Induced characters

$B < G$, θ - a true character of B

$\exists!$ class fn. θ^G of G such that

$$\langle \chi, \theta^G \rangle_G = \langle \chi_B, \theta \rangle_B$$

where χ_B = restriction of χ to B.

Clearly, θ^G is a true character.

$\longrightarrow$ It is called <u>the character of G induced from θ</u>.

<u>Example 0</u>: $B = \{e\}$, $\theta = 1$ (triv. char. of B)

$$\langle \chi, 1^G \rangle = \langle \chi_B, 1 \rangle$$
$$= \dim \chi.$$

$\therefore$ every irreducible occurs in 1^G as many times as its dimension. $\therefore$ 1^G is the char. of ${}_G\mathbb{C}[G]$.

<u>Example 1</u>: B arbitrary subgp, $\theta = 1$ (triv. char. of B)

$$\langle \chi, 1^G \rangle_B = \langle \chi_B, 1 \rangle_G$$
$$= \dim V_\chi^B$$

$V_\chi^B = \{ v \in V_\chi \mid \rho_\chi(b) v = v \quad \forall\, b \in B \}$.

Take $V = \mathbb{C}[B \backslash G]$ $\quad (\rho(g) f)(x) = f(xg)$.

~~What is~~ $\mathrm{Hom}_G(V_\chi, \mathbb{C}[B\backslash G]) \overset{?}{=} \mathrm{Hom}_B(V_\chi, 1)$

Given $\varphi : V_\chi \to \mathbb{C}[B \backslash G]$ G-hom, consider

$\lambda_\varphi : V_\chi \to \mathbb{C}$ defined by

$$\lambda_\varphi(v) = \varphi(v)(1)$$

Then $\lambda_\varphi(\rho_\chi(b) v) = \varphi(\rho_\chi(b) v)(1)$

$$= [\rho_\chi(b)\, \varphi(v)](1)$$
$$= \varphi(v)(b)$$
$$= \varphi(v)(1)$$

So $\lambda_\varphi \in \mathrm{Hom}_B(V_\chi, 1)$.

Conversely given $\lambda : \mathrm{Hom}_B(V_\lambda, 1)$, define $\varphi_\lambda : V_\lambda \to \mathbb{C}[G]$ by

$$\varphi_\lambda(v)(g) = \cancel{\lambda(v)}.\lambda(\rho_\lambda(g)v)$$

Then $\varphi_\lambda(v)(bg) = \lambda(\rho_\lambda(bg)v)$

$$= \lambda(\rho_\lambda(b)\rho_\lambda(g)v)$$

$$= \lambda(\rho_\lambda(g)v)$$

$$= \varphi_\lambda(v)(g).$$

$\therefore \varphi_\lambda(v) \in \mathbb{C}[B\backslash G] \ \forall \ v \in V_\lambda$.

$$\varphi_\lambda(\rho_\lambda(x)v)(g) = \lambda(\rho_\lambda(g)\rho_\lambda(x)v)$$

$$= \lambda(\rho_\lambda(gx)v)$$

$$= \varphi_\lambda(v)(gx)$$

$$= [\rho(x)\varphi_\lambda(v)](g).$$

$\therefore \varphi_\lambda \in \mathrm{Hom}_G(V_\lambda, \mathbb{C}[B\backslash G])$.

In general, the same trick holds:

Define $\mathbb{C}(B\backslash G; \rho_\theta) = \{f : G \to V_\theta \mid f(bg) = \rho_\theta(b)f(g) \ \forall b \in B, g \in G\}$

and for $f \in \mathbb{C}(B\backslash G; \rho_\theta)$ define

$$(\rho(g)f)(x) = f(xg).$$

Then $\mathrm{Hom}_G(V_\lambda, \mathbb{C}(B\backslash G; \rho_\theta)) = \mathrm{Hom}_B(V_\lambda, V_\theta)$

$\therefore \theta^G$ is the character of $\mathbb{C}(B\backslash G; \rho_\theta)$.

~~Example 2.3~~ <u>Understanding $\mathbb{C}[B\backslash G]$</u>:

~~Let X be a G set (set with G action $G \curvearrowright X$)~~

Let X be any finite set.

Then $\mathbb{C}[X] = \{f : X \to \mathbb{C}\}$ is a f.d.v.s / $\mathbb{C}$.

Given $k : X \times X \to \mathbb{C}$ define

$$T_k : \mathbb{C}[X] \to \mathbb{C}[X] \text{ by}$$

$$(T_k f)(x) = \sum_{y \in X} k(x,y)\, f(y).$$

Clearly, T_k is a linear map $\mathbb{C}[X] \to \mathbb{C}[X]$.

$$T_{k_1} \circ T_{k_2} f \;(x) = \sum_y k_1(x,y)\, T_{k_2} f(y)$$

$$= \sum_y \sum_z k_1(x,y) k_2(y,z) f(z)$$

$$= T_k f \;(x)$$

when $\displaystyle k(x,z) = \sum_y k_1(x,y) k_2(y,z) \stackrel{(*)}{=} k_1 * k_2(x,z)$

Theorem: ~~$k \mapsto T_k$ is an isomorphism~~ $\mathbb{C}[X \times X]$ becomes an algebra under $(*)$, and $k \mapsto T_k$ is an isomorphism of this algebra onto $\mathrm{End}_{\mathbb{C}} \mathbb{C}[X]$.

Pf: Enough to check injectivity.

k is called an [integral] kernel.

T_k : integral operator on $\mathbb{C}[X]$ with kernel k.

Now suppose X is a G-set (i.e., $\exists$ action $G \times X \to X$)

Then $\mathbb{C}[X]$ becomes a representation of G.

$$\rho(g) f(x) = f(g^{-1} \cdot x) \quad \text{— permutation rep}$$

Question: for which $k \in \mathbb{C}[X \times X]$

is $T_k \in \mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[X])$?

Ans: Need $\rho(g) T_k f = T_k \rho(g) f \quad \forall f$

$$\rho(g) T_k f(x) = T_k f(g^{-1}x)$$
$$= \sum_{y \in X} k(g^{-1}x, y) f(y).$$

$$T_k \rho(g) f(x) = \sum_{y \in X} k(x,y) T_k f(y)$$
$$= \sum_{y \in X} k(x,y) f(g^{-1}y)$$
$$= \sum_{y \in X} k(x, gy) f(y)$$

$\therefore$ must have $k(g^{-1}x, y) = k(x, gy)$ $\forall$ $x, y \in X$, $g \in G$.

In other words: $\boxed{k(x,y) = k(gx, gy)}$ $\forall$ $x, y \in X, g \in G$.

k is constant ~~under~~ on the orbits of the action of G on $X \times X$ given by
$$g \cdot (x, y) = (gx, gy).$$

$$\dim \mathrm{End}_G \mathbb{C}[X] = |G \backslash (X \times X)|$$

Example: $X = \{1, \dots, n\}$.

$G = S_n$.

~~Since S_n acts doubly transitively on~~
there a precisely two orbits of G on $X \times X$.

Note: $\mathbb{C}[X]$ always contains the trivial rep of G, given by constant functions.

Example: If G acts doubly transitively on X, then G has two simples in $\mathbb{C}[X]$, triv, and one of dim $|X|-1$.

<u>Defn</u>: The <u>relative position</u> of $(x, y) \in X \times X$ is its image in $G \backslash (X \times X)$.

<u>Example</u>: $X = G$, G acts by left translation.

$$G \backslash (X \times X) \xrightarrow{\sim} G$$

$$(x, y) \longmapsto y^{-1}x$$

[$y^{-1}x$ is a <u>complete</u> <u>invariant</u> of relative position in $G \backslash (G \times G)$.

(1) A fn. $f : X \times X \longrightarrow S$ is called an invariant of rel. pos. if $\exists\, \bar{f} \ni$

$$\begin{tikzcd} X \times X \arrow[r, "f"] \arrow[d] & S \\ G \backslash (X \times X) \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

(2) f is called a complete invariant of rel pos. if $\bar{f}$ is a bijection.

<u>Example</u>: V : fdvs. / k, $\dim V = n$, $0 < r < n$.

Let $X = Gr_r(V) = \{ W \subset V \mid \dim W = r \}$.

$GL(V)$ acts on X.

~~Lemma: If $\dim W_1 \cap W_2 = \dim W_1$~~

<u>Claim</u>: ~~$\dim (W_1 \cap W_2$~~ $(W_1, W_2) \mapsto \dim (W_1 \cap W_2)$ is a complete invariant of relative position in $X \times X$.

(20)

<u>Gelfand's trick</u>: If $(x,y) \mapsto (y,x)$ induces $X \times X \to X \times X$ the identity map on $G\backslash(X\times X)$ then $\mathbb{C}[X]$ has a multiplicity-free decomposition as a representation of G.

<u>Pf</u>: <u>Claim</u>: $\mathrm{End}_G \mathbb{C}[X]$ is commutative.

Why? Let $k^*(x,y) = k(y,x)$.

Then $k_1^* * k_2^*(x,y) = \sum_{z\in X} k_1^*(x,z)\, k_2^*(z,y)$

$$= \sum_{z\in X} k_1(z,x)\, k_2(y,z)$$

$$= \sum_{z\in X} k_2(y,z)\, k_1(z,x)$$

$$= (k_2 * k_1)(y,x)$$

$$= (k_2 * k_1)^*(x,y).$$

$\therefore$ $k \mapsto k^*$ is an iso. $\mathrm{End}_G \mathbb{C}[X] \to \mathrm{End}_G \mathbb{C}[X]^{\mathrm{opp}}$

But if hypothesis holds, then it is the identity map.

<u>Corollary</u>: Rep of $GL(V)$ on $\mathbb{C}[Gr_k(V)]$ is multiplicity free.