Summary of the previous lecture:

1. $\vec{e} = \vec{m}\, M^T J M$
2. $\vec{h} = \vec{m}\, M^T M$
3. $\vec{s} := \vec{m}\, M^T$
4. $\vec{p} = \vec{m}\, P \qquad P_{\lambda\mu} = tr(w_\mu;\ \mathbb{C}[X_\lambda])$, $\quad P_{\lambda\mu} \neq 0 \Rightarrow \lambda \leq \mu$, $\quad P_{\lambda\lambda} = \prod_{i=1}^{\infty} m_i(\lambda)!$
5. $\vec{p} = \vec{s}\, X \qquad X_{\lambda\mu} = tr(w_\mu;\ V_\lambda) = \chi^\lambda(\mu) = \chi^\lambda(w_\mu)$

The classical definition of Schur functions:

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ write

$$x^\alpha \text{ for } x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

~~Define~~ For $w \in S_n$, define

$$w(x^\alpha) = x_1^{\alpha_{w(1)}} \cdots x_n^{\alpha_{w(n)}}.$$

Define $a_\alpha = a_\alpha(x_1, \ldots, x_n)$

$$= \sum_{w \in S_n} \varepsilon(w)\, w(x^\alpha)$$

$$= \begin{vmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_n} \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_2^{\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} & x_n^{\alpha_2} & \cdots & x_n^{\alpha_n} \end{vmatrix}$$

Example: If $\cancel{\alpha =}\ \delta = (n-1, n-2, \ldots, 0)$, then a_δ is the Vandermonde determinant

$$a_\delta = \prod_{i<j} (x_i - x_j).$$

For general α, if we put $x_i = x_j$ for $i \neq j$ then a_α vanishes.

$\therefore$ a_α is divisible by $x_i - x_j$ $\forall$ $i \neq j$, and hence by a_δ (in $\mathbb{Z}[x_1, \ldots, x_n]$; why?)

Thus $\dfrac{a_\alpha}{a_\delta}$ is a symmetric polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$.

Define: $\sigma_\lambda = \dfrac{a_{\lambda+\delta}}{a_\delta}$. ~~for an~~ (Classical Schur poly)

Theorem: $\forall$ $\lambda \vdash n$, $\sigma_\lambda = s_\lambda$ (Schur polynomial)

Proof: Suffices to show that

$$\vec{e} = \vec{\sigma}\, J M$$

or

$$a_\delta \vec{e} = \vec{a}_{*+\delta}\, J M$$

or

$$a_\delta e_\mu = \sum_\lambda m_{\lambda'\mu}\, a_{\lambda+\delta}. \qquad (*)$$

Both sides of $(*)$ are skew-symmetric, so only monomials with distinct parts occur, and the polynomials are determined by monomials with strictly decreasing ~~indices~~ exponents, i.e., it is enough to compare coeffs. of $x^{\lambda+\delta}$, where λ is a partition.

Multiply a_δ by e_μ by successively multiplying by $e_{\mu_1}, e_{\mu_2}, \ldots$.

At each stage $a_\delta e_{\mu_1} e_{\mu_2} \cdots e_{\mu_k}$ is skew-sym.

So any monomial $x_1^{i_1} \ldots x_n^{i_n}$ appearing in it has distinct $\{i_1, \ldots, i_n\}$.

$$x_1^{i_1} \ldots x_n^{i_n} \underbrace{x_{m_1} \ldots x_{m_j}}_{\substack{\text{monomial} \\ \text{in } e_{\mu_{k+1}} \text{ (so } j = \mu_{k+1})}}$$

either two exponents become equal (in which case the monomial will cancel out) or the relative ordering of the exponents remains unchanged.

$\therefore$ to get $x^{\lambda+\delta}$ in $a_\delta e_\mu$, we must start with ~~an~~ ~~inc~~ a "decreasing monomial" in a_δ; the only one is x^δ.

Example: $n = 3$, $\mu = 2+1$

Start with $x_1^2 x_2 \, (x_1x_2 + x_1x_3 + x_2x_3)(x_1+x_2+x_3)$

$$\rightarrow = x_1^3 x_2^2 + \cancel{x_1^3 x_2 x_3} + \cancel{x_1^2 x_2^2 x_3} . (x_1+x_2+x_3)$$

$$\rightarrow = x_1^4 x_2^2 + \cancel{x_1^3 x_2^3} + x_1^3 x_2^2 x_3 \, .$$

from these monomials can reconstruct $a_\delta e_{2+1}$.

In how many ways can you get

$x^{\lambda+\delta}$ from x^{δ}

by multiplying monomials ~~from~~ x^{α_1}, x^{α_2} ...

from e_{μ_1}, e_{μ_2} ... ~~resp~~ respectively.

Define a SSYT $T(\alpha^1, \alpha^2, \dots)$ as follows:

"Column j of T contains an i if x_j occurs in x^{α^i}".

$n=4$, $\lambda = 5332$, $\lambda' = 44311$

$\lambda+\delta = 8542$

$\mu = 3222211$

$x^{\alpha^1} = x_1 x_2 x_3$

$x^{\alpha^2} = x_1 x_2$

$x^{\alpha^3} = x_3 x_4$

$x^{\alpha^4} = x_1 x_2$

$x^{\alpha^5} = x_1 x_4$

$x^{\alpha^6} = x_1$

$x^{\alpha^7} = x_3$

$$\left.\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & 5 \\ 4 & 4 & 7 & \\ 5 & & & \\ 6 & & & \end{array}\right\} = T(\alpha^1, \dots, \alpha^7).$$

and this gives rise to a bijective correspondence between ways of building up $x^{\lambda+\delta}$ from x^{δ} and SSYTs of shape λ' & type μ! QED.

~~Th.~~ Corollary: Let f be a symmetric poly in n variables, and suppose that $f = \sum_{\lambda \vdash n} b_\lambda s_n$. Then the coefficient of $x^{\lambda+\delta}$ in $a_\delta f$ is also b_λ.

Proof:

$$f = \sum_\lambda b_\lambda s_\lambda$$

$$a_\delta f = \sum b_\lambda a_\delta s_\lambda$$

$$= \sum b_\lambda a_{\lambda+\delta} \qquad \text{(alternating)}.$$

Note: In $a_{\lambda+\delta}$ there is only one monomial with decreasing exponents, namely $x^{\lambda+\delta}$.

$\therefore$ coeff. of $x^{\lambda+\delta}$ in $a_\delta f$ is b_λ (no other term will contribute). QED.

~~Cor.~~ Theorem (Frobenius character formula): The character of V_λ at ω_μ is the coefficient of ~~of~~ $x^{\lambda+\delta}$ in $a_\delta p_\mu$.

Proof:

$$p_\mu = \sum_{\lambda \vdash n} \operatorname{tr}(\omega_\mu; \mathbb{C}[X_\lambda])\, m_\lambda$$

$$= \sum_{\lambda \vdash n} \sum_{\nu \le \lambda} m_{\nu\lambda} \operatorname{tr}(\omega_\mu; V_\nu)\, m_\lambda$$

$$= \sum_\nu \cancel{\sum_{\lambda \ge \nu}} \operatorname{tr}(\omega_\mu; V_\nu) \sum_{\lambda \ge \nu} m_{\nu\lambda} m_\lambda$$

$$= \sum_\nu \operatorname{tr}(\omega_\mu; V_\nu)\, s_\nu.$$

Applying the corollary above to p_μ now gives the theorem.