Suppose X & Y are G-sets:

$$\mathrm{Hom}_G(\mathbb{C}[Y], \mathbb{C}[X]) = \{T_\varphi \mid \varphi : X \times Y \to \mathbb{C}\},$$

where $T_\varphi f(x) = \sum \varphi(x,y) f(y)$.

$T_\varphi \in \mathrm{Hom}_G(\mathbb{C}[Y], \mathbb{C}[X])$ iff

$$T_\varphi \circ \rho_Y(g) = \rho_X(g) \circ T_\varphi$$

$$(T_\varphi \circ \rho_Y(g) f)(x) = \sum \varphi(x,y) f(g^{-1} \cdot y)$$

$$= \sum \varphi(x, gy) f(y).$$

$$(\rho_X(g) \circ T_\varphi f)(x) = T_\varphi f(g^{-1}x)$$

$$= \sum \varphi(g^{-1}x, y) f(y)$$

$\therefore$ $T_\varphi \in \mathrm{Hom}_G(\mathbb{C}[Y], \mathbb{C}[X])$ if and only if

$$\varphi(x, gy) = \varphi(g^{-1}x, y) \quad \forall (x,y) \in X \times Y,\ g \in G$$

or $\varphi(gx, gy) = \varphi(x,y)$.

~~Special case~~

<u>Theorem</u>: $\mathrm{Hom}_G(\mathbb{C}[Y], \mathbb{C}[X]) = \{T_\varphi \mid \varphi : X \times Y \to \mathbb{C},\ \varphi(gx, gy) = \varphi(x,y)\}$.

<u>Example</u>: $X = X_k = \begin{cases} \{x \subset \underline{n} \mid |x| = k\} & [\lambda(x) := |x|] \\ \{x \in \mathbb{F}_q^n \mid \dim x = k\} & [\lambda(x) := \dim x] \end{cases}$

$Y = X_l$. If $k + l \le n$,

$$\dim \mathrm{Hom}_G(\mathbb{C}[X_l], \mathbb{C}[X_k]) = \min\{k, l\} + 1.$$

Decomposition of $\mathbb{C}[X_k]$

We consider two classes of examples simultaneously:

1) $G = S_n \quad X_k = \{ x \subset \underline{n} \mid |x| = k \} \quad r(x) := |x|$

2) $G = GL_n(\mathbb{F}_q) \quad X_k = \{ x \subset \mathbb{F}_q^n \mid \dim x = k \} \quad r(x) := \dim x$

We know:

<u>Lemma:</u> If $k + l \leq n$, then

$$(x, y) \longmapsto r(x \cap y)$$

defines a bijection $G \backslash X_k \times X_k \longrightarrow \{0, \dots, \min\{k, l\}\}$

~~Lemma~~ By Gelfand's trick, $\mathrm{End}_G \mathbb{C}[X_k]$ is commutative, and so for $2k \leq n$, $\mathbb{C}[X_k]$ is a sum of $k+1$ irreducible representations, each occuring with multiplicity one.

Since $X_0 = \{(0)\}$, $V_0 := \mathbb{C}[X_0]$ is the trivial rep. of G.

$\mathbb{C}[X_1]$ is a sum of two simples. (if $n \geq 2$)

But $\dim \mathrm{Hom}_G(\mathbb{C}[X_0], \mathbb{C}[X_1]) = 1$

So V_0 occurs in $\mathbb{C}[X_1]$. new representation!

We can write $\mathbb{C}[X_1] = V_0 \oplus V_1$

$\mathbb{C}[X_2]$ is a sum of three simples (if $n \geq 4$)

& $\dim \mathrm{Hom}_G(\mathbb{C}[X_1], \mathbb{C}[X_2]) = 2$

So V_0 & V_1 occur in $\mathbb{C}[X_2]$.

given rep.

We can write $\mathbb{C}[X_2] = V_0 \oplus V_1 \oplus V_2$

Continuing in this manner, we have:

<u>Theorem</u>: For $0 \le k \le \lfloor \frac{n}{2} \rfloor$ there exists irreducible representations V_0, V_k such that

$$\mathbb{C}[X_k] = \bigoplus_{0 \le i \le k} V_i$$

What about $k > \lfloor \frac{n}{2} \rfloor$?

<u>Lemma</u>: For each $0 \le k \le n$, there is an isomorphism of G-spaces $X_k \longrightarrow X_{n-k}$. (i.e., a bijective fn. φ

$\varphi : X_k \longrightarrow X_{n-k} \ni \varphi(g \cdot x) = g \cdot \varphi(x)$).

$\boxed{\therefore \ \mathbb{C}[X_k] \cong \mathbb{C}[X_{n-k}] \text{ as a representation of } G}$

<u>Note</u>: It is well-known that $|X_k|$ is a polynomial in q with positive coefficients.

It seems that $\dim V_k = |X_k| - |X_{k-1}|$ (for $k < \frac{n}{2}$)

Should also be a polynomial in q, and with non-negative coefficients!

Is there a representation theoretic interpretation?

More generally, lets calculate

$\mathrm{Hom}_G(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_K^G \sigma)$ $\quad (\rho, V), (\sigma, W)$

Theorem: Let

$$\Delta = \left\{ \varphi : G \times G \to \mathrm{Hom}(V,W) \;\middle|\; \begin{array}{ll} \varphi(kx, hy) = \sigma(k) \circ \varphi(x,y) \circ \rho(h^{-1}) & k \in K,\ x,y \in G,\ h \in H \\ \varphi(xg, yg) = \varphi(x,y) & x,y,g \in G \end{array} \right\}$$

$\forall\ \varphi \in \Delta$ define

$$T_\varphi f(x) = \sum_{y \in G} \varphi(x,y)(f(y)) \quad \forall\ f \in \mathrm{Ind}_H^G \rho.$$

Then $\mathrm{Hom}_G(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_K^G \sigma) = \{ T_\varphi \mid \varphi \in \Delta \}$.

Proof: • $T_\varphi f(kx) = \sum \varphi(kx, y)(f(y))$

$$= \sum \sigma(k)\, \varphi(x,y)(f(y))$$

$$= \sigma(k)\, T_\varphi f(x).$$

$\therefore\ T_\varphi \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_K^G \sigma)$.

• $$(T_\varphi \circ \rho^G(g) \circ f)(x) = \sum_y \varphi(x,y)\, f(yg)$$

$$= \sum \varphi(x, yg^{-1})\, f(y)$$

$$= \sum \varphi(xg, y)\, f(y)$$

$$= \cancel{\rho}\ \sigma\ T_\varphi f(xg) = \cancel{\rho}(\sigma^G(g) \circ T_\varphi \circ f)(x).$$

$\therefore\ T_\varphi \in \mathrm{Hom}_G(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_K^G \sigma)$.

To complete the proof, we shall recover φ from T_φ.

Given $g \in G$, $v \in V$, define $f_{g,v} \in \mathrm{Ind}_H^G \rho$ by

~~to be the unique fn~~ by

$$f_{g,v}(x) = \begin{cases} \rho(h)v & \text{if } x = hg \text{ for some } h \in H \\ 0 & \text{otherwise.} \end{cases}$$

Then
$$T_\varphi f_{g,v}(1) = \sum_{y \in G} \varphi(1, y) f_{g,v}(y)$$

$$= \sum_{h \in H} \varphi(1, hg)\rho(h)v$$

$$= \sum_{h \in H} \varphi(1, g)\rho(h)^{-1}\rho(h)v$$

$$= |H| \varphi(1, g)(v)$$

Accordingly, we should define, for $T \in \mathrm{Hom}_G(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_K^G \sigma)$

$$\varphi_T(x, y)(v) = \varphi(1, yx^{-1})(v)$$

$$= \frac{1}{|H|} T f_{yx^{-1}, v}(1).$$

We have already shown that

$$T_{\varphi_T} f_{g,v}(1) = T f_{g,v}(1)$$

$\forall\ g \in G$, $v \in V$.

~~But the~~ $f_{g,v}$'s ~~generate~~ span ~~$\mathrm{Ind}_H^G \rho$ (as a rep.)~~

~~So it remains~~

Using this, one can conclude $T_{\varphi_T} = T$.

The double coset trick

$$G/\cancel{K} \times (x, y) \longmapsto \cancel{}\, xy^{-1}$$

$$(K\backslash G \times H\backslash G)/_G \cong K\backslash G/H$$

More generally

$$\Lambda \cong \{\tilde{\varphi} : G \to \mathrm{Hom}(V,W) \mid \tilde{\varphi}(kxh) = \sigma(k)\circ\varphi(x)\circ\rho(h)\}$$

via $\tilde{\varphi}(x) = \varphi(x, 1)$

$$(\tilde{\varphi}(kxh) = \varphi(kxh, 1) = \varphi(kx, h^{-1}) = \dots \quad)$$

~~Eg~~ In the example of $\mathbb{C}[X_k]$, $X_k = \{x \subset \underline{n} : |x| = k\}$,

$$X_k = \cancel{S_n / S_k}\, S_k \backslash S_n$$

$$g^{-1}\cdot\{1, \dots, k\} \longleftarrow g$$

so $\mathbb{C}[X_k] = \mathrm{Ind}_{S_k}^{S_n} \underline{1}$

and our analysis of relative positions

means that $|S_k \backslash S_n / S_k| = k+1$ for $k \le \frac{n}{2}$,

which is not so intuitive.