Combinatorics of finite abelian groups and the Weil representation

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Motivation

We are interested in understanding the irreducible representations of $GL_n(\mathbf{Z}/p^k\mathbf{Z})$.

For $k = 1$, their characters were computed in (Green, *Trans. AMS*, 1955).

For $k > 1$, the problem is considered intractable.

It is still interesting if you ask the right questions.

When Clifford theory is applied to $GL_2(\mathbf{Z}/p^2\mathbf{Z})$, understanding the representation theory of groups of the form

$$\mathrm{A}ut_{k[t]}(k[t]/t^{\lambda_1} \oplus \cdots \oplus k[t]/t^{\lambda_l}) \quad (k = \mathbf{Z}/p\mathbf{Z})$$

also becomes important (Aubert et. al., *Israel J. Math.*, 2010 and Singla, *J. Algebra*, 2010).

Thus, we are interested in the representation theory of automorphism groups of finite abelian groups and their analogs for other Dedekind domains.

Thse results are motivated by the desire to understand how the Weil representation can be used to construct irreducible representations of such groups.

Weyl operators

A: finite abelian group *of odd order*
$\hat{A}$: Pontryagin dual of A (homomorphisms $A :\to U(1)$)
$K = A \times \hat{A}$: phase space

Weyl operator

For each $k = (x, \chi) \in A \times \hat{A} = K$, a unitary operator $W_k : L^2(A) \to L^2(A)$

$$W_k f(u) = \chi(u - x/2) f(u - x) \text{ for } f \in L^2(A),\ u \in A.$$

Heisenberg group

Composition of Weyl operators

$$W_k W_l = c(k, l) W_{k+l} \text{ for some } c(k, l) \in U(1).$$

The standard symplectic cocyle

If $k = (x, \chi)$ and $l = (y, \lambda)$ then

$$c(k, l) = \chi(y/2)\lambda(x/2)^{-1}$$

Heisenberg Group

$$H = \{zW_k : z \in U(1), k \in K\} \subset U(L^2(A)).$$

As defined, H comes with a unitary representation on $L^2(A)$ known as the *Schrödinger representation*.

Stone-von Neumann-Mackey Theorem

Theorem

1. *The Shrödinger representation is irreducible.*
2. *If $\rho : H \to U(\mathcal{H})$ is an irreducible unitary representation such that $\rho(zW_0) = z\mathrm{Id}_{\mathcal{H}}$ for every $z \in U(1)$,there exists a (unique up to scaling) isometry $W : L^2(A) \to \mathcal{H}$ such that*

$$WW_k = \rho(W_k)W$$

Weil representation

Symplectic group

$$Sp(K) := \{g : K \to K | c(gk, gl) = c(k, l)\}$$

For each $g \in Sp(K)$ define a representation of H by

$$\rho_g(k) = W_{g(k)}.$$

Then ρ_g is an irreducible unitary representation of H on $L^2(A)$ such that $\rho_g(cW_0) = c\mathrm{Id}_{L^2(A)}$.
By Stone-von Neumann, there exists (unique up to scaling) $W_g \in U(L^2(A))$ such that

$$W_g W_k W_g^* = W_{g(k)}.$$

Weil representation: $g \mapsto W_g$ is a representation of $Sp(K)$ on $L^2(A)$.

The problem

Investigate the decomposition

$$L^2(A) = \bigoplus_{\pi \in \widehat{Sp(K)}} m_\pi V_\pi.$$

$\widehat{Sp(K)}$: set of equivalence classes of irreducible representations.
m_π: multiplicity of π in the Weil representation.

Our results

1. $m_\pi \leq 1$ (multiplicity one)
2. Parametrization of the π's with $m_\pi = 1$
3. Computation of $\dim V_\pi$ for each such π

Examples

$A = (\mathbf{Z}/p\mathbf{Z})^n$

$$L^2(A) = [\text{Even functions}] \oplus [\text{Odd functions}]$$

(Gérardin, *J. Algebra*, 1977)

$A = (\mathbf{Z}/p^k\mathbf{Z})^n$

$$L^2(A) = \bigoplus_{r=0}^{\lfloor k/2 \rfloor} (U_{k-2r}^{+} \oplus U_{k-2r}^{-})$$

where

$$\dim U_{k-2r}^{\pm} = \begin{cases} (p^{n(k-2r)} - p^{n(k-2r-2)})/2 & \text{if } 0 < r < \lfloor k/2 \\ (\pm 1 + p^{n(k-r)})/2 & \text{if } r = \lfloor k/2 \rfloor \end{cases}$$

(Dipendra Prasad, *CRM lect. notes* 1993) for n even
(Cliff, McNeilly, Szechtman, *J. London Math. Soc.* 2000)

Main Idea

Understand $\mathrm{End}_{Sp(K)} L^2(A)$.

Useful fact

The Weyl operators W_k, $k \in K$ form a basis for $\mathrm{End}_{\mathbf{C}} L^2(A)$.

So every linear map $T : L^2(A) \to L^2(A)$ has an expansion

$$T = \sum_{k \in K} t_k W_k$$

In addition, T should commute with W_g for all $g \in Sp(K)$.
Applying the defining identity $W_g W_k W_g^* = W_{g(k)}$, we get

Theorem (Prasad, *J. Analysis* 2010)

T commutes with W_g for all $g \in Sp(K)$ if and only if $t_k = t_{g(k)}$ for all $g \in Sp(K)$, $k \in K$.

The condition can be rephrased as $k \mapsto t_k$ is constant on $Sp(K)$-orbits in K.

Understanding orbits

In order to understand $\mathrm{End}_{Sp(K)} L^2(A)$, we need to understand the $Sp(K)$-orbits in $K = A \times \hat{A}$. We begin with a simpler problem:

Problem

Understand the $\mathrm{A}ut(A)$-orbits in A

This problem has been studied for more than a 100 years:

- Miller (1905)
- Baer (1935)
- Birkhoff (1935)

They consider the set of orbits, with no additional structure.
We (Dutta and Prasad, preprint 2010) found it much more useful to think of the orbits as a partially ordered set.

Degeneration partial order

Definition (Degeneration)

For $x, y \in A$ say x degenerates to y, denotes $x \to y$, if there is an endomorphism ϕ of A such that $\phi(x) = y$.

Now assume

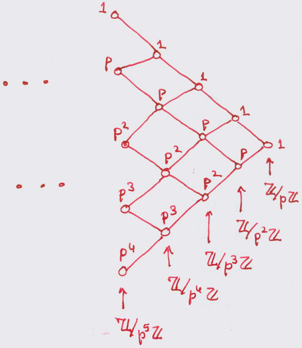
$$A = \mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_l}\mathbf{Z}.$$

so we can write any $x \in A$ as

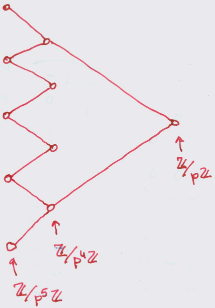
$$x = (x_1, \ldots, x_l)$$

with $x_i \in \mathbf{Z}/p^{\lambda_i}\mathbf{Z}$.

Construction of P_λ

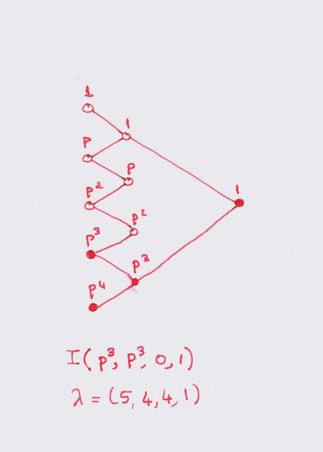


P



$P_{(5,4,4,1)}$

Orbits in A



- Given $x \in A$, plot it in P_λ.
- $I(x)$ is the order ideal generated by the plot in the poset P_λ.

Theorem (Dutta and Prasad)

- *For $x, y \in A$, x and y are in the same $\mathrm{Aut}(A)$-orbit if and only if $I(x) = I(y)$.*
- *The map $x \mapsto I(x)$ gives a bijection between the set of $\mathrm{Aut}(A)$-orbits in A and the partially ordered set of order ideals in P_λ.*

Thus the set of orbits can be thought of as the finite distributive lattice of order ideals in P_λ.

Back to Weil representations

$$Sp(K)\text{-orbits in } K \leftrightarrow \mathrm{Aut}(A)\text{-orbits in } A.$$

Since

$$\mathrm{End}_{Sp(K)} L^2(A) = \left\{ \sum_{k \in K} t_k W_k \mid k \mapsto t_k \text{ is constant on } Sp(K)\text{-orbits} \right\}$$

The characteristic functions of orbits form a basis of this algebra:

$$\mathcal{B} := \{ T_I : I \subset P_\lambda \text{ (ideal)} \}.$$

Multiplicity freeness is equivalent to showing that T_I commutes with T_J for any order ideals $I, J \subset P_\lambda$.
Unfortunately $T_I \circ T_J$ is hard to compute directly.

A tractable basis

If we let

$$\mathcal{T}_I = \sum_{J \subset I} T_J,$$

then the support of each $\mathcal{T}_I$ turns out to be a nice subgroup of K; for this reason $\mathcal{T}_I \circ \mathcal{T}_J$ can be calculated:

$$\mathcal{T}_I \circ \mathcal{T}_J = |K_{I \cap J}| \mathcal{T}_{(I \cap J)^\perp \cap (I \cup J)}$$

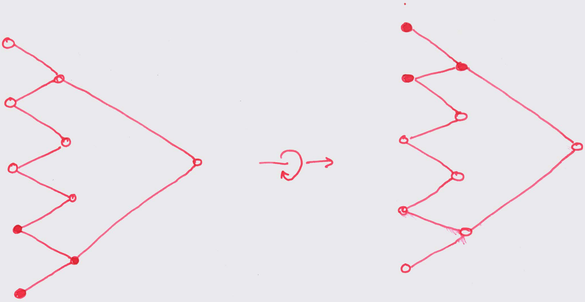
Since the transformation $T_I \mapsto \mathcal{T}_I$ is upper triangular, $\mathcal{T}_I$'s also form a basis of $\mathrm{End}_{Sp(K)} L^2(A)$.
This proves multiplicity one.

Reflection and small order ideals

The lattice of ideals in P_λ admits a reflection: $I \mapsto I^\perp$.

Definition (Small ideal)

An order ideal $I \subset P_\lambda$ is small if $I \subset I^\perp$.



I (Red dots)

$I^{\perp}$ (White dots)

Characteristic subgroups and invariant subspaces

For each order ideal $I \subset P_\lambda$,

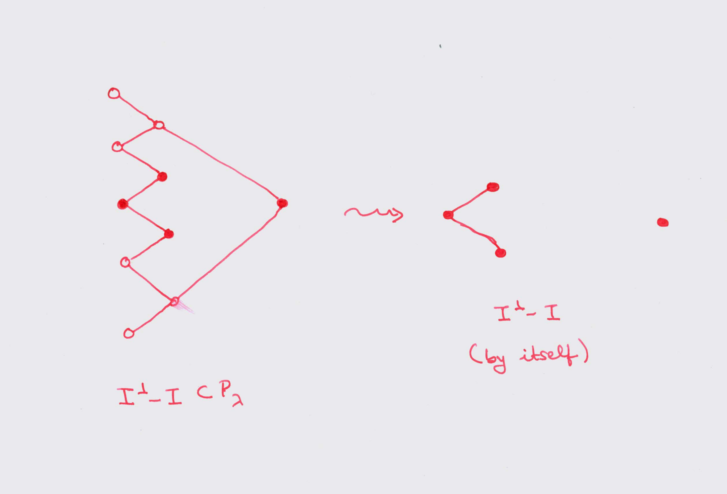
$$\begin{aligned} A_I : &= \{x \in A \mid I(x) \subset I\} \\ &= \coprod_{J \subset I} O_J \end{aligned}$$

forms a characteristic subgroup of A.

Theorem (Relatively easy)

For each small ideal I, the spaces of even and odd functions in $L^2(A_{I^\perp}/A_I)$ are $Sp(K)$-invariant subspaces of $L^2(A)$.

The poset $I^\perp - I$



A refinement

Theorem (A little harder)

For each small ideal I, and function $\Pi : h_0(I^\perp - I) \to \{\pm 1\}$ the subspace $L^2(A)_{I,\Pi}$ of $L^2(A_{I^\perp}/A_I)$ consisting of functions which are even in variables corresponding to coordinates coming from $\Pi^{-1}(1)$ and odd in variables coming from $\Pi^{-1}(-1)$ is $Sp(K)$-invariant.

These subspaces form a poset under inclusion, which can be characterized in terms of the combinatorial data (I, Π):
$L^2(A)_{I,\Pi} \subset L^2(A)_{I',\Pi'}$ if and only if

1. $I' \subset I$
2. For each $C' \in h_0(I'^\perp - I')$, $\Pi'(C') = \prod_{C \in h_0(I^\perp - I), C \subset C'} \Pi(C)$

These conditions define a poset structure Q_λ on the set of pairs (I, Π).

And these subspaces are enough

Theorem

For each $(I, \Pi) \in Q_\lambda$, *there exists a unique* $Sp(K)$*-irreducible subspace* $V_{I,\Pi}$ *of* $L^2(A)_{I,\Pi}$ *that is not contained in* $L^2(A)_{I',\Pi'}$ *for any* $(I', \Pi') < (I, \Pi)$. *Every irreducible* $Sp(K)$*-invariant subspace of* $L^2(A)$ *is of the form* $L^2(A)_{I,\Pi}$ *for some* $(I, \Pi) \in Q_\lambda$.

Calculating dimension

The dimension of each $L^2(A)_{I,\Pi}$ is not hard to calculate:

$$\dim L^2(A)_{I,\Pi} = \prod_{C \in h_0(I^\perp - I)} \frac{p^{[C]} + \phi(C)}{2}.$$

Since

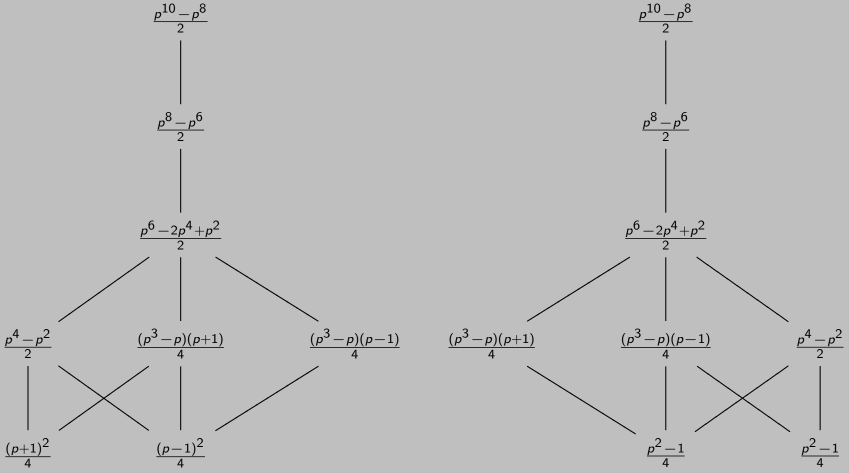
$$\dim L^2(A)_{I,\Pi} = \sum_{(I',\Pi') \leq (I,\Pi)} \dim V_{I',\Pi'},$$

$$\dim V_{I,\Pi} = \sum_{(I',\Pi') \leq (I,\Pi)} \mu((I,\Pi),(I',\Pi')) \dim L^2(A)_{I',\Pi'}$$

where μ is the Möbius function of Q_λ.
Working it all out gives:

$$\dim V_{I,\Pi} = \begin{cases} p^{[I^\perp - I]} \prod_{x \in \max I^\perp} (1 - p^{-2m(x)})/2 & \text{if } |I^\perp - I| \geq 2, \\ (p^{[I^\perp - I]} + \Pi)/2 & \text{if } |I^\perp - I| = 1, \\ 1 & \text{if } |I^\perp - I| = 0. \end{cases}$$

Example: $\lambda = (4, 3, 2, 1)$



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