k - field, G group.

<u>Representation</u> : (ρ, V). $\begin{cases} \circ\ V \text{ is a } k\text{-vector space} \\ \circ\ \rho : G \to GL_k(V) \text{ homom.} \end{cases}$

R - a k-algebra.

<u>Module</u> : $(\check{\rho}, V)$. $\begin{cases} \circ\ V \text{ is a } k\text{-vector space} \\ \circ\ \check{\rho} : R \to End_k(V) \text{ is a [linear] ring homom.} \end{cases}$

<u>Example ~~of k-algebra~~:</u> ~~Lassome~~

$k[G] :=$ vector space with basis $\{1_g \mid g \in G\}$.

$$1_g 1_h = 1_{gh}.$$

Suppose $\rho : G \to GL_k(V)$ is a rep.

Define $\check{\rho} : k[G] \to End_k(V)$ given by

$$\boxed{\check{\rho} : \sum a_g 1_g \mapsto \sum a_g \rho(g)} \quad (1)$$

Have: $(\check{\rho}, V)$ - $k[G]$-module.

Now suppose $\check{\rho} : k[G] \to End_k V$ is a module.

e - identity element of G.

1_e - unit in $k[G]$.

If $\check{\rho}(1_e) = id_V$, then

$$\check{\rho}(1_g)\check{\rho}(1_{g^{-1}}) = \check{\rho}(1_g 1_{g^{-1}}) = \check{\rho}(1_e) = id_V.$$

so $\check{\rho}(1_g)$ ~~invertive~~ $\in GL_k(V)$.

$$\boxed{\rho(g) := \check{\rho}(1_g)} \quad (2)$$

(1) & (2) define an equivalence between reps. of G and unital $k[G]$-modules.

$W \subset V$ is an invariant subspace if $\rho(g)W \subset W$ ~~(or $\tilde{\rho}(r)W \subset W$)~~ for all $g \in G$ (or $\tilde{\rho}(r)W \subset W$ $\forall\, r \in R$)

Defn: V is simple if it has no proper non-trivial invariant subspaces.

Defn: If (ρ_1, V_1) & (ρ_2, V_2) are reps of G (or R-modules) then

$$\mathrm{Hom}_G(V_1, V_2) = \{ T \in \mathrm{Hom}_k(V_1, V_2) \mid T(\rho_1(g)v) = \rho_2(g)\, Tv \ \ \forall\, g \in G,\ v \in V \}$$

$$\mathrm{Hom}_R(V_1, V_2) := \{ \mid T(\tilde{\rho}_1(g)v) = \tilde{\rho}_2(g)\, Tv \ \ \forall\, r \in R,\ v \in V \}$$

& We say T intertwines (ρ_1, V_1) with (ρ_2, V_2).

Lemma (Schur) Suppose k is algebraically closed, R a K-algebra, V simple. Then $\mathrm{End}_R V := \mathrm{Hom}_G(V, V) = k\,\mathrm{Id}$.

Pf: Let $T \in \mathrm{End}_G V$.

Then T has an eigenvalue λ.

$(T - \lambda I) \in \mathrm{End}_G V$

has non-trivial kernel.

Note: $W = \ker(T - \lambda I)$ is an invariant subspace

~~$\rho(g)(T - \lambda I)v = (T - \lambda I)\rho(g)v$~~

$w \in W \Rightarrow (T - \lambda I)\rho(g)w = \rho(g)(T - \lambda I)w = 0$

(by simplicity)

$\therefore W = V \Rightarrow T = \lambda\,\mathrm{Id}$. Q.E.D.

Defn: ~~Vg~~ $V_1 \cong V_2$ if $\exists$ an invertible intertwiner (its inverse will also be an intertwiner).

Lemma (Schur): If V_1 & V_2 are simple, then either $V_1 \cong V_2$ or $\mathrm{Hom}_G(V_1, V_2) = 0$.

Pf: If $\varphi \in \mathrm{Hom}(V_1, V_2)$, ~~$\varphi \neq 0$~~.

Then $\ker\varphi$ & $\mathrm{Im}\,\varphi$ are invariant subspaces.

$\varphi \neq 0 \Rightarrow \ker\varphi \neq V_1 \Rightarrow \ker\varphi = 0$

$\mathrm{Im}\,\varphi \neq 0 \Rightarrow \mathrm{Im}\,\varphi = V_2$

Cor: V_1, V_2 simple, then either $\mathrm{Hom}(V_1, V_2) = \mathbb{C}\phi$ for some φ or $\mathrm{Hom}(V_1, V_2) = 0$

Thm (Maschke): If (ρ, V) is a rep. of G, ~~and~~ char $k = \infty$ or char $k \nmid |G|$, $W \subset V$ is an invariant complement, then W admits an invariant complement ~~(note: this included char $k = \infty$)~~.

Ex: $k = \mathbb{F}_2$ ~~(field with two elements)~~ field with char. 2.

$G = \mathbb{Z}/2\mathbb{Z}$

$V = k[G]$

$\rho(g) 1_x = 1_{g+x}$

Suppose the line spanned by $a1_0 + b1_1$ is G-invariant.

Then $\rho(1)(a1_0 + b1_1) = b1_0 + a1_1$

$$= \alpha a 1_0 + \alpha b 1_1 \quad \text{for some } \alpha \in k$$

So $\alpha a = b$ & $\alpha b = a$, i.e., $\alpha^2 a = a \Rightarrow a = 1$

So the line spanned by $1_0 + 1_1$ is the only invariant subspace, and therefore does not admit a complement.

Proof of Maschke's theorem:

Take any ~~of~~ complement U of W, i.e., $V = U \oplus W$.

$x = x_u + x_w$.

Define $P_U(x) = x_U$ $\quad P_W(x) = x_W$. $\left(P_U = Id - P_W \right.$ & $\left. \ker P_W = U \right)$

~~Lemma~~ Lemma: U is invariant iff $\rho(g) P_W = P_W \rho(g)$ for all $g \in G$.

(equiv. $\rho(g) P_U = P_U \rho(g)$)

Pf: ~~Suppose it is invariant.~~ ~~$\ker P_W = U$~~

$\rho(g) P_W(x_w) = \rho(g) x_w$

$P_W \rho(g)(x_w) = \rho(g) x_w$ } since W is G-invar.

$\rho(g) P_W(x_U) = 0$

$P_W(\rho(g) x_U) = 0 \;\forall x_U \in U$ iff $\rho(g) x_U \in U \;\forall x_U \in U$.

If we start ~~ff~~ with $U \oplus W$ where U is not necessarily ~~invarg~~ invariant we can make it invariant:

Define: $\overline{P_W} = \frac{1}{|G|} \sum_{g \in G} \rho(g) P_W \rho(g)^{-1}$.

Then $\rho(x) \overline{P_W} \rho(x)^{-1} = \frac{1}{|G|} \sum_{g \in G} \rho(x) \rho(g) P_W \rho(g)^{-1} \rho(x)^{-1}$

$= \overline{P_W}$.

$\Rightarrow \rho(x) \overline{P_W} = \overline{P_W} \rho(x) \quad \forall x \in G$.

~~$\overline{P_W}^2 = \overline{P_W} \frac{1}{|G|^2} \sum_{g,h} \rho(g) P_W \rho(g)^{-1} \rho(h) P_W \rho(h)^{-1}$~~

$\overline{P_U} = I - \overline{P_W}$ | ~~$\operatorname{Im} \overline{P_U}$ is $\rho(g)$-invariant.~~

$\operatorname{Im} \overline{P_W} = \operatorname{Im} P_W$

Claim: $\bar{P}_W^2 = \bar{P}_W$.

If $x \in W$, then

$$\bar{P}_W x = \frac{1}{|G|}\sum \rho(g) P_W \rho(g)^{-1} x$$

$$= \frac{1}{|G|}\sum \rho(g)\rho(g)^{-1} x \qquad [\rho(g)^{-1}x \in W \Rightarrow P_W \rho(g)^{-1}x = \rho(g)^{-1}x]$$

$$= x.$$

If $x \in V$ then

$$\bar{P}_W^2 v = \bar{P}_W(\bar{P}_W v)$$

but $\bar{P}_W v \in W$

$\therefore \bar{P}_W \bar{P}_W v = \bar{P}_W v$

$\therefore \bar{P}_W^2 = \bar{P}_W$ (hence $\bar{P}_W$ is a projection)

Let $\bar{P}_U = 1 - \bar{P}_W$.

$\bar{U} = \operatorname{Im} \bar{P}_U$ is a G-invariant complement.

<u>Corollary</u> (Complete reducibility)

If (ρ, V) is any ~~irreducible~~ rep. of G and char $k = 0$ or char $k \nmid |G|$ then V can be written as a direct sum of simple invariant subspaces.

<u>Pf</u>: Easy.

~~Exam Applying~~ Schur's lemma(s)

Suppose $M = S_1^{\oplus m_1} \oplus$

$S_{11} \oplus \cdots \oplus S_{1m_1} \oplus S_{21} \oplus \cdots \oplus$

Applying Schur's lemma: $\mathrm{Hom}_R(\bigoplus_{i=1}^n V_i, \bigoplus_{j=1}^m W_j)$

$= \bigoplus_{i=1}^n \bigoplus_{j=1}^m \mathrm{Hom}_R(V_i, W_j)$

$M \cong \underbrace{S_1^{\oplus m_1}} \oplus \cdots \oplus \underbrace{S_k^{\oplus m_k}}$ isotypic pieces are unique.

$S_1, S_2, \ldots, S_k$ are pairwise non-isomorphic simples.

Then $\mathrm{End}_G M \cong \bigoplus_{i=1}^k M_{m_i}(k)$. (same for $\mathrm{End}_R M$)

Special case: ~~$M = \mathbb{C}[G]$.~~ $M = {}_R R$

$\rho_g 1_x = 1_{gx}$ $\tilde{\rho}(rx) = rx$

~~Think of $\mathbb{C}[G]$ as functions on G.~~

~~Claim:~~ $\mathrm{End}_G \mathbb{C}[G] = \mathbb{C}[G]^{opp}$ $\mathrm{End}_{\mathbb{C}[G]} \mathbb{C}[G]$

Pf: ~~Let~~ General fact: R any ring.

Then $\mathrm{End}_R R \cong R^{opp}$.

Pf: Let $\phi_r : x \mapsto xr$ $\phi_r \in \mathrm{End}_R R$.

$\phi_r \circ \phi_s (x) = \phi_s(x) r = xsr$

So $R^{opp} \longrightarrow \mathrm{End}_R R$.

Note: $\varphi_r(1) = r$.

Suppose $\varphi \in \mathrm{End}_R R$, $\varphi(1) = r$

Then $\varphi(x) = \varphi(x \cdot 1) = x\varphi(1) = xr$

So $\varphi = \varphi_r$.

Note: ~~$R \cong R^{opp}$ iff $\exists$~~ $\mathbb{C}[G] \cong \mathbb{C}[G]^{opp}$, $M_{m_r}(k) \cong M_{m_r}(k)^{opp}$

Corollary: $\mathbb{C}[G] = \bigoplus_{i=1}^{n} M_{m_r}(k)$

Defn: R is semisimple if ${}_R R$ can be written as a sum of simple modules.

Corollary: $R \cong \bigoplus_{i=1}^{n} M_{m_r}(k)$

Corollary: $\dim R = \sum_{r=1}^{n} m_r^2$.

Lemma: If V is a simple R-module (R semisimple) then V occurs in the decomposition of R into simples.

Pf: Take $v \neq 0$, $v \in V$.

Define $\varphi_v : R \to RV$ by

$\varphi_v(r) \longmapsto rv$

This must be a surjective homomorphism of R-modules.

$\therefore$ V occurs in R.

Corollary: $n \leq \dim R$

Exact value: $n = \dim(ZR)$.

Pf: $Z M_{m_r}(k) = 1$.

Example: $R =$ ~~$\mathbb{C}[G]$~~ $k[G]$ what is the centre?

$1_g \sum a_x 1_x = \left(\sum a_x 1_x\right) 1_g$

$a_{g^{-1}x} = a_{xg^{-1}} \quad \forall\ x, g \in G$

$a_x = a_{gxg^{-1}}$ — class fns.

$\therefore$ n = no. of conjugacy classes in G.

Example: $R = M_m(k)$

${}_R R \cong V^{\oplus m}$, where V is simple of dimension m.

Indeed, can take $V = k^m$.

$M_m(k)$ acts on the left of $M_m(k)$

$$A(v_1|v_2|\dots|v_m) = (Av_1|Av_2|\dots|Av_m)$$

Taking $k^m \to$ a column of $M_m(k)$

$$v \mapsto (0|\dots|0|v|0|\dots|0)$$

is a homom.

& $M_m(k)$ is a sum of k such copies.

Corollary: If R is semisimple, and

$R \cong \bigoplus_{i=1}^{n} M_{m_i}(k)$

then the simple R-modules have dimension

$m_1, \dots, m_n$ & mult. $m_1, \dots, m_n$.

Coroll. $|G| = \sum m_i^2$.

$R = \bigoplus_{i=1}^{n} \mathrm{End}_k S_i$ $S_1, \dots, S_n$ are the simple R-modules.

$1 = e_1 + \dots + e_n$.

where e_i = identity matrix of $\mathrm{End}_k S_i$.

$\to 1 = \sum_{i=1}^{n} e_i$ $\to e_i = e_i' + e_i''$ where e_i' & e_i'' are ... $\Rightarrow e_i' = 0$ or $e_i'' = 0$.

$\to e_i \in ZR$

$\to e_i e_j = \delta_{ij}$

e_i's are orthogonal primitive central idempotents.

(9)

Corollary: If R is a semisimple ring (k alg. closed), Decomp $1 = \varepsilon_1 + \cdots + \varepsilon_n$ into a sum of primitive central idempotents is unique.

Problem: In $\mathbb{C}[G]$,

$$1 = \varepsilon_1 + \cdots + \varepsilon_n$$

$$n = \text{no. of conj. classes.}$$

Can think of each $\varepsilon_i = \sum \cancel{f(g)}\, \chi_i(g)\, 1_g$.

Qn: what are these fns. $\chi_1, \dots, \chi_n$?

Formal properties

- ε_i central $\Rightarrow$ χ_i is a class fn.
- $\varepsilon_i \varepsilon_j = \delta_{ij}\varepsilon_i \Rightarrow \sum \chi_i(g^{-1})\chi_j(g) = \delta_{ij}\chi_i(e)$
- $\sum \chi_i(e) = 1$
- $\sum \chi_i(g) = 0$ if $g \neq e$.
- $\chi_1, \dots, \chi_n$ form a basis for class fns.

Explicit ~~decomposition~~ iso:

$$\bigoplus_{i=1}^{n} \mathrm{End}_G V_i \longrightarrow \mathbb{C}[G]$$

$$v \otimes \xi \longmapsto x \mapsto \xi(x) v.$$

$v \in V_i$, $\xi \in \tilde{V}_i$.

- $V \otimes \tilde{V} \cong \mathrm{End}_G V$.
- $V \otimes \tilde{V} \cong \mathrm{End}\, V$

$$\begin{tikzcd} V \otimes \tilde{V} \arrow[r, "\cong"] \arrow[d, "v\otimes\xi \mapsto \xi(v)"'] & \mathrm{End}\, V \arrow[dl, "tr"] \\ k & \end{tikzcd}$$

• under this iso, Id_V is obtained as follows:

let $v_1, \dots, v_n$ be any basis of V

$\xi_1, \dots, \xi_n$ be the dual basis of $\tilde{V}$.

Then $Id_V \longleftrightarrow \sum v_i \otimes \xi_i$.

Define $V_i \otimes \tilde{V}_i \longrightarrow \mathbb{C}[G]$ by

$$v \otimes \xi \longmapsto c_{v,\xi}(g)$$

where $c_{v,\xi}(g) = \xi(\rho_i(g) v)$.

<u>Note:</u> $V_i \otimes \tilde{V}_i$ is a simple $G \times G^{opp}$-module

$v \otimes \xi \mapsto c_{v,\xi}$ is not identically 0.

∴ it must be an iso. onto its image.

So it embeds $\dim \tilde{V}_i = \dim V_i$ copies of V_i in $\mathbb{C}[G]$.

$$\begin{tikzcd}
\bigoplus \mathrm{End}_{\mathbb{C}} V_i \arrow[r] \arrow[d, equal] & \mathbb{C}[G] \\
\bigoplus V_i \otimes \tilde{V}_i \arrow[ur] &
\end{tikzcd}$$

$(v \otimes \xi) \mapsto$ ~~$\xi(g)$~~ $g \mapsto \xi(\rho_i(g) v)$

$= \Theta(\rho_i(g)[v \otimes \xi])$

$$T \mapsto tr(T \circ \rho_i(g))$$

In partic. $Id \mapsto tr(\rho_i(g))$

this is called the character of ρ_i.

~~In particular~~,

$$\chi_i(g) = \cancel{T \circ \rho_i(g)} \quad tr(\rho_i(g))$$

$\therefore$ ~~upto a scalar,~~ $\cancel{\varepsilon_i(g) = tr(T \circ \rho_i(g))}$

$$\therefore \quad \varepsilon_i = \cancel{c_i tr(T \circ \rho_i(g))} \; c_i \chi_i$$

where $c_i \in \mathbb{C}$ for $i = 1, \ldots, n$.

Observe: $\mathbb{C}[G] = \bigoplus V_i^{\oplus d_i}$

where $d_i = \dim_{\mathbb{C}} V_i$.

So & $\operatorname{trace}(g ; \mathbb{C}[G]) = \sum d_i \, tr(\pi(g) ; V_i)$

$$|G| \delta_{ge} = |G| 1_{\mathbb{C}[G]} = \sum d_i \chi_i$$

$$\therefore \quad \text{or} \quad 1_{\mathbb{C}[G]} = \sum \frac{d_i}{|G|} \chi_i .$$

On the other hand

$$1_{\mathbb{C}[G]} = \sum \varepsilon_i$$

by linear independence of the ε_i's we have

$$\boxed{\varepsilon_i = \frac{d_i}{|G|} \chi_i}.$$

<u>Theorem</u>: Let G be a finite gp, (ρ, V) a rep $(\dim V < \infty)/\mathbb{C}$.
Then V admits a G invariant Hermitian inner product.

<u>Pf:</u> Let $\langle\, , \rangle$ be any inner product.

Define $(v, w) = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g) v, \rho(g) w \rangle$

Then $(\rho(x) v, \rho(x) w) = (v, w)$.

need to check positivity.

$$(x, x) = 0 \Rightarrow \tfrac{1}{|G|} \sum \langle \rho(g) x, \rho(g) x \rangle = 0.$$
$$\Rightarrow \langle x, x \rangle = 0$$
$$\Rightarrow x = 0. \qquad \text{QED.}$$

$\therefore$ with respect to an o.n.b, $\rho(g)$ ~~is a unitary~~ or $\rho(g)$ is unitary $\rho(g)^* = \rho(g)^{-1} = \rho(g^{-1})$.

$\therefore \overline{\operatorname{tr} \rho(g)} = \operatorname{tr} \rho(g^{-1})$.

$\therefore$ the identity

$\varepsilon_i^2 = \varepsilon_i$ becomes

$$\sum \frac{d_i^2}{|G|^2} \chi_i(g) \overline{\chi_i(g)} = \frac{d_i}{|G|} d_i$$

or $\frac{1}{|G|} \| \chi \|_{\ell^2(G)} = 1$

$\varepsilon_i \varepsilon_j = 0 \quad i \neq j$ becomes: $\langle \chi_i, \chi_j \rangle = 0$ for $i \neq j$

Schur's orthogonality rl.