COMBINATORICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 25/01/2011

(1) In either of the cases:
$$G = S_n, \quad X_k = \{x \subset \{1, \ldots, n\} : |x| = k\},$$
$$G = GL_n(\mathbf{F}_q), \quad X_k = \{x \subset \mathbf{F}_q^n : \dim x = k\}.$$
show that there exists an isomorphism of G-sets $X_k \to X_{n-k}$ for all $0 \leq k \leq n$. Conclude that $\mathbf{C}[X_k]$ and $\mathbf{C}[X_{n-k}]$ are isomorphic representations of G.

(2) With notation as in the previous problem, show that
$$|X_k| > |X_{k-1}| \text{ if } k \leq n/2.$$
(The sequence $|X_k|$, $k = 0, \ldots, n$ is *unimodal.*)

(3) Give a combinatorial proof[1] of the *Pascal identity* for Gaussian binomial coefficients:
$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q.$$
Deduce from it a second type of Pascal identity
$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q.$$
Optional challenge: give a combinatorial proof of the second Pascal identity.

(4) Use Pascal's identities above to prove the formal identities
$$\prod_{k=0}^{n-1} (1 + q^k t) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q t^k$$
$$\prod_{k=0}^{n-1} \frac{1}{1 - q^k t} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q t^k.$$

(5) From the second identity deduce that $\binom{n+k}{k}_q$ is a polynomial in q where the coefficient of q^i is the number of partitions with at most k parts, each part being no larger than n.

[1] A combinatorial proof is one which shows that two numbers are equal by giving a bijection between sets with those numbers as cardinalities.