Example: Let $\lambda, \mu$ be partitions of $n$, $\varepsilon$ be the sign char. $S_n \to \mathbb{C}$

$S_n \longrightarrow \{\pm 1\}$.

Question: What is $\dim \mathrm{Hom}_{S_n}(\mathrm{Ind}_{S_\lambda}^{S_n} 1, \mathrm{Ind}_{S_\mu}^{S_n} \varepsilon)$ ?

By our theorem on intertwiners, this is the dimension

of $\Delta = \left\{ k : S_n \to \mathbb{C} \;\middle|\; k(w_\mu w w_\lambda) = \varepsilon(w_\mu) k(w) \text{ for all } \begin{array}{l} w \in S_n \\ w_\mu \in S_\mu \\ w_\lambda \in S_\lambda \end{array} \right\}$

Such a function is completely determined by its values at $k(w)$, where $w$'s are reps. of $S_\mu \backslash S_n / S_\lambda$.

Defn: Say that $w$ <u>supports an intertwiner</u> if $\exists$ $k \in \Delta \ni k(w) \neq 0$. ~~(or rather $S_\mu w S_\lambda$)~~ (More accurately, $S_\mu w S_\lambda$ supports an intertwiner).

Suppose $w$ supports an intertwiner.

Then if $w_\mu$ & $w_\lambda$ are such that $w_\mu w w_\lambda = w$, then

$$k(w) = k(w_\mu w w_\lambda) = \varepsilon(w_\mu) k(w)$$

$\Rightarrow w_\mu \in A_n$

But $\exists\, w_\lambda \in W_\lambda \ni w_\mu w w_\lambda = w$ iff

$$w_\mu \in S_\mu \cap w S_\lambda w^{-1}$$

$\therefore$ if $w$ supports an intertwiner, then

$$S_\mu \cap w S_\lambda w^{-1} \subset A_n$$

Conversely, if $S_\mu \cap w S_\lambda w^{-1} \subset A_n$, define $k$ on $W_\mu w W_\lambda$ by $k(w_\mu w w_\lambda) = \varepsilon(w_\mu)$.

This is well-defined, for if $w_\mu w w_\lambda = w_\mu' w w_\lambda'$

then $w_\mu'^{-1} w_\mu w w_\lambda w_\lambda'^{-1} = w \Rightarrow w_\mu'^{-1} w_\mu \in S_\mu \cap w S_\lambda w^{-1} \subset A_n$

So $\varepsilon(w_\mu) = \varepsilon(w_\mu')$. ~~whence k(w~~

<u>Conclusion:</u> $\dim_{\mathbb{C}} \mathrm{Hom}_{S_n}(\mathrm{Ind}_{S_\lambda}^{S_n} 1, \mathrm{Ind}_{S_\mu}^{S_n} \varepsilon)$

$= \#\{ S_\mu w S_\lambda \mid S_\mu \cap w S_\lambda w^{-1} \subset A_n \}$.

<u>Lemma:</u> $S_\mu \cap w S_\lambda w^{-1} \subset A_n \iff S_\mu \cap w S_\lambda w^{-1} = \{id\}$.

<u>Pf:</u> $S_\mu$ and $w S_\lambda w^{-1}$ are ~~is the~~ subgroups which preserve the parts of a partitions ~~$S_\lambda$ is the~~ $P_\mu^\circ$ & $w P_\lambda^\circ$, so their intersection is precisely the subgroup which preserves the partition obtained by intersecting the parts of $P_\mu^\circ$ & $w P_\lambda^\circ$, hence also a subgroup of the form $S_{i_1} \times \dots \times S_{i_k} \subset S_n$. The only such subgp. contained in $A_n$ is trivial (since it is generated by transpositions).

<u>Recall:</u>

$X_\lambda$ = "flags of type $\lambda$ in $\underline{n}$"

= "partition of type $\lambda$ in $\underline{n}$"

$:= \{ \underline{n} = S_1 \sqcup \dots \sqcup S_\ell \mid |S_i| = \lambda_i \}$.

and $S_\mu \backslash S_n / S_\lambda \cong S_n \backslash X_\mu \times X_\lambda$

$S_\mu = \mathrm{Stab}_{S_n}(P_\mu^\circ)$ $\qquad w S_\lambda w^{-1} = \mathrm{Stab}_{S_n}(w P_\lambda^\circ)$

std. flag of shape $\mu$ $\qquad$ flag of shape $\lambda$.

$S_\mu \cap w S_\lambda w^{-1} =$ ~~$\{id\}$~~ $\mathrm{Stab}_{S_n}(P_\mu^\circ) \cap \mathrm{Stab}_{S_n}(w P_\lambda^\circ)$.

$\therefore$ $S_\mu \cap w S_\lambda w^{-1} = \{Id\}$ iff $\exists$ ~~a flags~~ partitions

~~$P, Q$, $P \in X_\mu$, $Q \in X_\lambda$~~

$P_\mu \in X_\mu$, $P_\lambda \in X_\lambda$ such that if

(*) $P_\lambda = S_1^\lambda \sqcup \dots \sqcup S_\ell^\lambda$, $P_\mu = S_1^\mu \sqcup \dots \sqcup S_m^\mu$, then $|S_i^\lambda \cap S_j^\mu| \le 1$ $\forall i, j$.

$|S_i^\lambda \cap S_j^\mu| \le 1$ for all $i,j$.

~~Theorem~~

Defn: Say that $P_\lambda$ is transversal to $P_\mu$ if the above condition holds. Write $P_\lambda \pitchfork P_\mu$.

Theorem: $\dim \mathrm{Hom}_{S_n}(\mathrm{Ind}_{S_\lambda}^{S_n} 1, \mathrm{Ind}_{S_\mu}^{S_n} \varepsilon)$

$$= \# \, \Delta S_n \backslash \{(P_\mu, P_\lambda) \in X_\mu \times X_\lambda \mid P_\mu \pitchfork P_\lambda\} =: e_{\lambda\mu}$$

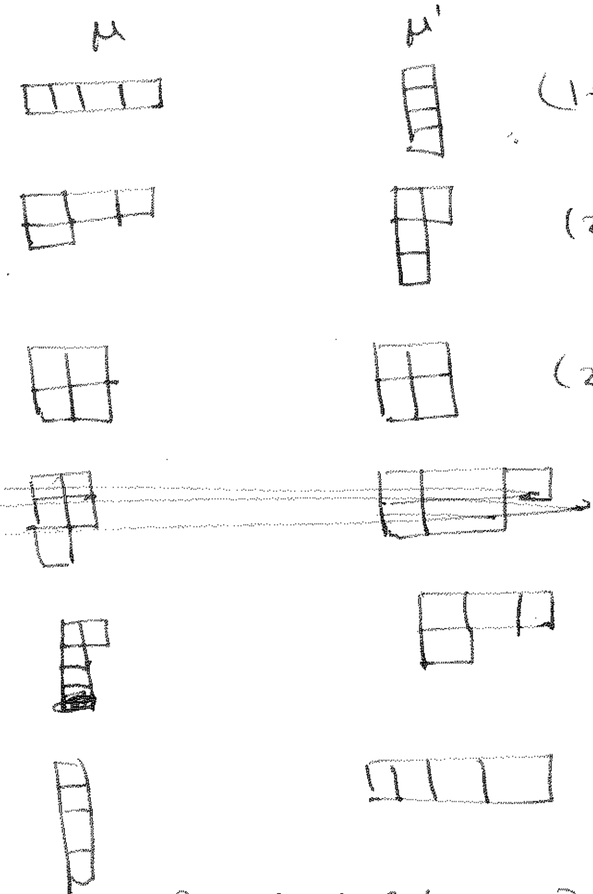
$$= \# \, S_\lambda \backslash \{P_\mu \in X_\mu \mid P_\mu \pitchfork P_\lambda^0\}$$

~~Lemma:~~

Defn: Given a partition $\mu$, $\mu'$ is the partition whose Ferrers diagram is the reflection of the Ferrers diagram of $\mu$ about its principal axis.

Example: $n=4$

| | $\mu$ | $\mu'$ | |
|---|---|---|---|
| $(4)$ | | | $(1+1+1+1)$ |
| $(3+1)$ | | | $(2+1+1)$ |
| $(2+2)$ | | | $(2+2)$. |
| ~~$(2+2+1)$~~ | | | |
| $(2+1+1)$ | | | $(3+1)$ |
| $(1+1+1+1)$ | | | $(4)$. |



$\mu'_1 = \#\{i \mid \mu_i \ge 1\}$, $\mu'_2 = \#\{i \mid \mu_i \ge 2\}$, etc..

~~Defn: $P_\lambda \pitchfork P_\mu$ if (*) holds ("$P_\lambda$ is transversal to $P_\mu$")~~

<u>Lemma:</u> Suppose $\exists$ a semistandard Young tableaux of shape $\mu'$ & type $\lambda$, then $\exists$ . $P_\mu \pitchfork P_\lambda^\circ$.

<u>Pf:</u> Label the boxes with 1's ~~by~~ frequentially by $\{1, \ldots, \lambda_1\}$, the boxes with 2's by $\{\lambda_1+1, \ldots, \lambda_1+\lambda_2\}$ etc.. Then $P_\mu$ is the partition obtained by frows ~~Column~~ columns of the resulting std. tableaux

<u>Example:</u>

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \rightsquigarrow \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 7 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

gives $P_\mu = \{1,4,6\} \amalg \{2,7\} \amalg \{3\} \amalg \{5\}$.

<u>Lemma:</u> Suppose $\exists$ $P_\lambda \pitchfork P_\mu$ then $\mu' \leq \lambda$, i.e., $\mu'_1 \geq \lambda_1$, $\mu'_1 + \mu'_2 \geq \lambda_1 + \lambda_2, \ldots$

<u>Pf:</u> Fill the parts of $P_\lambda$ into the rows of a ~~partition~~ ~~Young~~ Ferrers diagram of shape $\lambda$ and the parts of $P_\mu$ into the columns of a Ferrers diagram of shape $\mu'$.

<u>Example:</u> $P_\lambda = \{1,2,3\} \amalg \{4,5,6\}$
$P_\mu = \{2,5\} \amalg \{4,1\} \amalg \{3\} \amalg \{6\}$.

$$Y_\lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \qquad Y_{\mu'} = \begin{array}{|c|c|c|c|} \hline 2 & 4 & 3 & 6 \\ \hline 5 & 1 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|} \hline 2 & 1 & 3 & \\ \hline & \\ \cline{1-2} \end{array}$$

$P_\lambda \pitchfork P_\mu$ means that the ~~to~~ elements in the first row of $Y_\lambda$ occur in different columns of $Y_{\mu'}$.
"Float them up"; they will fit in the first row

$\therefore\ \mu_1' \geq \lambda_1$

Elements in the 2nd row of $Y_\lambda$ also occur in distinct cols. of $Y_\mu$, so "float them up". They will either come to lie just below an element from the first row of $Y_\lambda$ or in a new row; in any case in the first 2 rows of a Ferrers diagram of shape $\mu'$.

$\therefore\ \mu_1' + \mu_2' \geq \lambda_1 + \lambda_2$

etc...

<u>Corollary</u>: $\exists\ P_\lambda \pitchfork P_\mu$ iff $\mu' \leq \lambda$.

<u>Corollary</u>: $\mu' \leq \lambda' \iff \lambda \leq \mu$

~~Theo~~ <u>Corollary</u>: $\dim \mathrm{Hom}_{S_n}(\mathrm{Ind}_{S_\lambda}^{S_n} 1, \mathrm{Ind}_{S_\mu}^{S_n} \varepsilon) > 0$ iff

$\mu' \leq \lambda$ iff $\lambda' \leq \mu$.

<u>Lemma</u>: $\dim \mathrm{Hom}_{S_n}(\mathrm{Ind}_{S_\lambda}^{S_n} 1, \mathrm{Ind}_{S_{\lambda'}}^{S_n} \varepsilon) = 1$.

<u>Proof</u>: We will show that

$$\# \ S_\lambda \backslash \{P_{\lambda'} \in X_{\lambda'} \mid P_\lambda \pitchfork P_{\lambda'}\} = 1$$

Without loss of generality take $P_\lambda$ to be standard. What is $P_{\lambda'}$? Just fill the boxes in a Ferrers

~~The different~~ ~~$\{\lambda_1, \ldots$~~ $\{1, \ldots, \lambda_1\}$ must go into the $\lambda_1$ diff. parts of $P_{\lambda'}$. $\{\lambda_1+1, \ldots, \lambda_1+\lambda_2\}$ must go into diff parts of $P_{\lambda'}$. After rearranging

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & \lambda_1 \\ \hline \lambda_1+1 & \cdots & \lambda_1+\lambda_2 & \\ \hline \end{array}$$

$\{1, \ldots, \lambda_1\}$ & $\{\lambda_1+1, \ldots, \lambda_1+\lambda_2\}$

Can assume

diagram of shape $\lambda'$ along columns:

<u>Example:</u> $\lambda = (2,2,1,1)$

$P_\lambda = \{1,2\} \sqcup \{3,4\} \sqcup \{5\} \sqcup \{6\}$

$P_{\lambda'} = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \qquad \{1,3,5,6\} \sqcup \{2,4\}.$

Upto $S_\lambda$-action this is unique, for the nos. $\{1, \dots, \lambda_1\}$ must occupy the $\lambda_1$ distinct parts of $P'_\lambda$.

Then $\{\lambda_1+1, \dots, \lambda_1+\lambda_2\}$ must again occupy $\lambda_2$ distinct parts (each of which already has one of $\{1, \dots, \lambda_1\}$). After permuting, we can assume 1, & $\lambda_1+1$, 2 & $\lambda_1+2$, ... etc are in the same part.

Continuing in this way, we get $P_{\lambda'}$ described above.

<u>Theorem:</u> Let $\varepsilon_{\lambda\mu} = \dim \mathrm{Hom}_{S_n}(\mathrm{Ind}_{S_\lambda}^{S_n} 1, \mathrm{Ind}_{S_\mu}^{S_n} \varepsilon)$

Then ① $\varepsilon_{\lambda\mu} \neq 0 \iff \mu' \leq \lambda \iff \lambda' \leq \mu$.

② $\varepsilon_{\lambda\lambda'} = 1$

③ $\varepsilon_{\lambda\mu} = \varepsilon_{\mu\lambda}$

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<u>Corollary:</u> $\cancel{V_\lambda \otimes \varepsilon = V_{\lambda'}}$ $V_\lambda$ is the unique irrep. which occurs in $\mathbb{C}[X_\lambda]$ & $\mathbb{C}[X_{\lambda'}] \otimes \varepsilon$.

<u>Pf:</u> $\dim \mathrm{Hom}_{S_n}(\mathbb{C}[X_\mu], \mathbb{C}[X_{\lambda'}] \otimes \varepsilon) \neq 0$ iff $\lambda \leq \mu$

In particular, if $\mu < \lambda$, $\dim \mathrm{Hom}_{S_n}(\mathbb{C}[X_\mu], \mathbb{C}[X_{\lambda'}] \otimes \varepsilon) = 0$

$\Rightarrow V_\mu$ does not occur in $\mathbb{C}[X_{\lambda'}] \otimes \varepsilon$ for $\mu < \lambda$.

On the other hand,

$$\dim \mathrm{Hom}_{S_n}(\mathbb{C}[X_\lambda], \mathbb{C}[X_{\lambda'}] \otimes \varepsilon) = 1$$

~~So $V_\lambda \subset$~~

$$\|$$

$$\Big(\bigoplus_{\mu < \lambda} m_{\mu\lambda} V_\mu\Big) \oplus V_\lambda$$

$\Rightarrow$ $V_\lambda$ occurs exactly once in $\mathbb{C}[X_{\lambda'}] \otimes \varepsilon$.

This characterizes $V_\lambda$ as the unique irrep. of $S_n$ which occurs in $\mathbb{C}[X_\lambda]$ & $\mathbb{C}[X_{\lambda'}] \otimes \varepsilon$

So $V_\lambda \otimes \varepsilon$ is the unique irrep which occurs in $\mathbb{C}[X_\lambda] \otimes \varepsilon$ & $\mathbb{C}[X_{\lambda'}]$. Therefore

<u>Theorem:</u> $V_\lambda \otimes \varepsilon = V_{\lambda'}$.