COMBINATORICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 15TH FEBRUARY 2011

(1) Show that the derived group of the symmetric group S_n is the alternating group A_n.
(2) If $g \in \mathrm{GL}_n(\mathbf{F}_q)$ and $a \in \mathbf{F}_q^*$, then show that the conjugacy classes of g and ag have the same type.
(3) Prove or disprove: if $\rho \in \hat{G}$ and $\chi \in X^1(G)$, then ρ is isomorphic to $\rho \otimes \chi$ if and only if $\chi \equiv 1$.
(4) Recall that a type is an equivalence class of functions
$$f : \mathrm{Irr}(\mathbf{F}_q[t]) \to \Lambda$$
where $f(p) = \emptyset$ for all but finitely many p. The functions f_1 is said to be equivalent to f_2 if and only if there is a degree-preserving bijection $\phi : \mathrm{Irr}(\mathbf{F}_q[t]) \to \mathrm{Irr}(\mathbf{F}_q[t])$ such that $f_2 = f_1 \circ \phi$. Characterize the types corresponding to semisimple matrices, and the types which are similar to the companion matrix of their characteristic polynomials.
(5) Let ψ be any non-trivial homomorphism $\mathbf{F}_q \to \mathbf{C}^*$. Show that the map $\mathbf{F}_q \to X^1(\mathbf{F}_q)$ defined by $x \mapsto \psi_x$ where
$$\psi_x(y) = \psi(xy)$$
is an isomorphism of abelian groups.