Symmetrizer / Specht module

[Let] λ be a partition of S_n. Let Y be a Young [ta]bleau of shape λ.

[Exa]mple: $n = 3$, $\lambda = 2+1$ $\quad Y = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \\ \hline \end{array}$.

[Def]ine $S_Y : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ by

$$S_Y f(x) = \frac{1}{|\mathrm{Rowstab}|} \sum_{w \in \mathrm{Rowstab}(Y)} f(wx)$$

and $A_Y : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ by

$$A_Y f(x) = \frac{1}{|\cancel{\mathrm{Rowstab}}|} \sum_{w \in \mathrm{Colstab}(Y)} (-1)^w f(wx)$$

2 1 3

3 1 2

s_2 1 3 2

2 3 1

$s_1 s_2$

[T]he Young symmetrizer is the operator $A_Y S_Y : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$

[I]n the example:

$$S_Y = \cancel{(1\ 3\ 2) + (1\ 2} \left([3\ 2\ 1] + [1\ 2\ 3]\right)/2$$

$$A_Y = \left([1\ 2\ 3] - [1\ 3\ 2]\right)/2$$

$$A_Y S_Y = \left([3\ 2\ 1] + [1\ 2\ 3] - [2\ 3\ 1] - [1\ 3\ 2]\right)/4$$

$s_1 s_2 s_1$ $\quad$ 1 $\quad$ $s_1 s_2$ $\quad$ s_2

$s_2 s_1 s_2$ $\quad$ $s_1 s_2 s_1 s_2$

$\cancel{s_1 s_2} s_1 s_1$

$s_2 s_1 \cancel{s_3 s_2} s_1$

The matrix of this transformation is

	1	s_1	s_2	s_1s_2	s_2s_1	$s_1s_2s_1$
1	1	0	-1	0	-1	1
s_1	0	1	-1	1	-1	0
s_2	-1	0	1	0	1	-1
s_1s_2	-1	1	0	1	0	-1
s_2s_1	0	-1	1	-1	1	0
$s_1s_2s_1$	1	-1	0	-1	0	1
			$-c_1-c_2$	c_2	$-c_1-c_2$	c_1

The first two columns span the column space.

$\therefore$ image $A_Y S_Y$ has ~~rank~~ dim 2.

Lemma: $A_Y S_Y \neq 0$.

Pf: $A_Y S_Y \delta_1 = \dfrac{1}{|\text{Colstab } Y|} \dfrac{1}{|\text{Rowstab } Y|} \displaystyle\sum_{w \in \underbrace{\text{Colstab} \cap \text{Rowstab}}_{= \{I\}}} (-1)^w \neq 0.$

Now $\mathbb{C}[S_n]$ is a rep. of S_n by right translation:

$$w \cdot f(x) = f(xw)$$

$S_Y : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ is a homomorphism, and its image is iso. to $\mathbb{C}[X_\lambda]$ $\quad\therefore A_Y S_Y \mathbb{C}[S_n] \cong V_\lambda$

$A_{Y'} : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ is also a homom., and its image is iso to $\mathbb{C}[X_{\lambda'}] \otimes \varepsilon$.

Characteristic map: R_n = class fns. on S_n.

Define $ch : R_n \to \Lambda_n$ by

$$\chi^\lambda \longmapsto s_\lambda \quad \forall \lambda \vdash n \quad (\chi^\lambda(\omega) = tr(\omega; V_\lambda))$$

Lemma: (1) If $f(\omega) = tr(\omega; \mathbb{C}[X_\lambda])$ then $ch(f) = h_\lambda$

(2) If $f(\omega) = tr(\omega; \mathbb{C}[X_\lambda] \otimes \varepsilon)$ then $ch(f) = e_\lambda$

(3) For any ~~$f : S_n \to \mathbb{C}$~~, $f \in R_n$,

$$ch(f) = \frac{1}{n!} \sum f(\omega) p_{\rho(\omega)}$$

where $\rho(\omega)$ is the partition associated to the cycle decomposition of ω.

Pf: (1) & (2) are easy.

For (3) it suffices to show that

$$s_\lambda =: ch(\chi^\lambda) = \frac{1}{n!} \sum \chi^\lambda(\omega) p_{\rho(\omega)}$$

But

$$\frac{1}{n!} \sum_{\omega \in S_n} \chi^\lambda(\omega) p_{\rho(\omega)} = \frac{1}{n!} \sum_{\omega \in S_n} \chi^\lambda(\omega) \sum_{\mu \vdash n} tr(\omega; \mathbb{C}[X_\mu]) m_\mu$$

$$= \frac{1}{n!} \sum_{\omega \in S_n} \sum_{\mu \vdash n} \sum_{\nu \leq \mu} m_{\nu\mu} \chi^\lambda(\omega) \chi^\nu(\omega^{-1}) m_\mu$$

$$= \sum_{\nu \vdash n} \sum_{\mu \geq \nu} m_{\nu\mu} \delta_{\lambda\nu} m_\mu$$

$$= \sum_{\mu \geq \lambda} m_{\lambda\mu} m_\mu = s_\lambda \quad \text{as required.}$$

Let $R = \bigoplus_{n=0}^{\infty} R_n \quad (R_0 := \mathbb{C})$

$$\Lambda = \bigoplus_{n=0}^{\infty} \Lambda_n$$

Then $ch : R \to \Lambda$ is an iso of graded vector spaces.

Note: Λ is an algebra. This iso, therefore endows R with the structure of an algebra.

<u>Question</u>: What is the algebra structure on R?

~~Thm~~ $\operatorname{tr}(\omega; \mathbb{C}[X_\lambda]) \longleftrightarrow h_\lambda$

If $\lambda \vdash n$, $\mu \vdash k$, then $h_\lambda h_\mu = h_{\lambda \cup \mu}$, where $\lambda \cup \mu$ is the partition of $n+k$ obtained by arraying the parts of λ & μ in decreasing order.

$$h_{\lambda\cup\mu} \longleftrightarrow \operatorname{tr}(\omega; \mathbb{C}[X_{\lambda\cup\mu}]).$$

<u>Recall</u>: $\mathbb{C}[X_{\lambda\cup\mu}] \cong \mathbb{C}[X_{(\lambda,\mu)}]$

where (λ,μ) is the <u>composition</u> $(\lambda_1,\dots,\lambda_\ell,\mu_1,\dots,\mu_\ell)$.

$$\mathbb{C}[X_{(\lambda,\mu)}] = \operatorname{Ind}_{S_{(\lambda,\mu)}}^{S_{n+k}} 1 = \operatorname{Ind}_{S_\lambda \times S_\mu}^{S_{n+k}} 1$$

$$= \operatorname{Ind}_{S_n\times S_k}^{S_{n+k}} \operatorname{Ind}_{S_\lambda\times S_\mu}^{S_n\times S_k} 1 \quad \text{(induction in stages formula)}^{*}$$

$$= \operatorname{Ind}_{S_n\times S_k}^{S_{n+k}}\left[(\operatorname{Ind}_{S_\lambda}^{S_n} 1)\otimes(\operatorname{Ind}_{S_\mu}^{S_k} 1)\right] \quad \text{(induction product formula)}^{**}$$

$$= \operatorname{Ind}_{S_n\times S_k}^{S_{n+k}}\left(\mathbb{C}[X_\lambda]\otimes\mathbb{C}[X_\mu]\right)$$

$*$: Induction in stages formula:

$$H < K < G, \quad \operatorname{Ind}_K^G \operatorname{Ind}_H^K \sigma = \operatorname{Ind}_H^G \sigma.$$

<u>Pf</u>. $\cancel{\langle \operatorname{Ind}_H^G \sigma, \tau\rangle}$ $\langle \operatorname{Ind}_K^G \operatorname{Ind}_H^K \sigma, \tau\rangle = \langle \operatorname{Ind}_H^K \sigma, \tau|_K\rangle$

$$= \langle \sigma, \tau|_K|_H\rangle = \langle \sigma, \tau|_H\rangle = \langle \operatorname{Ind}_H^G \sigma, \tau\rangle.$$

$**$: Induction product formula. $H_1 < G_1$, $H_2 < G_2$

$$\operatorname{Ind}_{H_1\times H_2}^{G_1\times G_2} \sigma_1\otimes\sigma_2 = (\operatorname{Ind}_{H_1}^{G_1}\sigma_1)\otimes(\operatorname{Ind}_{H_2}^{G_2}\sigma_2)$$

Corollary: On R, define $f \circ g = \mathrm{Ind}_{S_n \times S_k}^{S_{n+k}} f \otimes g$.

Then ~~$f \circ g$ is~~ "$\circ$" is the ~~an~~ product ~~for~~ operation for an associative algebra structure on R.

$ch : R \to \Lambda$ is an iso. of algebras.

Note: R has an involution $\otimes \varepsilon$: $\tilde{f} = f \otimes \varepsilon$

Λ has the involution E $(s_\lambda \mapsto s_{\lambda'})$: $E(p) = \tilde{p}$.

Have: $ch(\tilde{f}) = \widetilde{ch(f)}$

Put an inner product structure on Λ by requiring that the s_λ be an o.n.b.

R has an inner product where the irreducible chars. of S_n, $n = 0, 1, \ldots$ form an onb.

$ch : R \to \Lambda$ is an isometry of Hilbert spaces.

Lemma: $\langle h_\lambda, m_\mu \rangle = \langle \vec{h}^T, \vec{m} \rangle_{\lambda\mu}$

$$= \langle M^T M \vec{m}^T, \vec{m} \rangle_{\lambda\mu}$$

$$= \langle M^T \vec{s}^T, \vec{m} \rangle_{\lambda\mu}$$

$$= \langle M^T \vec{s}^T, \vec{s} M^{T-1} \rangle_{\lambda\mu}$$

$$= \left(M^T \, \mathrm{Id} \, M^{T-1} \right)_{\lambda\mu} = \delta(\mathrm{Id})_{\lambda\mu} = \delta_{\lambda\mu}.$$

Note: In many textbooks (MacDonald, Stanley, ...) $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ is defined as ~~the~~ a bilinear pairing in Λ and then shown to be equivalent to ours.