COMBINATORICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 13/01/2011

(1) Suppose R and S are families of linear maps on finite dimensional complex vector spaces V and W respectively, for which there are no non-trivial proper invariant subspaces. Show that $U \otimes_{\mathbf{C}} V$ has no non-trivial proper subspace invariant under the linear maps $\{A \otimes B | A \in R,\ B \in S\}$.

(2) Let (ρ, V) be a representation of a finite group in a finite dimensional complex vector space. With respect to a fixed basis of V, let E_{ij} denote the matrices where all entries are 0 except the (i,j)th one, which is 1. Show that (ρ, V) is irreducible if and only if the matrix

$$\sum_{g \in G} \rho(g)^{-1} E_{ij} \rho(g)$$

is scalar for all (i,j). Hint: Use the following (fairly obvious) fact: if $T \in \mathrm{End}_G V$, then there exists $S \in \mathrm{End}_{\mathbf{C}} V$ such that

$$T = \sum_{g \in G} \rho(g)^{-1} S \rho(g).$$