Skew - tableau:

Let λ, μ be partitions with $\mu \subseteq \lambda$ (i.e., $\mu_i \le \lambda_i \ \forall i$). A semistandard tableau of skew shape λ/μ is an array $T = (T_{ij})$

$1 \le i \le \underbrace{\ell(\lambda)}_{\text{no. of parts of } \lambda}$, $\mu_i < j \le \lambda_i$ strictly

increasing with i and weakly increasing with j.

Example: $\lambda = (6,5,3,3)$ $\mu = (3,1)$

$$\begin{array}{|c|c|c|c|c|c|}\hline \times & \times & \times & 3 & 4 & 4 \\ \hline \times & 1 & 4 & 7 & 7 \\ \cline{1-5} 2 & 2 & 6 \\ \cline{1-3} 3 & 8 & 8 \\ \cline{1-3}\end{array} \qquad \begin{array}{cccccc} & & & \boxed{3} & \boxed{4} & \boxed{4} \\ & \boxed{1} & \boxed{4} & \boxed{7} & \boxed{7} & \\ \boxed{2} & \boxed{2} & \boxed{6} & & & \\ \boxed{3} & \boxed{8} & \boxed{8} & & & \end{array}$$

s.std. tableau of skew shape λ/μ & type

$(1,2;2,3,0,1,2,2)$

Recall: The Schur fn. s_λ was defined by ($\lambda \vdash n$)

$$s_\lambda = \sum_{\mu \vdash n} \underbrace{m_{\lambda\mu}}_{\text{no. of s.std. Young tab. of shape } \lambda \text{ type } \mu} m_\mu .$$

Defn: The skew-Schur fn. $s_{\lambda/\mu}$ is defined by

$$s_{\lambda/\nu} = \sum_{\mu \vdash n} m_{\lambda/\nu, \mu} \, m_{\bar{\mu}}$$

where $m_{\lambda/\nu, \mu} = \#\{\text{semi std. Young tableau of shape } \lambda/\nu \text{ \& type } \mu\}$

Recall: We had shown that $\forall\, \lambda \vdash n$

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

where $a_\alpha = \begin{vmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \cdots \\ \vdots & \vdots & \end{vmatrix}$ $\quad \alpha = (\alpha_1, \dots, \alpha_n)$

The proof was as follows: ~~to s~~ we had to show that

$$\cancel{a_{\lambda+\delta} \sum m_{\lambda\mu} m_\mu}\; a_\delta e_\mu = \sum K_{\lambda'\mu}\, a_{\lambda+\delta}$$

which we did by calculating the coeff. of $x^{\lambda+\delta}$ on both sides. A very similar argument gives:

Theorem: $\quad a_{\nu+\delta}\, e_\mu = \sum K_{\lambda'/\nu', \mu}\, a_{\lambda+\delta}$

Coeff. of $x^{\lambda+\delta}$ on RHS is $K_{\lambda'/\nu', \mu}$.

Coeff. of $x^{\lambda+\delta}$ on LHS is the no. of ~~paths~~ ways of starting with $x^{\nu+\delta}$ and multiplying by monomials $x^{\alpha_1}, x^{\alpha_2}, \dots$ (x^{α_i} is a monomial in e_{μ_i}) keeping indices strictly increasing.

The SSYT is defined by: Column j of ~~t~~ T contains an i if the x_j occurs in the monomial x^{α_i}.

(You may think of this as a part of the previous operation).

Corollary: $s_\nu e_\mu = \sum_\lambda \overset{m}{\cancel{k}}_{\lambda'/\nu;\mu} s_\lambda$. (*)

This allows us to prove a fundamental property of skew-Schur functions:

Theorem: $\forall\ f \in \Lambda$,

$$\langle f s_\nu, s_\lambda \rangle = \langle f, s_{\lambda/\nu} \rangle$$

Proof: Apply the involution E to (*) ~~and~~

$$s_{\nu'} h_\mu = \sum_\lambda \overset{m}{\cancel{k}}_{\lambda'/\nu';\mu} s_{\lambda'}$$

or
$$s_\nu h_\mu = \sum_\lambda \overset{m}{\cancel{k}}_{\lambda/\nu,\mu} s_\lambda \qquad (\dagger)$$

$$\therefore \quad \langle s_\nu h_\mu, s_\lambda \rangle = \cancel{k}\, m \cancel{k}_{\lambda/\nu,\mu} = \langle s_{\lambda/\nu}, h_\mu \rangle$$

$$[\ \langle h_\mu, m_\lambda \rangle = \delta_{\mu\lambda}\ ]$$

Since $\{h_\mu\}$ is a basis ~~done~~ ok.

Cor.: $\langle s_\mu s_\nu, s_\lambda \rangle = \langle s_\mu, s_{\lambda/\nu} \rangle$.

Cor: $E(s_{\lambda/\nu}) = s_{\lambda'/\nu'}$.

$$\langle s_{\lambda'/\nu'}, s_{\mu'} \rangle = \langle s_{\lambda'}, s_{\mu'} s_{\nu'} \rangle$$

$$\therefore \quad \langle E(s_{\lambda/\nu}), s_{\mu'} \rangle = \langle s_{\lambda/\nu}, s_\mu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle$$
$$= \langle E(s_\lambda), E(s_\mu s_\nu) \rangle = \langle s_{\lambda'}, s_{\mu'} s_{\nu'} \rangle.$$

QED.

If ν has only one part: $\nu = r$.

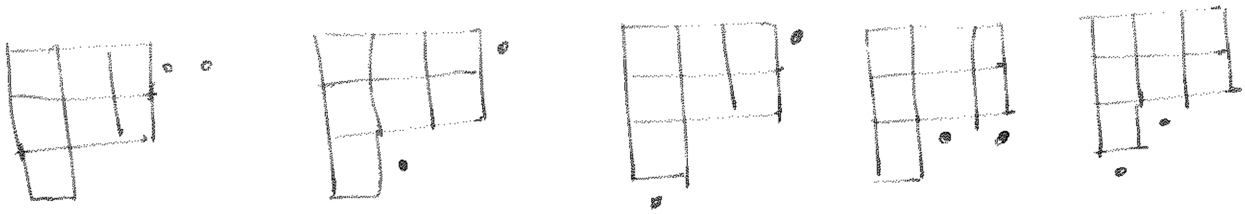
Then $s_\nu h_r = \sum_\lambda m_{\lambda/\nu, r} \, s_\lambda$.

$m_{\lambda/\nu, r} = 0$ unless the skew-shape λ/ν has at most one box in each column. Such skew shapes are called horizontal r-strips.

$$\therefore \quad s_\nu h_r = \sum_\lambda s_\lambda \qquad \text{(Pieri's rule)}$$

where λ/ν is a horizontal strip of size r.

For example: $\nu = (3,3,1)$ $r = 2$



$$s_{(3,3,1)} h_2 = s_{5,3,1} + s_{4,3,2} + s_{4311} + s_{333} + s_{3321}$$

Note: $\vec{h} = \vec{s} M \qquad h_\lambda = \sum_{\mu \leq \lambda} s_\mu m_{\mu\lambda}$

so $s_r = h_r$

Pieri's rule: $s_\nu s_r = \sum_\lambda s_\lambda$

λ/ν horiz. strip of size r.

Dually: $s_\nu s_{1^r} = \sum_\lambda s_\lambda$

λ/ν is a vertical strip of size r.

Jacobi - Trudi Identity:

It is an expansion of the Schur functions in terms of the complete symmetric polynomials. Therefore, it is an inversion of the M matrix.

Theorem: $s_{\lambda/\mu} = \det\left(h_{\lambda_i - \mu_j - i + j}\right)_{n\times n}$

(here $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$, and we set $h_0 = 1$ and $h_k = 0 \;\forall\, k<0$)

Proof: Let $c^{\lambda}_{\mu\nu} = \langle s_\lambda, s_\mu s_\nu\rangle$.

Then $s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda \qquad s_{\lambda/\mu} = \sum c^{\lambda}_{\mu\nu} s_\nu$.

Therefore $\sum_\lambda s_{\lambda/\mu}(x) s_\lambda(y) = \sum_{\lambda,\nu} c^{\lambda}_{\mu\nu} s_\nu(x) s_\lambda(y)$

$$= \sum_\nu s_\mu(y) s_\nu(y) s_\nu(x)$$

$$= s_\mu(y) \sum_\nu h_\nu(x) m_\nu(y)$$

Take $y = (y_1, \ldots, y_n)$, and multiply by $a_\delta(y)$.

$$\sum_\lambda s_{\lambda/\mu}(x)\, a_{\lambda+\delta}(y) = a_{\mu+\delta}(y) \sum_\nu h_\nu(x) m_\nu(y)$$

$$= \sum_{\alpha\in\mathbb{N}^n} h_\alpha(x) y^\alpha \sum_{w\in S_n} \varepsilon(w)\, y^{w(\mu+\delta)}$$

$$= \sum_{w\in S_n} \sum_\alpha \varepsilon(w) h_\alpha(x)\, y^{\alpha + w(\mu+\delta)}$$

Coeff. of $y^{\lambda+\delta}$:

$$s_{\lambda/\mu}(x) = \sum_{w\in S_n} \varepsilon(w)\, h_{\lambda+\delta - w(\mu+\delta)}(x) = \det\left(h_{\lambda_i - \mu_j - i + j}\right)$$

In particular,

$$s_\lambda(x) = \det\left(h_{\lambda_i - i + j}\right)$$

Applying the involution E gives

$$s_{\lambda/\mu}(x) = \det\left(e_{\lambda'_i - \mu'_j - i + j}\right)$$

"dual Jacobi-Trudi".