COMBINATORICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 18/01/2011

(1) Calculate the number of elements in each conjugacy class of S_4. For the action of S_4 on $\mathbf{C}^4$ by permutation matrices, determine the dimensions and multiplicities of irreducible representations using characters.

(2) Decompose the representation of S_3 induced from the nontrivial representation of $S_2 \times S_1$ (a subgroup of order two) into irreducible representations.

(3) Let V be a three-dimensional vector space over a field k. Consider the set X of pairs (P, l), where P is a plane and l is a line such that $l \subset P$. Then $GL(V)$ acts on $X \times X$ by $g((P, l), (P', l')) = ((gP, gl), (gP', Gl'))$. How many orbits does $GL(V)$ have on $X \times X$?

(4) Let V be an n-dimensional vector space over a finite field k. Let $X = \mathrm{Gr}_k(V)$, the set of k-dimensional subspaces of V, which is a $GL(V)$-space. Show that $(W, W') \mapsto \dim(W \cap W')$ is a complete invariant of relative position. What is the range of this map?