda_l$$

Take the first non-zero entry d_{li} and replace it by

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & d_{li}-1 & d_{l\,i+1} & \cdots & d_{lm} \end{bmatrix}$$

This is a refinement.

If you keep doing this, you will end up with $\lambda = (1^n)$.

Do a similar thing with μ.

Now given an $l \times m$ matrix $(d_{ij}) \ni \sum_j d_{ij} = \lambda_i$ & $\sum_i d_{ij} = \mu_j$ and d_{ij} are non-negative integers, refine it to a permutation matrix w.

Take $x = (x_1 \subset \dots \subset x_l)$ where

$$x_i = \text{span}\{e_1, \dots, e_{\lambda_1 + \dots + \lambda_i}\}$$

and $y = (y_1 \subset \dots \subset y_m)$ where

$$y_j = \text{span}\{e_{w^{-1}(1)}, \dots, e_{w^{-1}(\mu_1 + \dots + \mu_m)}\}.$$

Then

$$x_i \cap y_j = \text{span}\{e_\alpha \mid \alpha \le \lambda_1 + \dots + \lambda_i, \ w\alpha \le \mu_1 + \dots + \mu_j\}$$

$$x_{i-1} \cap y_j = \text{span}\{e_\alpha \mid \alpha \le \lambda_1 + \dots + \lambda_{i-1}, \ w\alpha \le \mu_1 + \dots + \mu_j\}$$

$$x_i \cap y_{j-1} = \text{span}\{e_\alpha \mid \alpha \le \lambda_1 + \dots + \lambda_i, \ w\alpha \le \mu_1 + \dots + \mu_{j-1}\}.$$

So $\dim \dfrac{x_i \cap y_j}{x_{i-1} \cap y_j + x_i \cap y_{j-1}} = \#\left\{\alpha \,\middle|\, \begin{matrix} \lambda_1 + \dots + \lambda_{i-1} < \alpha \le \lambda_1 + \dots + \lambda_i \\ \mu_1 + \dots + \mu_{j-1} < w\alpha \le \mu_1 + \dots + \mu_j \end{matrix}\right\}$

= sum of entries in the "$\lambda_i \times \mu_j$-block" of w, which, since w is a refinement of (d_{ij}) is exactly d_{ij}. We have proved:

Theorem: $(x, y) \longmapsto d(x,y)$ defines a bijection

$$G \backslash X_\lambda \times X_\mu \longrightarrow \underbrace{\left\{\begin{matrix}\text{non-neg. integer } l \times m \text{ matrices} \\ (d_{ij}) \text{ such that } \sum_j d_{ij} = \lambda_i, \ \sum_i d_{ij} = \mu_j\end{matrix}\right\}}_{\text{the set of } (\lambda, \mu)\text{-matrices}}.$$

$n = 3$:

~~dim Hom CG[X~~

$$|G \backslash X_\lambda \times X_\mu|$$

μ \ λ	3	1+2	1+1+1
3	1	1	1
1+2	1	2	3
1+1+1	1	3	6

$$\begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \\ \begin{matrix} 1 & \;2 \end{matrix} & \end{array} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbb{C}[X_3] = V_3$$

$$\mathbb{C}[X_{1+2}] = V_3 \oplus V_{1+2}$$

$$\mathbb{C}[X_{1+1+1}] = V_3 \oplus 2V_{1+2} \oplus V_{1+1+1}$$

$n=4$

	4	1+3	2+2	1+1+2	1+1+1+1
4	1	1	1	1	1
1+3	1	2	2	3	4
2+2	1	2	3	4	6
1+1+2	1	3	4	7	12
1+1+1+1	1	4	6	12	24

$(1+1+2,\ 1+3)$

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 1 \\ 1 \\ 2 \end{matrix} \\ \begin{matrix} 1 & \ \ 3 \end{matrix} & \end{array} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$(1+1+2,\ 2+2)$

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 1 \\ 1 \\ 2 \end{matrix} \\ \begin{matrix} 2 & \ \ 2 \end{matrix} & \end{array} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ & \\ & \end{pmatrix}$$

similarly, 2 more

$(1+1+2,\ 1+1+2)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ & & \\ & & \end{pmatrix}. \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

similarly 2 more

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$n=4$ could.

$(1+3,\ 1+1+1+1)$

$$\begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \\ 1 \quad 3 \end{array}$$ four of these

$(2+2,\ 1+1+1+1) \rightarrow \binom{4}{2}$

$(1+1+2,\ 1+1+1+1)$

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \\ 1 \quad 1 \quad 2 & \end{array} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ & & \\ & & \\ & & \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ & & \\ & & \\ & & \end{pmatrix}$$ similar to above.

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ & & \\ & & \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ & & \\ & & \end{pmatrix}$$

two more like

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$n=4$ contd.

$$\mathbb{C}[X_4] = V_0$$

$$\mathbb{C}[X_{1+3}] = V_0 \oplus V_{1+3}$$

$$\mathbb{C}[X_{2+2}] = V_0 \oplus V_{1+3} \oplus V_{2+2}$$

$$\mathbb{C}[X_{1+1+2}] = V_0 \oplus 2V_{1+3} \oplus V_{2+2} \oplus V_{1+1+2}$$

$$\mathbb{C}[X_{1+1+1+1}] = V_0 \oplus 3V_{1+3} \oplus 2V_{2+2} \oplus 3V_{1+1+2} \oplus V_{1+1+1+1}$$

So what we observe is that there is an ordering on λ's

- $\forall\ \lambda \vdash n$, $\exists!$ rep. V_λ such that

$$\mathbb{C}[X_\lambda] = V_\lambda \oplus \Big(\bigoplus_{\mu < \lambda} V_\mu\Big)$$

This allows us to define all the representations V_λ inductively in terms of the flag representations.

Question: to what extent can this be carried out for general n?