Standard subgps:

$G = GL_n(\mathbb{F}_q)$,

~~Thm~~ $\lambda = (\lambda_1, \dots, \lambda_\ell)$ a composition $\sum \lambda_i = n$

Let $\quad U_i = \operatorname{span}\{e_{\lambda_1 + \dots + \lambda_{i-1}+1}, \dots, e_{\lambda_1 + \dots + \lambda_i}\}$

$\quad\quad x_i = U_1 + \dots + U_i$

$P_\lambda := \{ g \in G \mid g(x_i) = x_i \ \forall i \}$ - <u>standard parabolic</u>

$M_\lambda := \{ g \in G \mid g(U_i) = U_i \ \forall i \}$ - <u>standard Levi</u>

$N_\lambda := \{ g \in P_\lambda \mid g \text{ induces } \frac{x_i}{x_{i-1}} \xrightarrow{id} \frac{x_i}{x_{i-1}} \}$ - <u>std unipotent</u>

$$P_\lambda = \begin{pmatrix} GL_{\lambda_1} & * & \cdots & * \\ & GL_{\lambda_2} & & \vdots \\ & & \ddots & * \\ & & & GL_{\lambda_\ell} \end{pmatrix}$$

"block upper triangular"

$$M_\lambda = \begin{pmatrix} GL_{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & GL_{\lambda_\ell} \end{pmatrix}$$ "block diagonal".

$\cong \prod_{i=1}^{\ell} GL_{\lambda_i}(\mathbb{F}_q)$.

$$N_\lambda = \begin{pmatrix} I_{\lambda_1} & & * \\ & \ddots & \\ 0 & & I_{\lambda_\ell} \end{pmatrix}$$ = maximal normal p-subgp of P_λ

=: unipotent radical of P_λ.

$M_\lambda \cong P_\lambda / N_\lambda$

Extreme example: $\lambda = (n)$.

$G_\lambda = G$, $M_\lambda = G$, $N_\lambda = \{I\}$.

Defn: An irreducible rep. (ρ, V) of G is said to be *cuspidal* if $V^{N_\lambda} = 0$ $\forall$ non-trivial composition λ
(the partition (n) of n is deemed to be triv.)

Propn: $V^{N_\lambda} = 0 \iff \mathrm{Hom}_G(V, \mathrm{Ind}_{P_\lambda}^G W) = 0$ $\forall$ rep. (σ, W) of P_λ whose restriction to N_λ is identity.

Pf: $\mathrm{Hom}_G(V, \mathrm{Ind}_{P_\lambda}^G W) = 0$ ~~$\mathrm{Hom}_{P_\lambda}(V, W)$~~ $\forall (\sigma, W) \ni \sigma|_{N_\lambda} = 1$.
Frob. rec.
$\iff \mathrm{Hom}_{P_\lambda}(V, W) = 0$. $\forall (\sigma, W) \ni \sigma|_{N_\lambda} = 1$.

$\iff V^{N_\lambda} = 0$.

Corollary: (ρ, V) is cuspidal iff $\mathrm{Hom}_G(V, \mathrm{Ind}_{P_\lambda}^G W) = 0$ $\forall$ rep. (σ, W) of P_λ whose restriction to N_λ is identity $\forall$ non-triv. partition $\lambda \vdash n$.

Defn: A rep. (ρ, V) of M_λ is a rep. of $\prod_{i=1}^{n} GL_{\lambda_i}(\mathbb{F}_q)$
so $(\rho, V) = \bigotimes_{i=1}^{\ell} (\rho_i, V_i)$ where (ρ_i, V_i) is a rep. of $GL_{\lambda_i}(\mathbb{F}_q)$.

Defn: A rep. (ρ, V) of M_λ is said to be cuspidal if (ρ_i, V_i) is cuspidal (in the earlier sense) $\forall\ i = 1, \dots, \ell$.

Harish Chandra's philosophy of cusp forms:

In order to determine all the irreps. of $GL_n(\mathbb{F}_q)$:

1. Determine the cuspidal reps. of $GL_m(\mathbb{F}_q)$ $\forall\, m \leq n$.
2. Decompose the reps. $\mathrm{Ind}_{P_\lambda}^G \sigma$ where σ is a cuspidal rep. of M_λ.

Let us begin with 2 (which is the easier part).

Recall: $\mathrm{Hom}_G(\mathrm{Ind}_{P_\lambda}^G \sigma, \mathrm{Ind}_{P_{\lambda'}}^G \sigma')$ consist of integral operators corresponding to kernels in

$$\Delta = \{ k : G \times G \to \mathrm{Hom}(V, V') \mid k(h'x'g, hxg) = \sigma(h')k(x', x)\sigma(h)^{-1}\ \forall\, h' \in P_{\lambda'},\ h \in P_\lambda,\ x, g \in G \}.$$

Each $k \in \Delta$ is determined by its value $k(w, 1)$ where w ranges over a set of permutation matrices obtained by stretching integer matrices with row & column sums λ & λ'.

Defn: Say that w intertwines σ & σ' if $\exists$ ~~$k \neq 0$~~ $k \in \Delta \ni k(w, 1) \neq 0$.

Lemma: ~~$w \in S_n$ intertwines σ & σ' $\Rightarrow$ $wM_\lambda w^{-1}$~~

Let $\lambda, \lambda' \vdash n$, σ, σ' be cuspidal reps. of M_λ and $M_{\lambda'}$ resp. ~~If An element~~ If $w \in S_n$ intertwines σ and σ' ~~$\Rightarrow$~~ then $wM_\lambda w^{-1} = M_{\lambda'}$.

Proof: Recall $U_i = \mathrm{span}\{ e_{\lambda_1 + \dots + \lambda_{i-1}+1}, \dots, e_{\lambda_1 + \dots + \lambda_i} \}$.

$U_i' = \mathrm{span}\{ e_{\lambda_1' + \dots + \lambda_{i-1}'+1}, \dots, e_{\lambda_1' + \dots + \lambda_i'} \}$.

$(\lambda' = (\lambda_1', \dots, \lambda_{\ell'}'))$

$\exists$ $wM_\lambda w^{-1} = M_{\lambda'}$ $\iff$ the collections of subspaces $\{w(U_i)\}$ and $\{U'_j\}$ are the same.

If they are not the same then $\exists$ $i \in \{1,\dots,\ell\}$ such that $w(U_i) = \bigoplus_j U'_j \cap w(U_i)$, with at least two non-trivial summands on the RHS.

Example: ~~$n = 4$, $\lambda = 3+1$, $\lambda' = 2+2$, $w =$~~

$n=3$, $\lambda = \lambda' = 2+1$, $w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$U_1 = \text{span}\{e_1, e_2\}$ $U_2 = \text{span}\{e_3\}$.

$w(U_1) = \text{span}\{e_1, e_3\}$, $w(U_2) = \text{span}\{e_2\}$.

$U_1' = \text{span}\{e_1, e_2\}$, $U_2' = \text{span}\{e_3\}$.

$w(U_1) = (w(U_1) \cap U_1') \oplus (w(U_1) \cap U_2')$.

$N_\lambda = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ $w^{-1} N_\lambda w = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix}$

$GL(U_1) \cap w^{-1} N_\lambda w$ is a standard unipotent subgroup of $GL(U_1)$.

But we could also have taken $w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

$w(U_1) = \text{span}\{e_2, e_3\}$, $w(U_2) = \text{span}\{e_2\}$.

$w(U_1) = (w(U_1) \cap U_1') \oplus (w(U_1) \cap U_2')$

$N_\lambda = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ $w^{-1} N_\lambda w = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}$

$GL(U_1) \cap w^{-1} N_\lambda w$ is not a standard unipotent.

<u>Defn:</u> A parabolic subgp. of G is any subgroup which is conjugate to P_λ for some $\lambda \vdash n$. Equivalently it is the stabilizer subgroup of a flag of subspaces of $\mathbb{F}_q^n$.

<u>Note:</u> Since the composition of a rep. with an inner auto gives rise to an isomorphic rep., a rep. is cuspidal iff ~~it is has no non-~~ $V^N = 0$ whenever N is the unipotent radical of a proper parabolic subgroup.

Now: in U_i we have a flag of subspaces

$$\{w^{-1}(x_j')\}_j \qquad (*)$$

(these need not be distinct for distinct j, but it does not matter), what we require is that at least one of these is a proper subspace, which follows from the assumption that more than one summand in $\bigoplus_j U_j' \cap w(U_i)$ is non-trivial)

$GL(U_i) \cap w^{-1} P_\lambda w$ is a parabolic subgp. of $GL(U_i)$ (since it stabilizes the flag $*$) and its unipotent radical is $GL(U_i) \cap w^{-1} N_\lambda w$.

Take $g \in GL(U_i) \cap w^{-1} N_\lambda w =: N$

$$k(w,1) = k(wg, g) = k(wgw^{-1}w, g) = \sigma'(wgw^{-1})\, k(w,1)\, \sigma(g)^{-1}$$

$$\sigma'(wgw^{-1})\, k(w,1)\, \sigma(g)^{-1} = k(w,1)\,\sigma(g)^{-1}$$

$\therefore\ k(w,1) \in \mathrm{Hom}_N(\sigma, 1^{\oplus d}) = 0$,

which proves the lemma.

Now assume that $w M_\lambda w^{-1} = M_{\lambda'}$. Then for $g \in M_{\lambda}$,

$$k(w,1) = k(wg, g) = k(wgw^{-1}w, g) = \cancel{\sigma'(wgw^{-1})}$$

$$= \sigma'(wgw^{-1})\, k(w,1)\, \sigma(g)^{-1}.$$

$\therefore\ k(w,1)$ intertwines the rep. σ of M_λ with the rep σ'^{w} of M_λ, where $\sigma'^{w}(g) := \sigma'(wgw^{-1})$.

In particular, by Schur's lemma, $k(w,1)$ is uniquely determined up to a scalar multiple.

Conclusion: If σ, σ' are cuspidal, then

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_{P_\lambda}^G \sigma, \mathrm{Ind}_{P_{\lambda'}}^G \sigma')$$

$$= \#\left\{ w \in S_{\lambda'} \backslash S_n / S_\lambda \;\middle|\; \begin{array}{l} w M_\lambda w^{-1} = M_{\lambda'} \text{ and} \\ \sigma \cong \sigma'^{w} \end{array} \right\}$$

Theorem: Either $\mathrm{Ind}_{P_\lambda}^G \sigma$ & $\mathrm{Ind}_{P_{\lambda'}}^G \sigma'$ are isomorphic or disjoint.

Proof: We will use the following result: if V_1 and V_2 are representations such that

$$\dim \mathrm{Hom}_G(V_1, V_2) = \dim \mathrm{End}_G V_1 = \dim \mathrm{End}_G V_2,$$

then V_1 and V_2 are isomorphic.

$\therefore$, it suffices to show that, if $\exists\ w_0 \in S_n \ni$ $w_0 M_\lambda w_0^{-1} = M_{\lambda'}$ & $\sigma \cong \sigma'^{w_0}$, then the three sets:

(1) $\{ w \in S_{\lambda'} \backslash S_n / S_{\lambda} \mid w M_\lambda w^{-1} = M_{\lambda'} \ \& \ \sigma \cong \sigma'^{w} \}$

(2) $\{ w \in S_{\lambda} \backslash S_n / S_{\lambda} \mid w M_\lambda w^{-1} = M_{\lambda} \ \& \ \sigma \cong \sigma^{w} \}$

(3) $\{ w \in S_{\lambda'} \backslash S_n / S_{\lambda'} \mid w M_{\lambda'} w^{-1} = M_{\lambda'} \ \& \ \sigma' \cong \sigma'^{w} \}$

have the same cardinality.

From (1) → (2) ~~(1) → (3)~~ : $w \mapsto w_0^{-1} w$

~~(2) → (2)~~ : $w \mapsto w w_0^{-1}$

(1) → (3) do indeed give rise to such

bijections (must check !!).

Thus to each irreducible rep. ρ of G, ~~one~~ there corresponds a ~~unique~~ pair (P_λ, σ) (unique upto $(P_\lambda, \sigma) \sim (P_{\lambda'}, \sigma')$ iff $\exists\, w \in S_n \ni$ $w P_\lambda w^{-1} = P_{\lambda'}$ & $\sigma'^{w} = \sigma$) $\ni$ ρ occurs in $\mathrm{Ind}_{P_\lambda}^{G} \sigma$. [When ρ is cuspidal, $\lambda = (n), \sigma = \rho$].