COMBINATORICS IN REPRESENTATION THEORY

MIDSEMESTER EXAMINATION

(1) Compute the character values of the representation $V_{(2,2,1)}$ of S_5.

(2) Write $\mathbf{n}$ for $\{1, \ldots, n\}$. Let $V = \mathbf{C}[\mathbf{n}]$. What is the decomposition of $V \otimes V$ into irreducibles as a representation of S_n? [Hint: express $V \otimes V$ as a permutation representation.]

(3) Show that

$$\prod_{i,j}(1 + x_i y_j) = \sum_{\lambda} m_\lambda(x) e_\lambda(y).$$

[Hint: the coefficient of $x^\lambda y^\mu$ is given by counting $(0, 1)$-matrices with row sums λ and column sums μ.]

(4) For each partition λ of n show that the number of permutations in S_n whose cycle decomposition is a partition of type λ is

$$\frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots}$$

where m_i is the number of times that i occurs in λ.

(5) Let $E : \Lambda_n \to \Lambda_n$ be the linear involution which takes s_λ to $s_{\lambda'}$. Show that $E(p_\lambda) = \epsilon(w_\lambda)p_\lambda$, where w_λ is any element whose cycle decomposition is of type λ and ϵ is the sign character.

Date: 9th March 2011, 9:00AM-11:00AM.