COMBINATORICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 26 MARCH 2011

(1) Prove that the polynomial representation ring of $\mathbf{C}^* = GL_1(\mathbf{C})$ is isomorphic to $\mathbf{Z}[x]$.
(2) Let $x = \mathrm{diag}(x_1, \ldots, x_n)$. Show that $\mathrm{trace}(x; \wedge^k \mathbf{C}^n)$ is e_k, the kth elementary symmetric polynomial in n variables (assume that $n \geq k$)[1].
(3) With notation as above, show that $\mathrm{trace}(x; \mathrm{Sym}^k \mathbf{C}^n)$ is h_k, the kth complete symmetric polynomial in n variables[2].
(4) Let $p(x_1, \ldots, x_n) \in \mathbf{C}[x_1, \ldots, x_m]$. Show that, if $p(x_1, \ldots, x_n) = 0$ whenever $|x_1| = \cdots = |x_n| = 1$, then p is identically 0.
(5) Assume that $U(n)$, the group of $n \times n$ unitary matrices has a translation invarant Borel measure (this can be shown by constructing a non-zero invariant differential form of top degree). Use this to show that every finite dimensional polynomial representation $\rho : U(n) \to \mathrm{GL}_{\mathbf{C}}(V)$ for which $g \mapsto \rho(g)v$ is a sum of simple representations.
(6) Show that $\{r_n(h_\lambda) | l(\lambda) \leq n\}$ is a basis for $\Lambda(n)$, the space of symmetric polynomials in n variables (recall that r_n is the specialization map $\Lambda(n) \to \Lambda$).
(7) Is $r_n(h_\lambda) = 0$ for all λ such that $l(\lambda) > n$?

[1] $\wedge^k \mathbf{C}^n$ denotes the space of alternating tensors of degree k on $\mathbf{C}^n$
[2] $\mathrm{Sym}^k \mathbf{C}^n$ denotes the space of symmetric tensors of degree k on $\mathbf{C}^n$