COMBINATORICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 11/01/2011

(1) Let $C(G;k)$ denote the space of all function $G \to k$. Define $\Phi : C(G;k) \to k[G]$ by
$$\Phi(f) = \sum_{g \in G} f(g) 1_g.$$
Note that Φ is an isomorphism of vector spaces. Let $f_1 * f_2$ denote the unique element of $C(G;k)$ for which
$$\Phi(f_1 * f_2) = \Phi(f_1)\Phi(f_2).$$
Show that
$$f_1 * f_2(g) = \sum_{xy=g} f_1(x) f_2(y) \text{ for all } g \in G.$$

(2) Let $G = \mathbf{Z}/n\mathbf{Z}$, $V = C(G;k)$. Define $(\rho(g)f)(x) = f(x-g)$ for all $f \in C(G;k)$, $x, g \in G$. Determine the one-dimensional invariant subspaces for the representation (ρ, V).

(3) Let X be any collection of $n \times n$ matrices which commute pairwise and have complex entries. Show that $\mathbf{C}^n$ has a basis with respect to which every matrix in X is upper triangular, and every diagonalizable matrix in X is diagonal. [Hint: Assume the result to be true for all $m < n$. If X consists only of scalar matrices, then there is nothing to prove. Otherwise there exists $A \in X$ which has a non-trivial proper eigenspace.]

(4) Show that any irreducible representation of an abelian group in a finite dimensional complex vector space is one-dimensional. For a finite abelian group, show that every indecomposable representation of this type is one-dimensional. Give an example of a indecomposable representation of a (necessarily infinite) abelian group which is not one-dimensional.

(5) Write down the character table for the symmetric group S_3 consisting of permutations of three objects.