GEOMETRIC THEORY OF INTERTWINERS

Let X be a set. A vector bundle over X is a collection of vector spaces $\{E_x : x \in X\}$.

→ If $\dim E_x = n \ \forall x \in X$, we say that the vector bundle is of <u>dimension n</u>. $E = \coprod_{x \in X} E_x$ is called the total space

~~Suppose G X is a G-space~~.

Define $E \xrightarrow{\pi} X$ by $\pi(v) = x$ if $v \in E_x$.

A <u>section</u> is a function $s : X \to E \ni \pi \circ s(x) = x \ \forall x \in X$.

The set of all sections forms a vector space under pointwise addition, which is denoted $\Gamma(X, E)$.

~~If G~~

If $Y \xrightarrow{f} X$ is a function, $(E \xrightarrow{\pi} X)$ is a vector bundle, then $(f^*E)_y := E_{f(y)}$ defines a vector bundle on Y.

If E_1 & E_2 are vector bundles on X, can define

$(E_1 \oplus E_2)_x = E_{1x} \oplus E_{2x}$ $\quad \mathrm{Hom}(E_1, E_2)_x = \mathrm{Hom}(E_{1x}, E_{2x})$

$(E_1 \otimes E_2)_x = E_{1x} \otimes E_{2x}$ $\quad (E_1^*)_x = (E_{1x})^*$

as vector bundles on X.

<u>Examples</u>: (1) $X \times V$ is a vector bundle over X, with "$V_x = V$" for all $x \in X$, called the trivial bundle.

(2) If X is an H-space and $\sigma : H \to GL(V)$ is a representation, let $E = H \backslash (X \times V)$, where H acts on $X \times V$ by $g(x, v) = (gx, \sigma(g)v)$.
or some gp.

Define $\pi: E \to H\backslash X$ by

$$\pi(H(x,v)) = H.x \quad \text{(which is well defined)}$$

E_x $\Gamma(H\backslash X, E)$ consists of functions $s: X \to V$ such that $s(gx) = \sigma(g)s(x)$. $E_{Hx} = H\backslash^{Hx \times V}_{H}$

(3) $H < G$, $\sigma: H \to GL(V)$ representation.
Then taking $X = G$ in ~~and~~ in example (2) we get $\Gamma(H\backslash G, E) = \{f: G \to V \mid f(hg) = \sigma(h)f(g)\}$

$$= \mathrm{Ind}_H^G V.$$

<u>Defn</u>: If X is a G-space and G acts on E in such a way that $g(E_x) = E_{gx}$, and the resulting map $E_x \to E_{gx}$ is linear, we say that E is a G-equivariant vector bundle.

<u>Example</u>: In Example (3) above, G acts on $H\backslash G$ by $g\cdot(Hx) = Hxg^{-1}$. $(x,v) \sim (hx, \pi(h)v)$

$$E_{Hx} = \coprod_{h\in H} V_{Hx} \Big/ (v_x \sim (\pi(h)v)_{hx})$$

The action of G on $G \times V$ given by

$$g\cdot(x,v) = (xg^{-1}, v)$$

descends to an action of G on E under which g takes E_{Hx} to $E_{Hxg^{-1}}$.

Lemma: If E is a G-equivariant vector bundle then $\Gamma(X, E)$ is a representation of G by

$$(\rho_E(g)s)(x) = g.s(g^{-1}.x).$$

Proof: straightforward.

In example ③,

$$(\rho(g)f)(x) = f(xg).$$

Thus we get the induced representation.

Conversely, if X is a transitive G-space and E is a G-equivariant vector bundle on X, then fix any point $x \in X$. Let $H = G_x$.

$V = E_x$ becomes a representation σ of H.

$$\Gamma(X, E) \cong \mathrm{Ind}_H^G \sigma \quad \text{as a representation of } G.$$

Thus induced representations correspond to G-equivariant vector bundles on transitive G-spaces.

Lemma: Let X and X' be sets, $E \xrightarrow{\pi} X$, $E' \xrightarrow{\pi'} X'$ be vector bundles. Then

$$\mathrm{Hom}_G(\Gamma(E, X), \Gamma(E', X')) = \{T_k \mid k \in \Gamma(X' \times X, \mathcal{H})\}$$

where $\mathcal{H} = \cancel{\pi^* E \otimes}\ \mathrm{Hom}(\pi^* E, \pi'^* E')$ is a vector bundle on $X' \times X$, and

$$T_k s(x') = \sum_{x \in X} k(x', x)(s(x))$$

Pf: Dimensions match, $k \mapsto T_k$ is linear, injective.

Note: $\mathcal{H}_{(x',x)} = \mathrm{Hom}(E_x, E'_{x'})$

$\mathcal{H}$ is also a G-equivariant vector bundle on $X' \times X$ if E and E' are: if $\varphi_{x',x} \in \mathcal{H}_{(x',x)}$

$$g \cdot \varphi_{x',x} = g \circ \varphi_{x',x} \circ g^{-1} \in \mathcal{H}_{(gx', gx)}$$

Note: $E_{gx} \xrightarrow{g^{-1}} E_x \xrightarrow{\varphi_x} E'_{x'} \xrightarrow{g} E'_{g \cdot x'}$

$\therefore\ \Gamma(X' \times X, \mathcal{H})$ is a representation of G.

$$\rho_{\mathcal{H}}(g) k(x', x) = g \circ k(g^{-1}x', g^{-1}x) \circ g^{-1}$$

Furthermore

$$T_{\rho_{\mathcal{H}}(g)k}\, s(x') = \sum_{x \in X} g \circ k(g^{-1}x', g^{-1}x) \circ g^{-1}(s(x))$$

$$= \sum_{x \in X} g \circ k(g^{-1}x', x)\, \rho_E(g^{-1})\, s(x)$$

$$= \rho_{E'}(g) \cancel{\Big(\sum_{x \in X} k(x', x)\Big)}\, T_k\, \rho_E(g^{-1})\, s(x')$$

$$\therefore\ T_{\rho_{\mathcal{H}}(g)k} = \rho_{E'}(g) \circ T_k \circ \rho_E(g^{-1}).$$

Corollary: T_k intertwines ρ_E with $\rho_{E'}$ iff

$k = \rho_{\mathcal{H}}(g) k \ \ \forall\ g \in G$, i.e.,

$$g \circ k(g^{-1}x', g^{-1}x) \circ g^{-1} = k(x', x)$$

$$\forall\ g \in G,\ x' \in X,\ x \in X.$$

(Raghavan's trick!!)

Suppose $E = {}_H\backslash^{G\times V}$, $E' = {}_{H'}\backslash^{G\times V'}$,

~~The map $G\times G\times \mathrm{Hom}(V,V') \to$~~

<u>Claim</u>: $\mathcal{H} \cong {}_{H'\times H}\backslash^{G\times G\times \mathrm{Hom}_{\mathbb{C}}(V,V')}$

where $\mathrm{Hom}_{\mathbb{C}}(V,V')$ is thought of as an $H'\times H$ module by $(h',h)\cdot\varphi = \rho\,\sigma'(h')\circ\varphi\circ\sigma(h)^{-1}$.

<u>Proof</u>: $E_{Hx} = {}_H\backslash^{Hx\times V}$ $\quad E'_{H'x'} = {}_{H'}\backslash^{H'x'\times V'}$

So $\mathrm{Hom}_{\mathbb{C}}(E_{Hx}, E'_{H'x'}) = {}_{H'\times H}\backslash^{H'x'\times Hx\times \mathrm{Hom}(V,V')}$ <u>QED.</u>

<u>Corollary</u>: $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Ind}_H^G V, \mathrm{Ind}_{H'}^{G} V') = \mathrm{Ind}_{H'\times H}^{G\times G}\mathrm{Hom}_{\mathbb{C}}(V,V')$ (as representations of $G\times G$)

But $\mathrm{Ind}_{H'\times H}^{G\times G}\mathrm{Hom}_{\mathbb{C}}(V,V')$

~~$= \{f : G\times G \to \mathrm{Hom}_{\mathbb{C}}(V,V') \mid f(h'g', hg) = \sigma'(h')f(g',g)\sigma(h)^{-1}\}$~~

$= \{f : G\times G \to \mathrm{Hom}_{\mathbb{C}}(V,V') \mid f(h'x', hx) = \sigma'(h')f(x',x)\sigma(h)^{-1}\ \forall h'\in H', h\in H, x',x\in G\} =: \Delta'$

<u>Corollary</u>: $\mathrm{Hom}_G(\mathrm{Ind}_H^G V, \mathrm{Ind}_{H'}^G V') = \left(\mathrm{Ind}_{H'\times H}^{G\times G}\mathrm{Hom}_{\mathbb{C}}(V,V')\right)^{\Delta G}$

$= \{f\in\Delta' \mid f(x'g, xg) = f(x',x)\ \forall x,x',g\in G\} =: \Delta$

Let $k(x) = f(x,1)$.

Then, if $f\in\Delta$, ~~then~~ $f(x,y) = f(xy^{-1},1) = k(xy^{-1})$.

$$\boxed{k(h'xh^{-1}) = f(h'x, h) = \sigma'(h')k(x)\sigma(h)}$$