Cohomological Detection of Algebraic Cycles

by

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CERTIFICATE

This is to certify that the Ph.D. thesis submitted by G.V.Ravindra to the University of Madras, entitled **Cohomological Detection of Algebraic Cycles** is a record of bonafide research work done by him under my supervision. The research work presented in this thesis has not formed the basis for the award to the candidate of any Degree, Diploma, Associateship, Fellowship or other similar titles.

It is further certified that the thesis represents independent work by the candidate and collaboration when existed was necessitated by the nature and scope of problems dealt with.

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Contents

CHAPTER 1

Preliminaries

0.1. Algebraic cycles. In this section we collect certain definitions and facts about algebraic cycles that will be used in the following chapters.

Let X be an algebraic variety (i.e., irreducible, reduced, algebraic scheme) defined over k, an algebraically closed field.

DEFINITION 1.1. A *p-cycle* on X is a finite formal sum

$$\sum n_i V_i$$

where the V_i are p-dimensional irreducible subvarieties of X, and the n_i are integers.

DEFINITION 1.2. *The group of p-cycles on* X, denoted by $Z_p(X)$, is the free abelian group on the p-dimensional subvarieties of X.

For any $(p+1)$-dimensional subvariety W of X, and any $r \in R(W)^*$, define a p-cycle $[div(r)]$ by

$$[div(r)] = \sum ord_V[r],$$

the sum over all codimension one subvarieties V of W; here $R(W)$ is the field of rational functions of W and $ord_V[r]$ is the order of vanishing of r along V and is well-defined even when W has singularities along V (see [**13**]).

DEFINITION 1.3. A p-cycle α is said to be *rationally equivalent to* 0, written $\alpha \sim_{rat} 0$, if there are a finite number of $(p+1)$-dimensional subvarieties W_i of X, and $r_i \in R(W_i)^*$, such that

$$\alpha = \sum [div(r_i)].$$

Since $[div(r_i)] = -[div(r_i^{-1})]$, the cycles rationally equivalent to zero form a subgroup $Rat_p(X)$ of $Z_p(X)$.

DEFINITION 1.4. The *Chow group* of p-dimensional cycle classes is the group of p-cycles modulo rational equivalence on X.

$$\mathrm{CH}_p(X) = Z_p(X)/Rat_p(X).$$

We denote by $\mathrm{CH}^p(X)$, the group of codimension p cycle classes, i.e., $\mathrm{CH}^p(X) := \mathrm{CH}_{dim X - p}(X)$.

0.1.1. *Flat Pull-backs and Proper Push-forwards.* For a flat morphism $f : X \to Y$ of varieties, there exists a well-defined functorial map

$$f^* : \mathrm{CH}^p(Y) \to \mathrm{CH}^p(X)$$

called the *pull-back* morphism which is defined as follows.

For an irreducible subvariety $V \subset Y$ of codimension p, let η be a generic point. Then the generic points $\{p_i\}$ of $f^{-1}(V)$ are all of codimension p (by flatness). Thus the coordinate ring of $f^{-1}(\eta)$ is an Artin ring A supported at the points $\{p_i\}$. Hence we may define,

$$f^*(V) = \textstyle\sum_i l(A_{p_i}).[\overline{p_i}]$$

where l denotes the length function associated to modules.

We then extend the above map to a map of $Z^p(X)$ by linearity. One can then check that this map descends to a map of Chow groups (see [**13**]).

For a proper morphism $f : X \to Y$ there exists a well-defined functorial map $f_* : \mathrm{CH}_p(X) \to \mathrm{CH}_p(Y)$ called the *push-forward* which is defined as follows.

For a subvariety $V \subset X$, let $W = f(V)$. We define the map at the level of cycles and refer to [**13**] to say that the map descends to a map of Chow groups.

$$\begin{aligned} V &\mapsto deg(f|V).W \quad if \quad dim(W) = dim(V) \\ &\mapsto \quad 0 \quad if \quad dim(W) < dim(V) \end{aligned}$$

As above, the map is extended by linearity to a map $f_* : Z_p(X) \to Z_p(Y)$.

0.2. Hodge Theory. In this section we state some facts about Hodge theory that will be used in later chapters.

DEFINITION 1.5. (see [**11**]) A *pure Hodge structure* of weight n is a pair $(V_{\mathbb{Z}}, \mathrm{F}^{\bullet})$ where (a) $V_{\mathbb{Z}}$ is a finitely generated abelian group.

(b) if $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$, then $\{\mathrm{F}^p V_{\mathbb{C}}\}_{p \in \mathbb{Z}}$ is a finite decreasing filtration of complex subspaces such that for all $p \in \mathbb{Z}$,

$$\mathrm{F}^p V_{\mathbb{C}} \cap \overline{\mathrm{F}^{n-p+1} V_{\mathbb{C}}} = (0)$$

and

$$\mathrm{F}^p V_{\mathbb{C}} \oplus \overline{\mathrm{F}^{n-p+1} V_{\mathbb{C}}} = V_{\mathbb{C}}$$

Here, if $W \subset V_{\mathbb{C}}$ is a complex subspace then $\overline{W} := \{\overline{w} | \ w \in W\}$. Equivalently, (b) can be written as

(c) if $V^{p,q} = \mathrm{F}^p V_{\mathbb{C}} \cap \overline{\mathrm{F}^q V_{\mathbb{C}}}$ where $p + q = n$, then

$$V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$$

and,

$$\overline{V^{p,q}} = V^{q,p}.$$

We can recover $F^p V_{\mathbb{C}}$ from $V^{p,q}$ by setting $F^p V_{\mathbb{C}} = \oplus_{p' \geq p} V^{p', n-p'}$.

EXAMPLE 1.6. Let X be a smooth complex projective variety. Then the singular cohomology groups $V^n_{\mathbb{Z}} := H^n(X; \mathbb{Z})$ come equipped with a pure Hodge structure of weight n. Here the complexification $V^n_{\mathbb{C}} = H^n(X; \mathbb{C})$, is the cohomology with $\mathbb{C}$-coefficients. The filtration is such that $V^{p,q}$ is naturally isomorphic to $H^q(X, \Omega^p_X)$ where the latter are the Hodge cohomology groups.

DEFINITION 1.7. A *morphism* $f : (V_{\mathbb{Z}}, F^{\bullet}) \to (V'_{\mathbb{Z}}, F'^{\bullet})$ of pure Hodge structures of weight $2n$ is a homomorphism $f : V_{\mathbb{Z}} \to V'_{\mathbb{Z}}$ of abelian groups such that if $f_{\mathbb{C}} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$ is the induced $\mathbb{C}$-map, then $f_{\mathbb{C}}(F^p V_{\mathbb{C}}) \subset F^{p+n} V'_{\mathbb{C}}$ for all $p \in \mathbb{Z}$.

EXAMPLE 1.8. Suppose $Y \hookrightarrow X$ is an inclusion of smooth projective varieties. Then the Gysin morphisms $f_* : H^i(Y; \mathbb{Z}) \to H^{i+2d}(X; \mathbb{Z})$ where d is the codimension of Y in X, is a morphism of pure Hodge structures of weight $2d$.

LEMMA 1.9 (Deligne, [11]). Pure Hodge structures with morphisms as described above form an additive category. Moreover pure Hodge structures of a given weight n form an abelian category.

DEFINITION 1.10. A mixed Hodge structure is a triple $V = (V_{\mathbb{Z}}, W_{\bullet}, F^{\bullet})$ which consists of

(a) a finitely generated abelian group $V_{\mathbb{Z}}$.

(b) a finite increasing filtration $W_{\bullet} V_{\mathbb{Q}}$ on $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes \mathbb{Q}$, by $\mathbb{Q}$-subspaces,

(c) a finite decreasing filtration $F^{\bullet} V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$ by $\mathbb{C}$-subspaces,

such that, for all n, $F^{\bullet}$ induces a pure $\mathbb{Q}$-Hodge structure of weight n on $\mathrm{Gr}^W_n V_{\mathbb{Q}} = W_n V_{\mathbb{Q}} / W_{n-1} V_{\mathbb{Q}}$.

A morphism of mixed Hodge structures is a homomorphism $f : V_{\mathbb{Z}} \to V'_{\mathbb{Z}}$ of abelian groups such that if $f_{\mathbb{Q}} : V_{\mathbb{Q}} \to V'_{\mathbb{Q}}$ and $f_{\mathbb{C}} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$ are the induced $\mathbb{Q}$-linear and $\mathbb{C}$-linear maps respectively, then $f_{\mathbb{Q}}(W_k V_{\mathbb{Q}}) \subset W_k V'_{\mathbb{Q}}$ for all $k \in \mathbb{Z}$ and $f_{\mathbb{C}}(F^p V_{\mathbb{Q}}) \subset F^p V'_{\mathbb{C}}$ for all $p \in \mathbb{Z}$.

EXAMPLE 1.11. Consider the following long exact sequence for cohomology for a pair $Y \subset X$ where X is a variety and Y is any subvariety.

$$\cdots \to \mathrm{H}^i(X, Y; \mathbb{Z}) \to \mathrm{H}^i(X, \mathbb{Z}) \to \mathrm{H}^i(Y; \mathbb{Z}) \to \dots$$

This is a sequence of mixed Hodge structures (see [**11**] for more details).

DEFINITION 1.12. The p-th *Intermediate Jacobian* of a Hodge structure H, $J^p(H)$ is

$$J^p(H) := \frac{H_{\mathbb{C}}}{F^p H_{\mathbb{C}} + H}$$

LEMMA 1.13. The 0-th intermediate Jacobian of a pure Hodge structure H of weight -1 admits the following identification:

$$J^0(H_{\mathbb{Z}} \otimes \mathbb{Q}) \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H_{\mathbb{Z}} \otimes \mathbb{Q})$$

where MHS stands for the category of mixed Hodge structures.

0.3. Cycle Class map. For a smooth variety X over $\mathbb{C}$, one can define a cycle class map [**13**]

$$\mathrm{CH}^p(X) \xrightarrow{cl} \mathrm{H}^{2p}(X; \mathbb{Z})$$

where the group on the right is the singular cohomology group with coefficients in $\mathbb{Z}$. For a subvariety $V \subset X$ of codimension p, $cl(V)$ is its cohomology class. The map is extended by linearity. That the map factors through rational equivalence is checked in [**13**].

It has been shown [13] that the cycle class map defined above is functorial for pullbacks under flat morphisms and push-forward under proper morphism. More precisely, if $f : X \to Y$ is a flat morphism of smooth varieties, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{CH}^p(Y) & \xrightarrow{cl_Y} & \mathrm{H}^{2p}(Y,\mathbb{Z}) \\ \downarrow f^* & & \downarrow f^* \\ \mathrm{CH}^p(X) & \xrightarrow{cl_X} & \mathrm{H}^{2p}(X,\mathbb{Z}) \end{array}$$

and the vertical morphism on the right is a morphism of Hodge structures.

Similarly one has a diagram for f_* for a proper morphism of smooth varieties $f : X \to Y$ with $d = dim(X) - dim(Y)$,

$$\begin{array}{ccc} \mathrm{CH}^p(X) & \xrightarrow{cl_X} & \mathrm{H}^{2p}(X,\mathbb{Z}) \\ \downarrow f_* & & \downarrow f_* \\ \mathrm{CH}^{p+d}(Y) & \xrightarrow{cl_Y} & \mathrm{H}^{2p+2d}(Y,\mathbb{Z}) \end{array}$$

where the vertical morphism on the right is the Gysin morphism which is a morphsim in the (see example 1.8) category of Hodge structures.

The cycle class map is not injective in general. For instance if C is a smooth projective irreducible curve, then the difference of any two distinct points $p - q$, is a 0-cycle whose image under the cycle class map goes to zero. However $p - q \neq 0$ in $\mathrm{CH}_0(C)$ unless $C \cong \mathbb{P}^1$.

Now define, $\mathrm{CH}^p(X)_{hom} := \mathrm{Ker}(cl)$. Though $\mathrm{H}^{2p}(X,\mathbb{C})$ fails to capture the cycles in this group, there is an Abel-Jacobi map(see [15]) into the p-th intermediate Jacobian :

$$\mathrm{CH}^p(X)_{hom} \xrightarrow{\phi} J^p(\mathrm{H}^{2p-1}(X))$$

For a cycle $\alpha \in \mathrm{CH}^p(X)_{hom}$, let Z denote the support of α. Consider the diagram:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{H}^{2p-1}(X) & \to & \mathrm{H}^{2p-1}(X \setminus Z) & \to & \mathrm{H}^{2p}_Z(X) & \to & \mathrm{H}^{2p}(X) \\ & \| & & \cup & & \uparrow \alpha & & \\ 0 \to & \mathrm{H}^{2p-1}(X) & \to & E & \to & \mathbb{Z}[\alpha] & \to & 0 \end{array}$$

Here the top exact sequence is a part of the long exact sequence for cohomology with supports. By purity, $\mathrm{H}^{2p-1}_Z(X) = 0$. The bottom sequence is just the pull-back. Thus α gives an element in $\mathrm{Ext}^1(\mathbb{Z}, \mathrm{H}^{2p-1}(X))$.

One can also define the above map by integration.

By definition, there exists a cycle Γ of real codimension $2p-1$ such that $[\alpha]$ is the boundary of Γ. This gives a map

$$\begin{array}{ccc} Z^p(X)_{hom} & \to & \mathrm{H}^{2p-1}(X, \mathbb{C})^* \\ Z & \mapsto & (\omega \mapsto \int_\Gamma) \end{array}$$

It has been checked in [**13**] that this induces a well defined map

$$\phi : \mathrm{CH}^p(X)_{hom} \to J^p(\mathrm{H}^{2p-1}(X))$$

The Abel-Jacobi map too is not injective in general. In fact in [**24**], Mumford proves that for a smooth complex projective surface S with $p_g \neq 0$, the kernel of the Abel-Jacobi map is infinite-dimensional in a certain sense.

0.4. Algebraic de Rham cohomology. Let X be a smooth variety of dimension n over a field k. Let $\Omega^1_{X/k}$ be the sheaf of Kahler differentials. Consider the de Rham complex, denoted by $\Omega^\bullet_{X/k}$:

$$0 \to \mathcal{O}_X \to \Omega^1_{X/k} \to \ldots\ldots \to \Omega^n_{X/k} \to 0$$

The hypercohomology $\mathbb{H}^\bullet(X, \Omega^\bullet_{X/k})$ of the above complex is referred to as algebraic de Rham cohomology [**18**] and is denoted by $\mathrm{H}^\bullet_{DR}(X/k)$. For each i, $\Omega^i_{X_L/L} \cong \Omega^i_{X/k} \otimes_k L$, when L is a field containing k and X_L

is the base change to L. It follows that the hypercohomology too has this base change property, namely $\mathrm{H}^i_{DR}(X_L/L) \cong \mathrm{H}^i_{DR}(X/k) \otimes_k L$.

There exists cycle class maps (see [**18**])

$$\mathrm{CH}^i(X) \otimes \mathbb{Q} \to \mathrm{H}^{2i}_{DR}(X/k)$$

which map into the i-th filtered pieces $F^i \mathrm{H}^{2i}_{DR}(X/k) := \mathbb{H}^{2i}(\Omega^{\geq i}_{X/k})$. The cycle class map above is functorial with respect to base change. That is to say that the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{CH}^p(X) & \to & \mathrm{H}^{2p}_{DR}(X/k) \\ \downarrow & & \downarrow \\ \mathrm{CH}^p(X_L) & \to & \mathrm{H}^{2p}_{DR}(X_L/L) \end{array}$$

0.5. Deligne-Beilinson cohomology. Let X be a smooth variety over $\mathbb{C}$. The $2p$ *Deligne-Beilinson cohomology* (see [**12**]) is

$$\mathrm{H}^{2p}_{Db}(X) = \mathbb{H}^{2p}((2\pi i)^p\mathbb{Z} \to \Omega^{<p}_X)$$

LEMMA 1.14. There is a natural exact sequence

$$0 \to J^p(X) \to \mathrm{H}^{2p}_{Db}(X) \to Hdg^p(X) \to 0$$

where $Hdg^p(X) = \mathrm{H}^{2p}(X, \mathbb{Z}) \cap \mathrm{H}^{p,p}(X)$.

PROOF. There is an exact sequence of complexes

$$0 \to \Omega^{<p}_X[-1] \to (\mathbb{Z} \to \Omega^{<p}_X) \to \mathbb{Z} \to 0$$

The long exact sequence for hypercohomolgy gives

$$\mathrm{H}^{2p-1}(X, \mathbb{Z}) \to \frac{\mathrm{H}^{2p-1}(X, \mathbb{C})}{F^p\,\mathrm{H}^{2p-1}(X, \mathbb{C})} \to \mathrm{H}^{2p}_{Db}(X) \to \mathrm{H}^{2p}(X, \mathbb{Z}) \to \frac{\mathrm{H}^{2p}(X, \mathbb{C})}{F^p\,\mathrm{H}^{2p}(X, \mathbb{C})}$$

The desired sequence now follows. □

There exists a cycle class map

$$\mathrm{CH}^p(X) \to \mathrm{H}^{2p}_{Db}(X)$$

which combines the usual cycle class map with the Abel-Jacobi map. See [**12**] for more details.

0.6. Variations of Hodge structures. One way to study the relations between algebraic cycles and Hodge theory is to look at their variation when X varies in a family. Suppose $f : \mathcal{X} \to S$ is a smooth family of complex projective varieties parametrised by a smooth complex variety S. In such a situation one can define a structure on the cohomology groups as they vary in the family.

Let $\mathcal{H}^k_{\mathbb{C}}$ denote the local system whose stalk at any point $s \in S$ is the cohomology group $\mathrm{H}^k(X_s, \mathbb{C})$ for. Let $\mathcal{F}^k$ denote the Hodge subbundle of the vector bundle $\mathcal{H}^k = \mathcal{H}^k_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{X}}$ whose fibre at any point $s \in S$ is

$$\mathcal{F}^p_s := \oplus_{p' \geq p} \mathrm{H}^{p',k-p'}(X_s)$$

$\{\mathcal{F}^p\}$ then defines a filtration on $\mathcal{H}^k$. Moreover the quotient

$$\mathcal{F}^p/\mathcal{F}^{p+1} = \mathcal{H}^{p,k-p}$$

where the term on the right is the vector bundle with fibre at any point $s \in S$ is equal to $\mathrm{H}^{p,k-p}(\mathcal{X}_s)$.

THEOREM 1.15 (Griffiths). With notation as above,

1. $\mathcal{H}^k_{\mathbb{C}}$ is a local system and hence $\mathcal{H}^k$ comes equipped with a flat holomorphic connection ∇ which is called the Gauss-Manin connection. Thus we have a complex of bundles:

$$0 \to \mathcal{H}^k \xrightarrow{\nabla} \Omega^1 \otimes \mathcal{H}^k \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Omega^{dimS} \otimes \mathcal{H}^k \to 0$$

The complex above is referred to as the de Rham complex of the bundle $\mathcal{H}^k$.

2. The subbundles $\mathcal{F}^p$ are holomorphic subbundles and satisfy the following transversality with respect to the Gauss-Manin connection

$$\nabla(\mathcal{F}^p) \subset \Omega^1_S \otimes \mathcal{F}^{p-1}$$

Moreover the following is a subcomplex of the de Rham complex of $\mathcal{H}^k$

$$0 \to \mathcal{F}^p \to \Omega^1_S \otimes \mathcal{F}^{p-1} \to \dots$$

We denote this subcomplex as $\mathrm{F}^p(\mathcal{H}^k \otimes \Omega^\bullet_S)$.

3. The $\mathcal{O}_S$-linear map $\overline{\nabla}$

$$\mathcal{F}^p/\mathcal{F}^{p+1} \xrightarrow{\overline{\nabla}} \Omega^1_S \otimes (\mathcal{F}^{p-1}/\mathcal{F}^p)$$

obtained from ∇ by passing to the quotient, gives for any $s \in S$ a map:

$$T_sS \to \mathrm{Hom}(\mathrm{H}^{k-p}(\Omega^p_{\mathcal{X}_s}), \mathrm{H}^{k-p+1}(\Omega^{p-1}_{\mathcal{X}_s}))$$

which can be identified with the composite

$$T_sS \xrightarrow{\kappa} \mathrm{H}^1(T\mathcal{X}_s) \to \mathrm{Hom}(\mathrm{H}^{k-p}(\Omega^p_{\mathcal{X}_s}), \mathrm{H}^{k-p+1}(\Omega^{p-1}_{\mathcal{X}_s}))$$

where κ is the Kodaira-Spencer map [**23**] and the last map is given by cup product.

REMARK 1.16. Any local system which satisfies the conditions in the theorem above defines what is called a Variation of Hodge structures.

Let $f : \mathcal{X} \to S$ be a smooth family as above. Denote by $J^p(\mathcal{X}_s)$, the intermediate Jacobian $J^p(\mathrm{H}^{2p-1}(\mathcal{X}_s, \mathbb{Q}))$. The family of intermediate Jacobians $(J^p(\mathcal{X}_s))_{s\in S}$ has a natural complex structure, for which the sheaf of holomorphic sections is

$$\mathcal{J}^p := \mathcal{H}^{2p-1}/\mathcal{F}^p + \mathcal{H}^{2p-1}_{\mathbb{Z}}$$

Let $\mathcal{Z} \subset \mathcal{X}$ be a codimension p-cycle, whose support is flat over S and such that $\mathcal{Z}_s \subset \mathcal{X}_s$ is homologous to zero for each $s \in S$. The cycle $\mathcal{Z}$ then defines a normal function $\nu_{\mathcal{Z}} \in \mathcal{J}^p$ defined by

$$\nu_{\mathcal{Z}}(s) = \Phi_{\mathcal{X}_s}(\mathcal{Z}_s)$$

There is an analog of Griffiths transversality for the normal function $\nu_{\mathcal{Z}}$.

LEMMA 1.17. The normal function defined above is a holomorphic section of the Jacobian bundle. Moreover if $\tilde{\nu}_{\mathcal{Z}}$ is a lift of $\nu_{\mathcal{Z}}$ to a section of $\mathcal{H}^{2p-1}$, then it satisfies $\nabla\tilde{\nu}_{\mathcal{Z}} \in \mathcal{F}^{p-1} \otimes \Omega^1_S$

DEFINITION 1.18. The intermediate Jacobian bundle as defined above sits in a short exact sequence

$$0 \to \mathcal{H}^{2p-1}_{\mathbb{Z}} \to \mathcal{H}^{2p-1}/\mathcal{F}^p \to \mathcal{J} \to 0$$

The long exact sequence of cohomology then gives a coboundary map

$$\mathrm{H}^0(S, \mathcal{J}) \xrightarrow{\partial} \mathrm{H}^1(S, \mathcal{H}^{2p-1}_{\mathbb{Z}})$$

The image of any normal function ν under ∂ is called the cohomology class or the obstruction class of the normal function ν.

0.7. Some spectral sequences. In this section we describe some generalities regarding spectral sequences.

Let $(K^\bullet, d)$ be filtered complex (of abelian groups) filtered by a decreasing sequence of complexes

$$K^\bullet = F^0K^\bullet \supset F^1K^\bullet \supset \cdots \supset F^nK^\bullet \supset F^{n+1}K^\bullet = (0)$$

PROPOSITION 1.19. [17] Let $K^\bullet$ be a filtered complex. Then there exists a spectral sequence $\{E_r\}$ with

$$E_r^{p,q} = \frac{\{a \in F^pK^{p+q} |\ d(a) \in F^{p+r}K^{p+q+1}\}}{d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}$$

Note that,

$$E_1^{p,q} = (\mathrm{H}^{p+q}(\mathrm{Gr}^p K^\bullet))$$

and

$$E_\infty^{p,q} = \mathrm{Gr}^p(\mathrm{H}^{p+q}(K^\bullet))$$

EXAMPLE 1.20. The de Rham complex of $\mathcal{H}^k$ with its filtration by $\mathrm{F}^p(\mathcal{H}^{2k-1} \otimes \Omega_S^\bullet)$ is an example of a filtered complex.

In this thesis we shall be interested in two types of spectral sequences.

DEFINITION 1.21. Let $f : X \to S$ be a smooth family of projective varieties with S smooth. Let $\mathcal{F}$ be any sheaf on X. Define the q-th direct image sheaf of $\mathcal{F}$, $\mathrm{R}^q f_* \mathcal{F}$, which is the sheaf associated to the presheaf

$$U \to \mathrm{H}^q(f^{-1}(U), \mathcal{F})$$

The Leray spectral sequence is a spectral sequence $\{E_r\}$ with

$$E_2^{p,q} = \mathrm{H}^p(Y, \mathrm{R}^q f_* \mathcal{F}) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{F})$$

DEFINITION 1.22. Let $(\mathcal{K}^\bullet, d)$ be a complex of sheaves on X. The hypercohomology $\mathbb{H}^\bullet(X, \mathcal{K}^\bullet)$ has two spectral sequences $'E$ and $''E$ abutting to it with

$$'E_2^{p,q} = \mathrm{H}^p(X, \mathcal{H}^q(\mathcal{K}^\bullet)) \qquad ''E_2^{p,q} = \mathrm{H}_d^q(\mathrm{H}^p(X, \mathcal{K}^\bullet))$$

Here $\mathcal{H}^q(\mathcal{K}^\bullet)$ is the q-th cohomology of the complex of sheaves $(\mathcal{K}^\bullet, d)$ and $\mathrm{H}_d^q(\mathrm{H}^p(X, \mathcal{K}^\bullet))$ is the q-th cohomology of the complex

$$\mathrm{H}^p(X, \mathcal{K}^0) \xrightarrow{d} \mathrm{H}^p(X, \mathcal{K}^1) \xrightarrow{d} \ldots$$

CHAPTER 2

Statement of the Problem

1. The process of spreading out

We fix once and for all an embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$.

Let X be a smooth complex projective variety. Then X has an embedding in a projective space $\mathbb{P}^N$ for some N. Let K be the subfield of $\mathbb{C}$ generated over $\overline{\mathbb{Q}}$ by the coefficients of the defining polynomials of X. This is a field of finite transcendence degree over $\overline{\mathbb{Q}}$. Let X_K be the subvariety of $\mathbb{P}^N_K$ defined by the same polynomials as X. Then we have a Cartesian diagram

$$\begin{array}{ccc} X & \rightarrow & X_K \\ \downarrow & & \downarrow \\ Spec\mathbb{C} & \rightarrow & SpecK \end{array}$$

We shall refer to such a diagram as a model for X.

One can similarly find a model for a given cycle $\xi \in \mathrm{CH}^p(X)$. Let $\sum n_i Z_i$ be a representative for the cycle ξ. Let K now stand for the field generated by the coefficients of the defining polynomials for X and the various Z_i's. Then one obtains a Cartesian diagram as above. We let $Z_{i,K}$ denote the subvarieties of X_K defined by the polynomials which define the Z_i's. Then $\xi_K = \sum n_i Z_{i,K}$ gives a model for ξ.

The two varieties X and X_K are (naturally) quite closely related both cycle theoretically as well as cohomologically. Since singular cohomology only makes sense for complex varieties we shall use the formalism of cycle class maps into algebraic de Rham cohomology. Of course for smooth complex varieties there is a comparison theorem which identifies the singular cohomology with the algebraic de Rham cohomology [18]. These relations are given by the following lemmas:

LEMMA 2.1. Let X_K be a variety defined over an algebraically closed field K and let L be an algebraically closed field containing K. Define $X_L := X_K \times_{SpecK} SpecL$. Let f denote the map $X_L \to X_K$. Then the map $f^* : \mathrm{CH}^p(X_K) \to \mathrm{CH}^p(X_L)$ is is an injection.

PROOF. (see [7]) □

COROLLARY 2.2. Let X be defined over $\mathbb{C}$. Then

$$\lim_{\to} \mathrm{CH}^d(X_L) \tilde{\to} \mathrm{CH}^d(X)$$

where the limit is over all models of X over algebraically closed subfields of finite transcendence degree in $\mathbb{C}$.

PROOF. We note that

1. Injectivity follows since each of the maps are injective by previous lemma.
2. Surjectivity follows from the fact that any algebraic cycle in X has a representative which is defined over a field of finite transcendence degree and hence it lives in X_L for some L as explained above.

□

Recall from the previous section that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^p(X_K) & \to & \mathrm{H}^{2p}_{DR}(X_K/K) \\ \downarrow & & \downarrow \\ \mathrm{CH}^p(X) & \to & \mathrm{H}^{2p}_{DR}(X/\mathbb{C}) \cong \mathrm{H}^{2p}(X,\mathbb{C}) \end{array} \tag{1}$$

This is a statement about the functoriality of the cycle class map into de Rham cohomology and its compatibility with the comparison theorem for singular and de Rham cohomology.

Diagram (1) and Corollary 2.2 imply that one can think of the cycle class map for a complex projective variety X as a limit of cycle class maps over its various models. Thus a cycle ξ and its image under the cycle class map $cl(\xi)$ should be thought of as the image ξ_K and $cl_K(\xi_K)$ respectively where the subscript K implies that everything is happening in some model X_K of X.

As mentioned in the previous chapter, one way to study algebraic cycles and their relation with cohomology theories is to look at their variation as the variety varies in a family. One way to do this is the following.

Let X be the variety that we wish to study. Let R be the $\overline{\mathbb{Q}}$-algebra, generated by the coefficients of the polynomials which define X. One can define a $\overline{\mathbb{Q}}$-variety $\mathcal{X} \subset \mathbb{P}^N_R$ by considering the zero locus of the defining polynomials of X. The map $\mathcal{X} \to Spec\overline{\mathbb{Q}}$ then factors through $S := SpecR$ and we have a family $\mathcal{X} \to S$. Since the quotient field of R is K, we get that $\mathcal{X} \times_S SpecK = X_K$ by definition. Thus we have the following diagram.

$$\begin{array}{ccc} X_K & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ SpecK & \to & S \\ & & \downarrow \\ & & Spec\overline{\mathbb{Q}} \end{array}$$

The variety $X \to Spec\mathbb{C}$, now maps to this family yielding a Cartesian diagram

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec\mathbb{C} & \to & S \\ & & \downarrow \\ & & Spec\overline{\mathbb{Q}} \end{array}$$

Here the lower horizontal map factors through $SpecK$ where K is the field over which X is defined. This process of constructing a family is called *spreading out* the variety. Note that by shrinking S if necessary, we can assume that the family $\mathcal{X} \to S$ is a smooth family over a smooth base.

One can similarly spread out cycles along with the variety so that as the variety deforms in the family, so does the cycle. Let ξ be the element of the Chow group that we wish to spread out. Let $\xi = \sum_i n_i Z_i$ be a representative of ξ. Suppose that $Z = \cup Z_i$ is the support of the ξ. Then we have a diagram:

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec\mathbb{C} & \to & S \\ & & \downarrow \\ & & Spec\overline{\mathbb{Q}} \end{array} \tag{2}$$

In this case let R be the $\overline{\mathbb{Q}}$-algebra generated by the coefficients of the defining polynomials of X and the subvarieties Z_i and the lower horizontal map factors through $SpecK$ where K is the field over which X and Z are defined.

The entire exercise can be carried out for finitely many pairs (X_i, ξ_i) for $i = 1 \ldots n$ where X_i is a variety and ξ_i a cycle on it. We give a brief description.

Let $\mathcal{X} \to S$ be a spreadout diagram for X as above. For each i we have a diagram

$$(3) \qquad \begin{array}{ccc} X_i & \to & \mathcal{X}_i \\ \downarrow & & \downarrow \\ Spec\mathbf{C} & \to & S_i \end{array}$$

Let $S^n := S_1 \times_S \cdots \times_S S_n$ and $\mathcal{X}^n$ be the pullback of $\mathcal{X}$ under the map $S^n \to S$. The spread out cycles Ξ_i via the pullback of the projection morphisms lift to cycles in the family $\mathcal{X}^n \to S^n$ which is thus a spreading out for this finite collection of varieties and cycles. The idea here is essentially that these different cycles are defined (a priori) over distinct fields and therefore can be captured in a bigger field containing these fields as subfields.

The well definedness of various objects is checked as explained above.

In what follows we shall assume that by shrinking S if necessary, the family $\mathcal{X} \to S$ is a smooth family over a smooth base.

An analog of Corollary 2.2 is the following

LEMMA 2.3. For any smooth projective variety X,

$$\lim_{\to} \mathrm{CH}^d(\mathcal{X}) \xrightarrow{\simeq} \mathrm{CH}^d(X)$$

where the limit is taken over all diagrams $(*)$ as above.

PROOF. Surjectivity is obvious. To prove injectivity, note that if a cycle $\Xi \in \mathrm{CH}^d(\mathcal{X})$ maps to zero in the direct limit then it means that in particular Ξ restricts to zero in the generic fibre of the family $\mathcal{X} \to S$. This in turn implies that Ξ is supported in the inverse image of a proper closed subset W of S. . Let $S^* = S \setminus W$ and $\mathcal{X}^*$ be the pullback of $\mathcal{X} \to S$ over S^*. Then $\mathcal{X}^* \to S^*$ also gives a diagram and Ξ is zero as a cycle in $\mathcal{X}^*$. □

1.1. Cycle class maps. As earlier, we need to show the compatibility of the process of spreading out with the formalism of the cycle class map. Note that since we view $\mathcal{X}$ as a $\overline{\mathbb{Q}}$-variety, there is a cycle class map

$$\mathrm{CH}^p(\mathcal{X}) \to \mathrm{H}^{2p}_{DR}(\mathcal{X}/\overline{\mathbb{Q}})$$

Let $\xi \in CH^p(X)$ and suppose that by Lemma 2.3, it is the image of a cycle Ξ in some family $\mathcal{X}$. Now for the family $\mathcal{X} \to S$ there is a spectral sequence whose E_2 terms are $E_2^{p,q} = \mathbb{H}^p(S, \Omega^\bullet_S \otimes \mathrm{H}^q_{DR}(\mathcal{X}/S))$ and which abuts to $\mathrm{H}^{p+q}_{DR}(\mathcal{X}/\overline{\mathbb{Q}})$. Since S is affine this spectral sequence degenerates at E_2. Thus we have a projection map

$$\mathrm{H}^{2p}_{DR}(\mathcal{X}/\overline{\mathbb{Q}}) \to \mathrm{E}^{0,2p} \hookrightarrow \mathrm{H}^{2p}_{DR}(\mathcal{X}/S)$$

$\mathrm{H}^{2p}_{DR}(\mathcal{X}/S)$ is a module over R whose localisation at the generic point of S namely $SpecK$ is $\mathrm{H}^{2p}_{DR}(X_K/K)$ where X_K is the generic fibre of the family $\mathcal{X} \to S$. There is a commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}^p(\mathcal{X}) & \to & \mathrm{H}^{2p}(\mathcal{X}/S) \\ \downarrow & & \downarrow \\ \mathrm{CH}^p(X_K) & \to & \mathrm{H}^{2p}(X_K/K) \end{array}$$

Using the embeddding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, we have a comparison isomorphism

$$\mathrm{H}^{2p}_{DR}(\mathcal{X}/\overline{\mathbb{Q}}) \otimes \mathbb{C} = \mathrm{H}^{2p}_{DR}(\mathcal{X}_{\mathbb{C}}/\mathbb{C}) \cong \mathrm{H}^{2p}_{sing}(\mathcal{X}(\mathbb{C}), \mathbb{C})$$

Consider all diagrams such as

$$(4) \qquad \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec\mathbb{C} & \to & S \end{array}$$

For each such diagram we have a cycle class map into the Deligne-Beilinson cohomology, since we have chosen an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$,

$$\mathrm{CH}^p(\mathcal{X}) \xrightarrow{cl_{Db}} \mathrm{H}^{2p}_{Db}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(p))$$

Taking limits over all diagrams such as (4), we have a cycle class map

$$\mathrm{CH}^p(X) \xrightarrow{cl_{ADb}} \mathrm{H}^{2p}_{ADb}(X, \mathbb{Q}) := \varinjlim \mathrm{H}^{2p}_{Db}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(p))$$

We refer to the latter cohomology group as the Absolute Deligne Beilinson cohomology.

Moreover one has the following commutative diagrams.

$$(5) \qquad \begin{array}{ccc} \mathrm{CH}^p(\mathcal{X}) & \xrightarrow{cl_{Db}} & \mathrm{H}^{2p}_{Db}(\mathcal{X}(\mathbb{C}), \mathbb{Q}) \\ \searrow cl_{sing} & & \swarrow \\ & \mathrm{H}^{2p}(\mathcal{X}(\mathbb{C}), \mathbb{Q}) & \end{array}$$

$$(6) \qquad \begin{array}{ccc} \mathrm{CH}^p(\mathcal{X}) & \xrightarrow{cl_{Db}} & \mathrm{H}^{2p}_{Db}(\mathcal{X}(\mathbb{C}), \mathbb{C}) \\ \downarrow cl_{DR} & & \downarrow \\ \mathrm{H}^{2p}_{DR}(\mathcal{X}/\overline{\mathbb{Q}}) \otimes \mathbb{C} & \cong & \mathrm{H}^{2p}_{sing}(\mathcal{X}(\mathbb{C}), \mathbb{C}) \end{array}$$

2. The Conjecture

We now describe a conjecture for detecting nullhomologous cycles on any smooth projective variety X over $\mathbb{C}$.

As explained earlier, the cycle class map is not injective in general. For cycles in the kernel of this map one can associate an invariant in the intermediate Jacobian via the Abel-Jacobi map. However, Mumford showed that for a smooth projective complex surface with $p_g \neq 0$ there are non-trivial elements in the kernel of the Abel-Jacobi map. In fact he proved that this kernel is "infinite-dimensional" in a certain sense. So far there is no satisfactory "cohomological" construct which can detect cycles in the kernel of the Abel-Jacobi map. However conjecturally, at least for varieties defined over number fields, it is believed that the

Abel-Jacobi map is enough to detect all cycles which are in the kernel of the cycle class map. The precise conjecture in this context, is the following:

CONJECTURE 2.4 (Bloch-Beilinson). Let X be a smooth projective variety over a number field. Then the Abel Jacobi map

$$\mathrm{CH}^p(X)_{hom} \otimes \mathbb{Q} \to J^p(X(\mathbb{C})) := \frac{\mathrm{H}^{2p-1}(X(\mathbb{C}); \mathbb{C})}{F^p + \mathrm{H}^{2p-1}(X(\mathbb{C}); \mathbb{Z})}$$

is injective.

More generally,

CONJECTURE 2.5. The cycle class map into Deligne-Beilinson cohomology is injective for a smooth variety defined over a number field (see for eg. [**29**]).

There are examples [**32**] of non-trivial nullhomologous cycles in a surface defined over a field of transcendence degree one which show that the conjecture above is tight.

By results of the preceding section one can lift any given cycle to a cycle in a variety defined over $\overline{\mathbb{Q}}$. The conjecture of Bloch and Beilinson then can be used to formulate the following

CONJECTURE 2.6 (Asakura-Paranjape-M.Saito). [**2, 3, 30**] The cycle map cl_{ADb} into Absolute Deligne Beilinson cohomology is injective.

REMARK 2.7. Notice that the above conjecture is true if the Bloch-Beilinson conjecture stated above is true. Conjecture 2.5 implies that any given cycle lifts to a cohomologically detectable cycle in some $\overline{\mathbb{Q}}$ variety $\mathcal{X}$. Since for every diagram the cycle class map into Deligne Beilinson cohomology is injective, this implies that the limiting map is injective.

3. Motivation

The principal motivation for the above conjecture comes from the Beilinson-Bloch conjectures on the filtrations on Chow groups.

Consider a smooth projective curve C over $\mathbb{C}$. Then the Chow groups of C are well-understood. It is well-known that $\mathrm{CH}^0(C) \cong \mathrm{Pic}^0(C)$ where $\mathrm{Pic}^0(X)$ is an abelian variety. Similar results do not hold true for surfaces as shown by Mumford. Moreover in the case of a curve the isomorphisms

$$\mathrm{Pic}^0(C) \cong J(C) \cong \mathrm{Ext}^1_{MHS}(\mathbb{Q}(-p), \mathrm{H}^1(C, \mathbb{Q}))$$

meant that $\mathrm{H}^1(C, \mathbb{Q})$ (more precisely a quotient of its complexification) captured the nullhomologous cycles. It was therefore expected that for higher dimensional varieties, cycles in $\mathrm{CH}^p(X)$ which are not detected by either the cycle class map or the Abel-Jacobi map would be detected in the cohomology groups $\mathrm{H}^i(X, \mathbb{Q})$ for $i = 0 \ldots 2p-2$ in a similar fashion as above. However the absence of non-trivial higher Ext's in the category of mixed Hodge structures meant that one could not detect *all* cycles by just using Hodge theory.

This led to the following :

CONJECTURE 2.8 (Beilinson-Bloch). There exists a category of mixed motives $\mathcal{MM}$ containing the category $\mathcal{M}_k$ of Grothendieck motives such that for X smooth projective over $k \hookrightarrow \mathbb{C}$, there is a filtration on the Chow groups satisfying the following:

1. $F^r\,\mathrm{CH}^p(X)_{\mathbb{Q}} \bullet F^s\,\mathrm{CH}^p(X)_{\mathbb{Q}} \hookrightarrow F^{r+s}\,\mathrm{CH}^p(X)_{\mathbb{Q}}$ under the intersection product $\bullet$.
2. $F^\bullet$ is respected by f^* and f_* for morphisms $f : X \to Y$;

3. (assuming the algebraicity of the Kunneth components of the diagonal)

$$\mathrm{Gr}^{\nu}_{F}\,\mathrm{CH}^{p}(X)_{\mathbb{Q}} \cong \mathrm{Ext}^{\nu}_{\mathcal{MM}_k}(1, h^{2p-\nu}(X)(p)),$$

where $1 = h(speck)$ is the trivial motive and $h^{2p-\nu}(X)$ is the Grothendieck motive corresponding to the (Weil) cohomology group $\mathrm{H}^{2p-\nu}(X)$.

Moreover the isomorphisms in 3 above are compatible via the various comparison theorems with those in the category of mixed Hodge structures and the category of l-adic representations.

M.Saito has constructed a category $\mathrm{MHM}(S)$, for any variety S, whose objects are mixed Hodge modules which can be described as follows. To each $\overline{\mathbb{Q}}$ family $\mathcal{X} \to S$ there exists an object $\mathrm{R}^{i}(\mathcal{X}/S)$ in $\mathrm{MHM}(S)$ such that its pull back via the map $Spec\mathbb{C} \to S$ is the natural mixed Hodge structure on the cohomology $\mathrm{H}^{i}(X, \mathbb{Q})$ of X. In addition he has shown that this category has higher Ext's: for instance if S is a curve then Ext^2 is non-trivial. Furthermore, there exists a spectral sequence

$$E_1^{a,b} = \mathrm{Ext}^{a}_{MHM(S)}(\mathbb{Q}(c), \mathrm{R}^{b}(\mathcal{X}/S)) \Rightarrow \mathrm{Ext}^{a+b}_{MHM(\mathcal{X})}(\mathbb{Q}(c), \mathbb{Q})$$

One of the most important feature of this category from our point of view is that when $a = 2p - k$, $b = k$ and $c = -p$, the term on the right hand side can be identified with $\mathrm{H}^{2p}_{\mathcal{D}b}(\mathcal{X}, \mathbb{Q}(p))$. It follows from the following lemma of Jannsen that Conjecture 2.8 is equivalent to the conjecture about the injectivity of the cycle class map into Absolute-Deligne cohomology.

Lemma 2.9. If the cycle class map into the Deligne-Beilinson cohomology is injective, then the filtration on the Chow groups obtained by restricting the Hochschild-Serre filtration on the cohomology groups is the conjectural Bloch-Beilinson filtration.

PROOF. (see [**22**]) □

4. Statement of Problem

The objective of this thesis is twofold. The first is to give a simpler, more unified treatment for all the diverse examples of non-trivial nullhomologous cycles. This then would allow us to construct new examples of such cycles. In fact the assumptions we make have been checked in the various papers in specific examples. The second is that we have been able to employ the method of detection explained above to detect various of these nullhomologous cycles. More precisely we have checked with the exception of cycles over arithmetic varieties

> For every known nullhomologous cycle on varieties over $\mathbb{C}$ there exists a spread out diagram such as 4 above such that the cycle spreads out to a homologically non-trivial cycle under the map cl_{sing} in diagram (5).

4.1. Examples of nullhomologous cycles. We now list various examples of non-trivial nullhomologous cycles that we have studied.

1. *The example of Griffiths* [**15**]: Let X be a general quintic hypersurface in $\mathbb{P}^4$. Then the difference of any two lines l_1 and l_2 is a non-trivial nullhomologous cycle.
2. *The example of Nori* [**25**]: Let X be a smooth projective variety and Y a smooth complete intersection of sufficiently high degree in X. Then if $\xi \in \mathrm{CH}^p(X)$ is not homologically trivial, its restriction to Y is not algebraically equivalent to zero. In particular, if

$cl(\xi)$ is in the kernel of the map $H^{2p}(X, \mathbb{Q}) \to H^{2p}(Y, \mathbb{Q})$, then ξ restricted to Y gives a homologically trivial cycle.

3. *The examples of Clemens* [**10**]: Let X be a quintic hypersurface in $\mathbb{P}^4$. Then X contains countably many non-trivial cycles whose image in the Griffiths group generates a subgroup of infinite rank over $\mathbb{Q}$. This result has been extended for arbitrary K-trivial complete intersection threefolds (see [**27**]).
4. *The examples of Voisin* [**33**]: Let X be a general quintic hypersurface in $\mathbb{P}^4$. Then there exists a countably infinite number of cycles which are linearly independent in the Griffiths group tensor $\mathbb{Q}$.
5. *The examples of Bardelli-Mueller-Stach* [**5**]: Let Y be a smooth Fano fourfold and X a general member of the linear system $|-K_Y|$. If $G^2(X)$ is the Griffiths group of codimension 2 cycles, then $G^2(X) \otimes \mathbb{Q}$ is not a finitely generated group.
6. *The example of Albano-Collino* [**1**] : For a general cubic threefold the Griffiths group of codimension 3 cycles is infinitely generated.
7. *The example of Ceresa* [**8**]: Let C be a general curve of genus $g \geq 3$. If C^- denotes the image of C in the Jacobian $J(C)$ under multiplication by -1, then $C - C^-$ is not algebraically equivalent to zero.
8. *The example of Nori* [**26**]: Let B be a generic threefold abelian variety. Then $G^2(B) \otimes \mathbb{Q}$ is an infinite dimensional vector space over $\mathbb{Q}$. The same result was proved by Bardelli [**4**].

The above results merely are representatives of the general method used in the detection of such cycles. We now give a broad classification based on the general methods used in constructing these cycles. Detailed descriptions follow in the subsequent chapters.

1. *The method of Connectivity*: The main step involved in this is to prove a statement about connectivity between a variety and a certain family of subvarieties. This then implies that if a cycle on the variety is homologically non-trivial then its restriction to a generic subvariety in the family is non-trivial. This has been used by Griffiths [**15**] and Nori [**25**] to show the non-triviality of the nullhomologous cycles.
2. *The method of Degeneration*: The cycles of Clemens and Paranjape are detected using this method. It is shown that if a degree zero cycle deforms in a family in such a way that it passes through an ordinary double point singularity of the degenerate fibre then it is actually non-trivial in a neighbourhood of this degeneration.
3. *The infinitesimal method*: The non-triviality of cycles is established by showing that the infinitesimal invariant of the normal functions defined by these cycles as they vary in the universal family containing X is not identically zero. The cycles in ([**4**], [**5**], [**33**], [**1**]) are detected using this method.

CHAPTER 3

The method of Connectivity

The first instance of cohomology being used directly to detect cycles appears in a paper of M.Nori ([**25**]). The following is based on this work.

1. The basic setup

Let X be a smooth projective variety, and suppose that dimension of X is $n+h$. Let S be a smooth scheme parametrising subvarieties of X of codimension h i.e., let $A := X \times S$ and B be a closed subvariety of A such that $B \to S$ is flat. Let $s \in S$ be the geometric generic point of S and $Y = B_s$ be the fibre over it.

Let ξ be a cycle on X. By flat pull back this gives a cycle on A. By restricting this cycle to B and then to Y we obtain a cycle η on Y. In particular, we have the following

PROPOSITION 3.1. In the above situation, suppose $\mathrm{H}^i(A, B; \mathbb{Q}) = 0$ for $i = 2p-1, 2p$. For any codimenion p homologically non-trivial cycle ξ on X its restriction to Y is not rationally equivalent to zero.

PROOF. The Kunneth decomposition

$$\mathrm{H}^i(X \times S) = \oplus_j \mathrm{H}^j(X) \otimes \mathrm{H}^{i-j}(S)$$

gives an injection

$$\mathrm{H}^{2p}(X) \hookrightarrow \mathrm{H}^{2p}(A_S)$$

Consider the following composition:

$$\begin{array}{ccccc} \mathrm{H}^{2p}(X;\mathbb{Q}) & \hookrightarrow & \mathrm{H}^{2p}(A_S;\mathbb{Q}) & \cong & \mathrm{H}^{2p}(B_S;\mathbb{Q}) \\ [\xi] & \mapsto & [p_X^*(\xi)] & \mapsto & [\bar{\eta}] \end{array}$$

where the first map is an injection by Kunneth decomposition and the second one is an isomorphism by hypothesis. Since η spreads out to $\bar{\eta}$ this implies that η is not rationally equivalent to zero. □

There are examples due to Nori [25] and Griffiths [15] among others where certain stronger results about the non-triviality of η can be deduced from additional connectivity hypothesis over A and B. In this chapter we will explore these examples.

2. Detecting nullhomologous cycles

For any smooth morphism $T \to S$, we denote by A_T and B_T, the base change of A and B respectively. Then the principal connectivity hypothesis is

Assumption 3.2. With notation as above,

$$F^k \mathrm{H}^{n+k}(A_T, B_T; \mathbb{C}) = 0 \qquad for \quad k \leq n$$

where $F^\bullet$ is the Hodge filtration on the cohomology groups of the pair as introduced in [11] (see also Chapter1).

Following [25], we define $\Omega^\bullet_{A,B}$ as the kernel of the map

$$\Omega^\bullet_A \to \Omega^\bullet_B \to 0$$

The long exact sequence of hypercohomology groups

$$(7) \quad \ldots \mathbb{H}^i(A, \Omega^\bullet_{A,B}) \to \mathbb{H}^i(A, \Omega^\bullet_A) \to \mathbb{H}^i(B, \Omega^\bullet_B) \to \mathbb{H}^{i+1}(A, \Omega^\bullet_{A,B}) \ldots$$

can be identified with the long exact sequence for singular cohomology with coefficients in $\mathbb{C}$.

$$\text{(8)} \qquad \ldots \mathrm{H}^i(A,B) \to \mathrm{H}^i(A) \to \mathrm{H}^i(B) \to \mathrm{H}^{i+1}(A,B) \ldots$$

via the comparison theorem between singular and de Rham cohomology.

The spectral sequence $E_1^{p,q} = \mathrm{H}^p(A, \Omega^q_{A,B})$ induces a filtration on $\mathbb{H}^\bullet(A, \Omega^\bullet_{A,B})$, which we denote by $G^p\,\mathrm{H}^\bullet(A,B,\mathbb{C})$. This filtration in turn contains the Hodge filtration defined by Deligne [11]

$$G^p\,\mathrm{H}^\bullet(A,B;\mathbb{C}) \supset F^p\,\mathrm{H}^\bullet(A,B;\mathbb{C})$$

Thus Assumption 3.2 follows from

Assumption 3.3. $G^k\,\mathrm{H}^{n+k}(A_T, B_T) = 0 \qquad \forall k \le n$

By using the Leray spectral sequence the above assumption follows if

Assumption 3.4. $\mathrm{H}^p(A_T, \Omega^q_{A_T,B_T})$ vanish for $p \le k$ and $p + q \le n + k$.

From the Leray spectral sequence, it then suffices to check that the following is true.

Assumption 3.5. Let p_T be the projection from $A_T \to T$. Then

$$\mathrm{R}^p\,p_{T*}\Omega^q_{A_T,B_T} = 0$$

Assumption 3.2 implies the following vanishing of cohomology of pairs.

Lemma 3.6. $\mathrm{H}^i(A_T, B_T; \mathbb{Q}) = 0 \qquad$ *for* $\; i \le 2n$

PROOF. (see [**25**]) Let W denote the weight filtration on the cohomology of the pair. Suppose $gr_i^W \mathrm{H}^{n+k}(A_T, B_T)$ is non-zero for some i, then this implies it has a non-vanishing $\mathrm{H}^{p,q}$. Assumption 3.2 implies that $p \leq k-1$ and therefore using the fact that $\mathrm{H}^{p,q} = \overline{\mathrm{H}^{q,p}}$, we have $q \leq k-1$. Consequently $i \leq 2(n-1)$. Since A_T and B_T are smooth varieties, we have $gr_i^W \mathrm{H}^{n+k}(A_T) = 0$ if $i \leq n+1$ and $gr_i^W \mathrm{H}^{n+k-1}(B_T = 0$ if $i \leq n+k-1$. Thus

$$gr_i^W \mathrm{H}^{n+k}(A_T, B_T) = 0 \quad if \quad i < n+k-1$$

Thus we see that $2(k-1) \geq n+k-1$ or equivalently $k \geq n+1$. In particular $\mathrm{H}^{n+k}(A_T, B_T, \mathbb{Q}) \neq 0$ implies $n+k \geq 2n+1$. □

Lemma 3.6 has the following stronger implications for algebraic cycles.

THEOREM 3.7 (Nori). *Let X and Y be as above. Let ξ be a codimension d algebraic cycle on X, whose cohomology class $[\xi]$ lies in $\mathrm{P}^{2d}(X, Y; \mathbb{Q}) := \mathrm{Ker}(\mathrm{H}^{2d}(X; \mathbb{Q}) \to \mathrm{H}^{2d}(Y; \mathbb{Q}))$. Also assume $d < n$. Then if $[\xi]$ is a non-zero element, then its restriction to Y is homologically trivial but not algebraically equivalent to zero.*

PROOF. (see [**25**]) We prove the contrapositive of the above statement. So we assume that the restriction of ξ to Y is algebraically equivalent to zero. This then by definition means

$$\eta = u(C_s, \alpha_s, \beta_s) := p_Y^*(\alpha_s . p_{C_s}^*(\beta_s))$$

where C_s is a smooth projective curve defined over $\overline{k}$, where $\overline{k}$ is the algebraic closure of the function field of S, $\beta_s \in \mathrm{CH}_0(C_s)_{hom}$ and $\alpha_s \in \mathrm{CH}^d(C_s \times Y)$ and p_Y and p_{C_s} are the two projections from $C_s \times Y$ to the two factors Y and C_s respectively. As mentioned in the previous chapter, all this data lives in a field L of finite degree over k. Let T be the variety with generic point $SpecL$ such that the map $T \to S$ is etale and corresponds to the field extension $k \subset L$ at the generic points. This procedure then gives the following:

1. $T \to S$ etale.
2. a smooth projective morphism $\mathcal{C} \to T$
3. an algebraic cycle $\beta \in \mathrm{CH}_t(\mathcal{C})_{hom}$ for $t = dimT$, and
4. a correspondence $\alpha \in \mathrm{CH}^d(B_\mathcal{C})$

Moreover $e\pi_X^*\xi = u_*(\beta.v^*\alpha)$ in $\mathrm{CH}^d(B_T)$ where $\pi_X : B_T \to X$, $u : B_\mathcal{C} \to B_T$ (this is checked by looking at the fibres on either side and noting that they are equal) and $v : B_\mathcal{C} \to \mathcal{C}$ are the "spreadout" of the given morphisms.

From Lemma 3.6, we have that

$$\mathrm{H}^j(A_\mathcal{C}; \mathbb{Q}) \to \mathrm{H}^j(B_\mathcal{C}, \mathbb{Q})$$

is an isomorphism for $j \leq 2n-1$ and because $d < n$, we obtain $\beta' \in \mathrm{H}^{2d}(A_\mathcal{C}, \mathbb{Q})$ whose image in $\mathrm{H}^{2d}(B_\mathcal{C}, \mathbb{Q})$ is the cohomology class associated to β. As above, we have

$$ep_X^*[\xi] = (1_X \times f)_*(\beta'.p_\mathcal{C}^*[\alpha])$$

where as usual $p_X : A_T \to X$ and $p_\mathcal{C} : A_\mathcal{C} \to \mathcal{C}$ are the spreading out of the given maps.

Since for any $t \in T(\mathbb{C})$, X embeds into A_T via $X \mapsto X \times \{t\}$, we get by checking at the fibres that

$$e[\xi] = (p_1)_*((\beta'|X \times f^{-1}(t).p_2^*([\alpha]|f^{-1}(t)))$$

Since $[\beta]|f^{-1}(t) = 0$ by assumption, ξ is homologically trivial. □

CHAPTER 4

The method of Degeneration

1. The Setup and Notation

Let X be a smooth projective variety of dimension $2d-1$ and let C be a codimension d subvariety of X. Let $f : \mathcal{X} \to S$ be a flat family where $\mathcal{X}$, S are smooth with X as the geometric generic fibre. Further suppose that S is a curve and the family has special fibre X_0 over $s_0 \in S$ containing only ordinary double points $\{p_i\}$ as singularities. Let $\mathcal{C} \to T$ be a smooth family with geometric generic fibre C such that there exists a diagram:

$$\begin{array}{ccc} \mathcal{C} & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\delta} & S \end{array} \tag{9}$$

Assume that the map $T \to S$ is a finite cover with simple ramification at $t_0 \in T$ where $\delta(t_0) = s_0$.

We consider the following two cases.

1. The special fibre C_0 of the family $\mathcal{C}$ passes through exactly one of the double points, denoted p_0, in the fibre X_0.
2. The special fibre C_0 of the family $\mathcal{C}$ misses the double points in the fibre X_0.

Let $\tilde{X}_0$ be the blow up of the double point p_0 and let E denote the exceptional fibre over the point p_0.

Let $\mathcal{X}_T := \mathcal{X} \times_S T$ be the pull back of $\mathcal{X}$ to a family over T. Then there is a lifting $\mathcal{C} \xrightarrow{i} \mathcal{X}_T$. Now define $\hat{\mathcal{X}}_T$ as the normalisation of $\mathcal{X}_T$. This normalisation still contains singularities at the ordinary double points in the fibre over t_0.

Let $\mathcal{Y} \to \hat{\mathcal{X}}_T$ be the blowing up at the ordinary double point p_0. The special fibre of $\mathcal{Y} \to T$ at t_0 is the union of $\tilde{X}_0$ and a smooth quadric Q such that $\tilde{X}_0$ meets Q transversally along E. Let $\tilde{\mathcal{C}}$ be the strict transform under the blow up map. If C_0 passes through the double point p_0 then $\tilde{\mathcal{C}}$ intersects $\tilde{X}_0$ in a projective space $\mathbb{P}^{d-1} \subset E$.

We make the following assumption in order to simplify our arguments.

Assumption 4.1. For a general fibre X in $\mathcal{X} \to S$, $\mathrm{H}^{2d}(X, \mathbb{Z}) \cong \mathbb{Z}$ so that the cycle class map $\mathrm{CH}^d(X) \to \mathrm{H}^{2d}(X, \mathbb{Z})$ is the degree map.

Let A be a relatively ample class on $\mathcal{X} \to S$. Let $\mathcal{H}$ be A^d. Then $\Xi = \mathcal{C} - deg(C).\mathcal{H}$ is a cycle on $\hat{\mathcal{X}}_T$ whose restriction $\xi := C - deg(C).H$ is a nullhomologous cycle. Let $\tilde{\mathcal{C}}$ be the strict transform of $\mathcal{C}$ under the map $\mathcal{Y} \to \hat{\mathcal{X}}_T$, where the latter is the normalisation of $\mathcal{X}$ and let $\tilde{\Xi} := \tilde{\mathcal{C}} - deg(C).\mathcal{H}$.

We shall now show that the cycle $\tilde{\Xi}$ can be detected in the cohomology of the total space $\mathcal{Y}$.

2. Detecting nullhomologous cycles

We shall now show that the cycle ξ is non-trivial. Our method here is entirely topological and avoids any use of Hodge theory. We shall work locally over a disc Δ around the point $0 \in S$ in the base locus. We denote by $\tilde{\Delta}$ the component of the inverse image of Δ in T containing t_0. Having reduced the situation to such a neighbourhood, we note that the special fibre X_0 is a deformation retract of the family $\mathcal{X} \times_S \Delta \to \Delta$. Without loss of generality, we can assume that X_0 contains exactly one ordinary double point. We then have that in the blow up family $\mathcal{Y} \times_T \tilde{\Delta} \to \tilde{\Delta}$, $\tilde{X}_0 \cup Q$ is a deformation retract of $\mathcal{Y} \times_T \tilde{\Delta}$.

THEOREM 4.2. We work with notation as above. Let $\mathcal{N} := (\mathcal{Y} \setminus \tilde{X}_0) \times_T \tilde{\Delta}$.

1. Suppose that the special fibre C_0 passes through the ordinary double point p_0, then $\tilde{\Xi} := \tilde{\mathcal{C}} - (degC).\mathcal{H}$ is non-trivial 2-torsion in the cohomology of $\mathcal{N}$.
2. If C_0 does not pass through p_0 in X_0 then $\tilde{\Xi}$ is trivial in the cohomology of $\mathcal{N}$.

PROOF. (1) Consider the following diagram:

$$\begin{array}{ccc} & \mathrm{H}^{2d-1}(E,\mathbb{Z}) = 0 & \\ & \downarrow & \\ \ldots \to \mathrm{H}^{2d-2}(\tilde{X}_0,\mathbb{Z})) \xrightarrow{i_*} & \mathrm{H}^{2d}(\mathcal{Y}_{\tilde{\Delta}},\mathbb{Z}) & \xrightarrow{j^*} \mathrm{H}^{2d}(\mathcal{N};\mathbb{Z}) \to \ldots \\ & \downarrow (i_Q^*, i^*) & \\ & \mathrm{H}^{2d}(Q;\mathbb{Z}) \oplus \mathrm{H}^{2d}(\tilde{X}_0,\mathbb{Z}) & \\ & \downarrow & \\ & \mathrm{H}^{2d}(E;\mathbb{Z}) & \end{array}$$

where the horizontal sequence is the Gysin sequence and the vertical one is the Mayer-Vietoris sequence. The composite $i^* \circ i_*$ in the diagram

$$\mathrm{H}^{2d-2}(\tilde{X}_0,\mathbb{Z}) \xrightarrow{i_*} \mathrm{H}^{2d}(\mathcal{Y}_{\tilde{\Delta}},\mathbb{Z}) \xrightarrow{i^*} \mathrm{H}^{2d}(\tilde{X}_0,\mathbb{Z})$$

is $\cup c_1(N_{\tilde{X}_0/\mathcal{Y}})$, the cup product with the first Chern class of the normal bundle of $\tilde{X}_0$ in $\mathcal{Y}$. Assumption 4.1 then implies that this is just multiplication by the degree of $N_{\tilde{X}_0/\mathcal{Y}}$.

The degree of the normal bundle $N_{X_t/\mathcal{Y}}$ for the fibre over any point $t \in T$ is just the degree of the fibre X_t. By the adjunction formula this is just $\mathcal{O}_{\mathcal{Y}_{\tilde{\Delta}}}(X_t)_{|X_t}$. Note for $t \in \tilde{\Delta}$ this is trivial. Thus at $t = t_0$, we have $N_{X_0 \cup Q/\mathcal{Y}_{\tilde{\Delta}}}$ is trivial. This implies that (again by the adjunction formula) $\mathcal{O}_{\mathcal{Y}_{\tilde{\Delta}}}(\tilde{X}_0) \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\Delta}}}(Q)$ is trivial. Since Q has degree -2, X_0 has degree $+2$. Thus we conclude that $N_{\tilde{X}_0/\mathcal{Y}}$ has degree 2 and therefore the composite $i^* \circ i_* =$ multiplication by 2.

We use the identification $\mathrm{H}^{2d}(\tilde{X}_0; \mathbb{Z}) \cong \mathbb{Z}.H \oplus \mathbb{Z}.L'$ where H is the codimension d linear section and L' is the push-forward of the codimension d class coming from the exceptional fibre with generator $\mathbb{P}^{d-1}$ which can be identified with $\tilde{\mathcal{C}} \cap \tilde{X}_0$.

Since degree Ξ is zero and $\tilde{\mathcal{C}}$ intersects $\tilde{X}_0$ in a $\mathbb{P}^{d-1}$ whose class is L', $\tilde{\Xi} \mapsto (0, 1)$ under the restriction map (i_Q^*, i^*). This implies that $\tilde{\Xi}$ does not lie in the image of i_* and hence belongs to $\mathrm{Coker}(i_*) \hookrightarrow \mathrm{H}^{2d}(\mathcal{N}; \mathbb{Z})$. On the other hand, since the composition $i^* \circ i_* = \times 2$ this implies that $2\tilde{\Xi} \in \mathrm{Im}(i_*)$ and hence $\tilde{\Xi}$ is 2-torsion in $\mathrm{H}^{2d}(\mathcal{N}; \mathbb{Z})$.

(2) Suppose on the other hand that C_0 does not pass through the ordinary double point. Since Ξ does not pass through the double point of the singular fibre, it is isomorphic to its pullback in $\mathcal{Y}$. Therefore its support has empty intersection with the exceptional fibre Q. Hence its restriction to $\mathcal{N}$ is zero. □

COROLLARY 4.3. In the case where C_0 passes through the ordinary double point of the degenerate fibre, the cycle ξ is a non-trivial nullhomologous cycle.

PROOF. The cycle ξ spreads out to a cycle Ξ on the family $\mathcal{X} \to \Delta$. The cycle Ξ pulls back to a cycle $\tilde{\Xi}$ in $\mathcal{Y}$ which is homologically non-trivial. Since $\tilde{\Xi}$ has non-zero image under the cycle class map it is non-zero. This implies that Ξ is non-trivial and hence so is ξ. □

2.1. More cycles. We wish now to generalise the above situation to one where there are finitely many cycles on a variety X, the generic fibre of a flat family $\mathcal{X} \to S$ with only ordinary double points occurring as singularities. For this we assume that for $i = 1 \ldots l$, there are diagrams:

$$\begin{array}{ccc} \mathcal{C}_i & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ T_i & \xrightarrow{\delta_i} & S \end{array} \tag{10}$$

Here δ_i is finite map of smooth varieties such that its branch locus is contained in the singular locus of the family $\mathcal{X} \to S$. Moreover, $\mathcal{C}_i \to T_i$ is a smooth family with generic fibre C_i which is a codimension d subvariety of X. Let B_i (respectively R_i) be the branch loci (resp. the ramification loci) of the maps δ_i. We denote by S_i' (resp. T_i') the complement of B_i in S (resp. R_i in T_i). We further assume that

Assumption 4.4. The branch loci for the maps δ_i in Diagram 10 are distinct. Moreover for each i, $\mathcal{C}_i$ misses the singularities in the fibres outside its branch locus.

We now wish to detect the cycles $\{C_i\}$ in the general fibre X of the family $\mathcal{X} \to S$.

For any integer $1 \leq i \leq l$, one can now construct finitely many diagrams in the following manner: Since $\mathcal{X} \to S$ is a flat family with generic fibre smooth, this implies that over a Zariski dense set the family $\mathcal{X} \to S$ restricts to a smooth family. We choose a curve S_i which passes through exactly one of the points of the branch locus of δ_i and which misses the branch loci of δ_j for $j \neq i$.

We then have the following diagrams:

$$\begin{array}{ccc} \mathcal{C}_i & \to & \mathcal{X}_i \\ \downarrow & & \downarrow \\ T_i & \xrightarrow{\delta_i} & S_i \end{array}$$

and for $j \neq i$,

$$\begin{array}{ccc} \mathcal{C}_{j,i} & \to & \mathcal{X}_i \\ \downarrow & & \downarrow \\ S_i & \cong & S_i \end{array}$$

Here the families $\mathcal{C}_{j,i}$, $\mathcal{C}_i$, $\mathcal{X}_i$ are the restrictions of the families in diagram (10) to the curve S_i.

Thus we have the following situation.

1. For $j \neq i$ the family $\mathcal{C}_{j,i}$ completely misses the singular locus of $\mathcal{X}_i$.
2. The map $T_i \to S_i$ is a double cover which is ramified at one point.
3. For the family of cycles $\mathcal{C}_i$, there exists a point $0 \in S_i$ such that the special fibre C_0 at 0 passes through one of the ordinary double points of the (singular) fibre of $\mathcal{X}_i \to S_i$.

On a general member X of the family $\mathcal{X}_i \to S_i$, we then have finitely many codimension d cycles $\xi_j = C_j - deg(C_j).H$ where C_j is a general member of the family $\mathcal{C}_{j,i}$ and H is a codimension d linear section in X. We shall now study the relations between these cycles.

2.2. Relations between cycles. We now wish to study the relations between the cycles Ξ_i defined by ξ_i in X as it varies in the family $\mathcal{X} \to S$.

THEOREM 4.5. *The cycles Ξ_i are linearly independent modulo 2 in $\mathrm{CH}^d(\mathcal{X})$, the Chow group of codimension d cycles.*

PROOF. Suppose there exists a relation

$$\sum_i n_i \Xi_i = 0$$

For $i = i_0$, consider the restriction of the above sum to $\mathcal{X}_{i_0}$. Consider its image in the cohomology of $\mathcal{N}_{i_0}$. By Theorem 4.2 we know that Ξ_i vanishes for $i \neq i_0$ since these do not intersect the singularities in the special fibre of $\mathcal{X}_{i_0} \to \Delta_{i_0}$. This implies that $n_{i_0}.\Xi_{i_0} = 0$. Since its image is 2-torsion, this implies that n_{i_0} is divisible by 2. Similarly arguing, we see that 2 divides n_i for all i. Hence the cycles are linearly independent modulo 2. □

3. Torsion in chow groups

We shall now make some remarks about the rank of $\mathrm{CH}^d(\mathcal{X})$. We make note of the following useful lemma.

LEMMA 4.6. If G is an abelian group such that its torsion subgroup G_{tor} is a subgroup of $(\mathbb{Q}/\mathbb{Z})^r$, then we have

$$rank_{\mathbb{Q}}(G \otimes \mathbb{Q}) + r \geq rank_{\mathbb{Z}/2\mathbb{Z}}(G \otimes \mathbb{Z}/2\mathbb{Z})$$

PROOF. (see **[10]**) □

According to above lemma, we need to show that $\mathrm{CH}^d(\mathcal{X})_{tors}$, the torsion subgroup is a subgroup of $(\mathbb{Q}/\mathbb{Z})^r$ for some r. We shall now briefly describe certain results of Colliot-Thelene et al **[9]** which were inspired by Bloch's construction of a cycle class map on the torsion algebraic cycles into cohomology with finite coefficients. We start by collecting some basic results.

Let X be a smooth projective variety defined over an algebraically closed field k and let l be a prime different from k.

Let $\mathcal{H}^q(\mu_{l^\nu}^{\otimes r})$ denote the Zariski sheaf associated to the presheaf

$$U \to \mathrm{H}^q_{et}(U, \mu_{l^\nu}^{\otimes r})$$

where $U \hookrightarrow X$ is a Zariski open set, q, ν, r are integers, and $\mu_{l^\nu}^{\otimes r}$ denote the etale sheaf of the l-th roots of unity on X, tensored with itself r times. Note that

$$\begin{aligned} \mu_{l^\nu}^{\otimes r} &:= \mathbb{Z}/l^\nu\mathbb{Z} & r = 0 \\ &:= Hom_X(\mathbb{Z}/l^\nu\mathbb{Z}, \mu_{l^\nu}^{\otimes -r}) & r < 0 \end{aligned}$$

The Leray spectral sequence associated to the morphism of sites $X_{et} \to X_{Zar} \to Speck$ gives a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(X, \mathcal{H}^q(\mu_{l^\nu}^{\otimes r})) \Rightarrow \mathrm{H}^{p+q}_{et}(X, \mu_{l^\nu}^{\otimes r})$$

Let X^c denote the set of codimension c points of X. The following then is a flasque resolution of the sheaf $\mathcal{H}^q(\mu_{l^\nu}^{\otimes r})$:

$$0 \to \mathcal{H}^q(\mu_{l^\nu}^{\otimes r}) \to \coprod_{x \in X^0} (\iota_x)_* \mathrm{H}^q(k(x), \mu_{l^\nu}^{\otimes r}) \to \ldots$$

$$\to \coprod_{x \in X^{q-1}} (\iota_x)_* \mathrm{H}^1(k(x), \mu_{l^\nu}^{\otimes r-q+1}) \to \coprod_{x \in X^q} (\iota_x)_* \mathrm{H}^0(k(x), \mu_{l^\nu}^{\otimes r-q}) \to 0$$

where $(\iota_x)_*(A)$ for an abelian group A denotes the constant sheaf A denotes the constant sheaf A supported on the Zariski closure of the point x. It then immediately follows that

LEMMA 4.7 (Bloch-Ogus). With notation as above, $\mathrm{H}^p(X, \mathcal{H}^q(\mu_{l^\nu}^{\otimes r})) = 0$ for p < q. In particular, for $i \geq 1$, there is a boundary map

$$\mathrm{H}^{i-1}(X, \mathcal{H}^i(\mu_{l^\nu}^{\otimes i})) \xrightarrow{?} \mathrm{H}^{2i-1}(X, \mu_{l^\nu}^{\otimes i})$$

Let

$$\mathbb{Q}_l/\mathbb{Z}_l(r) := \lim_{\leftarrow \nu} \mu_{l_\nu}^{\otimes r}$$

Then one has a long exact sequence of cohomology:

$$\ldots \to \mathrm{H}^{n-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(m)) \to \mathrm{H}^n(X, \mathbb{Z}_l(m)) \to \mathrm{H}^n(X, \mathbb{Q}_l(m)) \to \mathrm{H}^n(X, \mathbb{Q}_l/\mathbb{Z}_l(m)) \to \ldots$$

Bloch [6] has constructed the following cycle class map.

$$\lambda_l^d : \mathrm{CH}^d(X)(l) \to \mathrm{H}^{2d-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(d))$$

where

$$\mathrm{CH}^d(X)(l) := \lim_{\to_\nu} {}_{l^\nu}\mathrm{CH}(X)$$

and

$$\mathrm{H}^{2d-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(d)) := \lim_{\leftarrow \nu} \mathrm{H}^{2d-1}_{et}(X, \mu_{l^\nu})$$

LEMMA 4.8 (Colliot-Thelene et al.). (see [9]) Let k be a field, l a prime different from the characteristic of the field k and X be a smooth algebraic k-variety. One then has the following diagram which is commutative upto a sign:

$$\begin{array}{ccc} {}_{l^\nu}\mathrm{CH}^i(X) & \xrightarrow{\rho_\nu} & \mathrm{H}^{2i}(X; \mu_{l^{\nu'}}^{\otimes i}) \\ \uparrow \alpha & \searrow \lambda_l^d & \uparrow \beta \\ \mathrm{H}^{i-1}(X, \mathcal{H}^i(\mu_{l^\nu}^{\otimes i})) & \xrightarrow{\gamma} & \mathrm{H}^{2i-1}(X, \mu_{l^\nu}^{\otimes i}) \end{array}$$

where the map ρ is the cycle class map, β is the Bockstein morphism which is defined as the boundary map of the long exact sequence corresponding to the sequence

$$1 \to \mu_m^{\otimes r} \to \mu_{mn}^{\otimes r} \to \mu_n^{\otimes r} \to 1$$

and γ is the map obtained from the Bloch-Ogus spectral sequence $E_2^{p,q} \Rightarrow \mathrm{H}^{2i-1}(X; \mu_{l^\nu}^{\otimes r})$ where $E_2^{p,q} := \mathrm{H}^p(X, \mathcal{H}^q(\mu_{l^\nu}^{\otimes r}))$, the map α is defined and is onto via the Merkurjev -Suslin theorem.

In what follows we suppose $k = \mathbb{C}$.

Taking direct limits over ν and inverse limit over ν' the above diagram reads as follows:

$$\begin{array}{ccc} \mathrm{CH}^i(X)(l) & \xrightarrow{\rho} & \mathrm{H}^{2i}(X; \mathbb{Z}_l) \\ \uparrow \alpha & \searrow \lambda_l^d & \uparrow \beta \\ \mathrm{H}^{i-1}(X, \mathcal{H}^i(\mathbb{Q}_l/\mathbb{Z}_l(i))) & \xrightarrow{\gamma} & \mathrm{H}^{2i-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(i)) \end{array}$$

We note the following obvious corollary:

COROLLARY 4.9. For X smooth projective variety over $\mathbb{C}$, we have

$$\mathrm{CH}^i(X)_{tors} \subset (\mathbb{Q}/\mathbb{Z})^r$$

for some r.

PROOF. Since $\mathrm{H}^{2i-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(i))$ is a free $\mathbb{Q}_l/\mathbb{Z}_l(i)$-module, this implies that it is isomorphic to a direct sum of copies of $\mathbb{Q}_l/\mathbb{Z}_l(i)$. Therefore the statement is true for any l. By identifying $\mathbb{Q}_l/\mathbb{Z}_l(i)$ with $\mathbb{Q}/\mathbb{Z}$, we conclude by taking union over all l, that the torsion subgroup is contained in $(\mathbb{Q}/\mathbb{Z})^r$. □

We now apply Lemma 4.6 to conclude that $rank_{\mathbb{Q}}(\mathrm{CH}^d(X) \otimes \mathbb{Q})$ is bounded from below. We can use this result for instance to produce varieties with large Chow groups.

COROLLARY 4.10. Suppose there exists a countable number of diagrams as in 9 satisfying the assumptions above. Then the cycles $\{\Xi_i\}$ generate a subgroup whose rank when tensored with $\mathbb{Q}$ is infinite.

CHAPTER 5

The Infinitesimal method

1. Some invariants

Let $f : \mathcal{X} \to S$ be a smooth family of projective varieties with S smooth defined over the field of complex number $\mathbb{C}$. We assume that S is affine. Let $\mathrm{CH}^d(\mathcal{X}/S)_{hom} \subset \mathrm{CH}^d(\mathcal{X})$ be the Chow group of (relative) algebraic cycles which are fibrewise homologically trivial. We are interested in the various cohomological invariants associated to such a cycle, say $\mathcal{Z}$.

Consider the cycle class map

$$\mathrm{CH}^d(\mathcal{X}) \xrightarrow{cl_{dR}} \mathbb{H}^{2d}(\mathcal{X}; \Omega^\bullet_{\mathcal{X}}) \cong \mathrm{H}^{2d}(\mathcal{X}; \mathbb{C})$$

Here the isomorphism is given by the comparison theorem.

Let $[\mathcal{Z}] = cl_{dR}(\mathcal{Z})$ denote the cohomology class of the cycle $\mathcal{Z}$. By Grothendieck, $[\mathcal{Z}]$ maps into the d-th filtered piece $F^d\mathbb{H}^{2d}(\mathcal{X}; \Omega^\bullet_{\mathcal{X}})$ where $F^\bullet$ stands for the Hodge filtration.

The Leray spectral sequence gives a filtration whose graded pieces are $E_\infty^{p,2d-p}$. Since S is affine, we have $E_2 = E_\infty$ where the E_2 terms are

$$E_2^{p,q} = \mathrm{H}^p(S, \mathcal{H}^q_{\mathbb{C}})$$

Here $\mathcal{H}^q_{\mathbb{C}}$ is the local system which has stalk $\mathrm{H}^q(\mathcal{X}_s, \mathbb{C})$ at any point $s \in S$. Since Z is fibrewise homologically trivial, $cl(\mathcal{Z})$ maps to zero in $\mathrm{H}^0(S, \mathcal{H}^{2d}_{\mathbb{C}})$. It then defines the obstruction class $\partial\nu_{\mathcal{Z}} \in \mathrm{H}^1(S, \mathcal{H}^{2d-1}_{\mathbb{C}})$.

We shall be interested in an infinitesimal invariant associated to $\partial\nu_Z$ which we shall denote by $\delta_1\nu_Z$.

Since the projection maps into various graded pieces respects the Hodge filtration, we have the following:

$$\begin{array}{ccc} F^d\mathbb{H}^{2d}(\mathcal{X},\Omega^\bullet_{\mathcal{X}}) & \to & F^d\,\mathrm{H}^1(S,\mathcal{H}^{2d-1}_{\mathbb{C}}) \\ cl_{dR}(\mathcal{Z}) & \mapsto & \partial\nu_{\mathcal{Z}} \end{array}$$

Since S is affine, we have

$$F^d\,\mathrm{H}^1(S,\mathcal{H}^{2d-1}_{\mathbb{C}}) = \mathbb{H}^1(S, F^d(\mathcal{H}^{2d-1}_{\mathbb{C}}\otimes\Omega^\bullet_S))$$

where $F^d(\mathcal{H}^{2d-1}_{\mathbb{C}}\otimes\Omega^\bullet_S)$ is defined as follows:

Consider the de Rham complex associated to the local system $\mathcal{H}^{2d-1}_{\mathbb{C}}$:

$$0\to\mathcal{H}^{2d-1}\to\Omega^1_S\otimes\mathcal{H}^{2d-1}\to\cdots\to\Omega^{dimS}_S\otimes\mathcal{H}^{2d-1}\to 0$$

Then the k-th piece in the filtration of the above complex is the complex $F^k(\mathcal{H}^{2d-1}\otimes\Omega^\bullet)$ defined by

$$0\to\mathcal{F}^k\mathcal{H}^{2d-1}\to\Omega^1_S\otimes\mathcal{F}^{k-1}\mathcal{H}^{2d-1}\to\Omega^2_S\otimes\mathcal{F}^{k-2}\mathcal{H}^{2d-1}\to\ldots$$

$$\to\Omega^{dimS}_S\otimes\mathcal{F}^{k-dimS}\mathcal{H}^{2d-1}\to 0$$

The graded pieces of this are the complexes $KS^{k,2d-1}(\mathcal{X}/S)$:

$$0\to\mathcal{H}^{k,2d-1-k}\to\mathcal{H}^{k-1,2d-1-k}\otimes\Omega^1_S\to\cdots\to\mathcal{H}^{k-dimS,2d-1+dimS-k}\otimes\Omega^{dimS}_S\to 0$$

The short exact sequence of complexes

$$0\to F^{k+1}(\mathcal{H}^{2d-1}\otimes\Omega^\bullet_S)\to F^k(\mathcal{H}^{2d-1}\otimes\Omega^\bullet_S)\to KS^{k,2d-1}\to 0$$

gives a map

$$\mathbb{H}^1(S,F^k(\mathcal{H}^{2d-1}\otimes\Omega^\bullet_S))\to\mathbb{H}^1(S,KS^{k,2d-1})$$

We define $\delta_1\nu_{\mathcal{Z}}$ as the image of $\partial\nu_{\mathcal{Z}}$ under the map

$$\mathbb{H}^1(S, F^d(\mathcal{H}^{2d-1}\otimes\Omega_S^\bullet)) \to \mathbb{H}^1(S, KS^{d,2d-1})$$

By shrinking S if necessary we may assume that the complex $KS^{d,2d-1}(\mathcal{X}/S)$ is a complex of vector bundles and thus $\delta_1\nu_{\mathcal{Z}}$ is a section of a vector bundle $\mathbb{H}^1(S, KS^{d,2d-1})$.

2. Detecting cycles

Let X be a smooth projective variety over $\mathbb{C}$. Assume that there exists a smooth family $f : \mathcal{X} \to S$ with X as a general member. Assume further that the base S is also smooth. On a general member such as X, we wish to detect cycles that are nullhomologous by showing that they spread out to the family $\mathcal{X}$ and have non-zero images in the cohomology of $\mathcal{X}$.

To do this we consider diagrams such as the following:

$$\begin{array}{ccc} \mathcal{Z} & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & S \end{array} \tag{11}$$

where

1. $\mathcal{Z} \to S$ is a flat family of varieties with Z as generic fibre.
2. The map $T \to S$ is quasi-finite and etale onto its image and for each $t \in T$, $Z_t \in \mathrm{CH}^n(X_{\phi(t)})_{hom}$.

Let $\mathcal{X}_T = \mathcal{X} \times_S T$. Then $\mathcal{Z} \in \mathrm{CH}^n(\mathcal{X}_T/T)_{hom}$.

The additional assumption required to prove the non-triviality of Z is

Assumption 5.1. The infinitesimal invariant $\delta_1\nu_{\mathcal{Z}}$ associated to the cycle $\mathcal{Z}$ is non-zero.

COROLLARY 5.2. Under the above assumptions the cycle Z is non-trivial as an element of $\mathrm{CH}^n(X)_{hom}$, the Chow group of homologically trivial cycle classes.

PROOF. Consider the diagram below.

$$\begin{array}{ccc} X & \rightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec\mathbb{C} & \rightarrow & T \end{array}$$

The cycle Z defines $\mathcal{Z} \in \mathrm{CH}^n(\mathcal{X}_T/T)_{hom}$. Since by assumption, the infinitesimal invariant is non-zero, this implies that the obstruction class $\partial\nu_{\mathcal{Z}}$ and hence the cohomology class $[\mathcal{Z}]$ are non- zero. □

More cycles. Assume now there exists a finite number of diagrams such as (11) i.e., for $i = 1 \ldots l$

$$\begin{array}{ccc} \mathcal{Z}_i & \rightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ T_i & \xrightarrow{\phi_i} & S \end{array}$$

Here $T_i \to S$ is quasi-finite and etale onto its image, $\mathcal{Z}_i \to T_i$ is a flat family of subvarieties with generic fibre Z_i such that for any $t \in T_i$, $\mathcal{Z}_{i,\phi_i(t)}$ is a codimension n homologically trivial cycle in $X_{\phi_i(t)}$, the fibre of $\mathcal{X} \to S$ over $\phi_i(t)$.

Let $T_{(l)} := T_1 \times_S \cdots \times_S T_l$ and $\mathcal{X}_{(l)}$ be the pullback of $\mathcal{X}$ under the etale map $T_{(l)} := T_1 \times_S \cdots \times_S T_l \to S$. The cycles $\mathcal{Z}_i$ pull back via projection maps from $\mathcal{X}_{(l)}$. We shall refer to the pull-back cycles by $\tilde{\mathcal{Z}}_i$.

We shall now show that for the collection of cycles Z_i, we can study the relations between them.

Denote by K the function field of S and by L_i the function field of T_i. Since $T_i \to S$ is etale, L_i is a finite extension of K. The infinitesimal invariant $\delta_1\nu_{\mathcal{Z}_i}$ is a section of the vector bundle $\mathbb{H}^1(S, KS^{d,2d-1}(\mathcal{X}/S))\otimes_{\mathcal{O}_S}$

$\mathcal{O}_{T_i}$ on T_i. By localising at the generic point, this defines an element

$$\delta_1\nu_i \in \mathrm{H}^1(SpecK, KS^{d,2d-1}(\mathcal{X}_K)) \otimes_K L_i$$

We now make the following

ASSUMPTION 5.3. With notation as above

1. The fields L_i are linearly disjoint over K.
2. The infinitesimal invariants $\delta_1\nu_{Z_i}$ for $i = 1 \ldots l$ do not descend to sections over S.

The above assumption then immediately implies that

PROPOSITION 5.4. Given any l, and diagrams as in (11) for $i = 1, \ldots, l$ the cycles $\{Z_i\}$ in the Chow group of a general member X of the family $\mathcal{X}_l \to T_{(l)}$ are linearly independent.

PROOF. Suppose there exists a relation

$$\sum_i n_i Z_i = 0$$

This then spreads out to a relation $\sum_i n_i \tilde{\mathcal{Z}}_i = 0$. Under the cycle class map the image $\sum_i n_i cl_{dR}(\tilde{\mathcal{Z}}_i) = 0$. This then implies that $\sum_i n_i \delta_1 \nu_{\tilde{\mathcal{Z}}_i} = 0$. The generic point of the variety $T_{(l)}$ is $SpecL$ where L is the compositum of the fields L_i. The stalk at $SpecL$ of the above relation is

$$\sum_i n_i \delta_1 \nu_i = 0$$

Suppose $n_{i_0} \neq 0$. Then $\delta_1\nu_{i_0}$ depends on the remaining $\delta_1\nu_i$'s. This means that $\delta_1\nu_{i_0}$ is defined over the intersection of L_{i_0} and the compositum of the fields $\{L_i\}_{i \neq i_0}$. Since L_i are linearly disjoint this intersection is just K. Thus $\delta_1\nu_i$ is defined over K which contradicts our assumption. Hence we have $n_i = 0$ for all i. □

3. Another example

Let $\mathcal{X} \to S$ and $\mathcal{Y} \to T$ be two smooth families over smooth varieties S and T respectively. We denote by X and Y the generic varieties of the families $\mathcal{X} \to S$ and $\mathcal{Y} \to T$ respectively. Let $U \hookrightarrow S \times T$ be the subvariety parametrising pairs of varieties (X', Y') such that there exists an algebraic correspondence $\Gamma \in \mathrm{CH}^*(\mathcal{X} \times \mathcal{Y})$ which induces a functorial diagram:

$$\begin{array}{ccc} \mathrm{CH}^m(Y') & \to & \mathrm{H}^{2m}(Y';\mathbb{Z}) \\ \downarrow \Gamma_* & & \downarrow \Gamma_* \\ \mathrm{CH}^n(X') & \to & \mathrm{H}^{2n}(X';\mathbb{Z}) \end{array}$$

where the vertical arrows are the maps induced by the correspondence Γ and the horizontal ones are the cycle class maps.

3.0.1. *Defining the Noether-Lefschetz locus.* Suppose we have a diagram

$$\begin{array}{ccc} \mathcal{Z} & \to & \mathcal{Y} \\ \downarrow & & \downarrow \\ C & \xrightarrow{\alpha} & T \end{array} \tag{12}$$

Here α is a quasi finite map of smooth varieties, $\mathcal{Z} \to C$ is a flat family with generic fibre Z, a subvariety of codimension m of Y. Further assume that the image of Z under the composition

$$\mathrm{CH}^m(Y) \xrightarrow{\Gamma_*} \mathrm{CH}^n(X) \xrightarrow{cl_X} \mathrm{H}^{2n}(X;\mathbb{Z})$$

is zero. Then Z defines a homologically trivial cycle on X via the correspondence Γ. We shall also refer to C as the Noether-Lefschetz locus of Z. The reason for this is that one can understand the variety $\mathcal{Z}$ by studying its cohomology class ξ. This is because in the deformation locus of Z, ξ too deforms and since ξ is the cohomology class of Z this means that ξ remains of type (m, m). We make this precise.

Let $\mathcal{H}$ be the vector bundle on T whose fibre at any point $t \in T$ is the cohomology group $\mathrm{H}^{2m}(Y_t, \mathbb{C})$. We denote by $\mathcal{F}^\bullet$ the Hodge subbundles of $\mathcal{H}$. Then the locus of deformation of $\xi = cl(Z)$, which we shall denote by T_ξ has the following description.

Let ∇ denote the Gauss-Manin connection on the flat bundle $\mathcal{H}$. Griffiths transversality (cf. Preliminaries) implies that $\nabla(\mathcal{F}^m) \subset \Omega^1_{T_\xi} \otimes \mathcal{F}^{m-1}$. To say that $\xi \in F^m\mathrm{H}^{2m}(Y_t;\mathbb{C})$ $\forall t \in T_\xi$, implies that if we consider the induced map ∇ on $\mathcal{F}^m/\mathcal{F}^{m+1}$ then contracting $\nabla(\xi)$ with any tangent vector v is zero. In other words T_ξ is in the kernel of the composite map

$$\mathcal{F}^m/\mathcal{F}^{m+1} \xrightarrow{\nabla} \Omega^1_{T_\xi} \otimes \mathcal{F}^{m-1}/\mathcal{F}^m \xrightarrow{\cup v} \mathcal{F}^{m-1}/\mathcal{F}^m$$

It is then clear that T_ξ is defined by the vanishing of $h^{m-1,m+1} := dim_{\mathbb{C}}\,\mathrm{H}^{m-1,m+1}$ analytic equations.

Let $t_0 \in T_\xi$ be a point. The tangent space of T_ξ at t_0 admits the following cohomological description:

$$\mathcal{T}_{t_0}(T_\xi) := \mathrm{Ker}(\mathcal{T}_{t_0}(T) \xrightarrow{\kappa} \mathrm{H}^1(Y, \mathcal{T}_Y) \xrightarrow{\cup\xi =:\bar{\xi}} \mathrm{H}^{m-1,m+1}(Y))$$

where the map κ is the Kodaira-Spencer map and $\bar{\xi}$ as defined is the cup product with the class ξ. Since $\mathcal{T}_{t_0}(T_\xi)$ is locally given by $h^{m-1,m+1}$ equations, it is clear then that T_ξ is smooth at any point Y iff the composite $\bar{\xi} \circ \kappa$ is surjective.

At this point, we shall need the following

ASSUMPTION 5.5. With notation as above,

1. T_ξ is smooth.
2. The map of tangent spaces

$$\mathcal{T}_c(C) \to \mathcal{T}_{t_0}(T_\xi)$$

is a surjection for $c \in C$ such that $\alpha(c) = t_0$.

Assumption 5.5 (2) implies that T_ξ is in fact given by the vanishing of polynomials and hence is algebraic. This holds for instance if the Hodge (m, m)-conjecture is true.

3.0.2. *Nullhomologous cycles on X.* Define $\mathcal{Y}_\xi := \mathcal{Y} \times_T T_\xi$, the pull-back family on T_ξ. Let p and q be the projection maps from $U \to S$ and $U \to T$ respectively. Define $M_\xi := q^{-1}(T_\xi)$. Let p_ξ denote the restriction of p to the subspace M_ξ.

Assumption 5.6. The projection map $p_\xi : M_\xi \to S$ is etale.

The map p_ξ then induces a cycle $\mathcal{Z}'$ on the family $\mathcal{X}$ via the correspondence Γ. This implies that on the generic fibre X we get a homologically trivial cycle Z' which is the image of Z under Γ. Since the cycle Z deforms along every direction in T_ξ, so does Z' in the variety S_ξ. Thus we are in the situation of diagram (11) i.e.,

$$\begin{array}{ccc} \mathcal{Z}' & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ M_\xi & \to & S \end{array}$$

3.0.3. Consider the family $(\mathcal{X} \times \mathcal{Y}) \times_U M_\xi$. We then have

$$\begin{array}{ccc} \mathrm{H}^{2m}(\mathcal{Y} \times_{T_\xi} M_\xi) & \xrightarrow{\Gamma_\bullet} & \mathrm{H}^{2n}(\mathcal{X} \times_S M_\xi) \\ \mathcal{Z} & \mapsto & \Gamma_*(\mathcal{Z}) \end{array}$$

Since $\Gamma_\bullet(\mathcal{Z})$ is fibrewise homologically trivial, it maps to its obstruction class, $\partial\nu_{\Gamma_\bullet(\mathcal{Z})}$ in $\mathrm{H}^1(M_\xi, \mathcal{H}^{2n-1})$. Moreover, since $M_\xi \to S$ is etale, the infinitesimal invariant

$$\delta_1\nu_{\Gamma_\bullet(\mathcal{Z})} \in \mathcal{H}^1(S, KS^{n,2n-1}(\mathcal{X}/S)) \otimes \mathcal{O}_{M_\xi}$$

3.1. Linear Independence. The assumptions 5.5 and 5.6 can be seen to be open conditions on the cycle ξ if we assume the Hodge

(m,m)-conjecture holds for Y. We wish to understand how the infinitesimal invariant $\delta_1\nu_{\Gamma,\mathcal{Z}}$ varies with the cycle ξ.

We further specialise to the case where Y is a smooth hyperplane section of X and that Γ is the correspondence induced by the inclusion $Y \hookrightarrow X$. Thus $n = m+1$ in the notation of preceding sections.

We work over a point $o := (t_0, s_0) \in U$, and X and Y denotes the fibres over the point s_0 and t_0 in S and T respectively. Consider the short exact sequence

$$0 \to \Omega^i_X \to \Omega^i_X(logY) \to \Omega^{i-1}_Y \to 0 \tag{13}$$

We have the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^m(X, \Omega^{m+1}_X) & \xrightarrow{d} & \Omega^1_{U,o} \otimes \mathrm{H}^{m+1}(X, \Omega^m_X) \\
\downarrow & & \downarrow \\
\mathrm{H}^m(X, \Omega^{m+1}_X(logY)) & \xrightarrow{d} & \Omega^1_{U,o} \otimes \mathrm{H}^{m+1}(X, \Omega^m_X(logY)) \\
\downarrow res & & \downarrow res \\
\mathrm{H}^m(Y, \Omega^m_Y) & \xrightarrow{d} & \Omega^1_{U,o} \otimes \mathrm{H}^{m+1}(Y, \Omega^{m-1}_Y) \\
\downarrow & & \downarrow \\
\mathrm{H}^{m+1}(X, \Omega^{m+1}_X) & \xrightarrow{d} & \Omega^1_{U,o} \otimes \mathrm{H}^m(X, \Omega^{m+2}_X)
\end{array} \tag{14}$$

Here the vertical sequence is the long exact sequence associated to the short exact sequence (13) above and the horizontal maps are the differentials occuring in the Kodaira-Spencer complex at a point $(t_0, s_0) \in U$.

The cycle $\xi = cl(Z)$ is primitive by definition. Hence it belongs to $K := \mathrm{Ker}(\mathrm{H}^m(Y, \Omega^m_Y) \to \mathrm{H}^{m+1}(X, \Omega^{m+1}_X))$. Choose a splitting $K \to \mathrm{H}^m(X, \Omega^{m+1}_X(logY))$ and let $\bar{\xi}$ be the image of ξ in $\mathrm{H}^m(X, \Omega^{m+1}_X(logY))$. Let η be a cohomology class in a neighbourhood of ξ in K. Then $d(\bar{\eta})$ gives an element of $\Omega^1_{U,o} \otimes \mathrm{H}^{m+1}(X, \Omega^m_X(logY))$ or equivalently a map

$$T_oU \xrightarrow{d(\bar{\eta})} \mathrm{H}^{m+1}(X, \Omega^m_X(logY))$$

We consider the following diagram:

$$
\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & T_f & & \\
 & & \downarrow & & \\
T_oM & \to & T_oU & \xrightarrow{d(\tilde{\eta})} & \mathrm{H}^{m+1}(X,\Omega^m_X(logY)) \\
 & \cong\searrow & \downarrow & \searrow\cup\eta & \downarrow res \\
 & & T_{s_0}S & & \mathrm{H}^{m+1}(Y,\Omega^{m-1}_Y) \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}
\tag{15}
$$

Here by assumption, the map $\cup\eta$ is surjective and T_oM is its kernel. T_f is such that the vertical sequence is exact. This then induces a map $g : T_{s_0}S \to \mathrm{H}^{m+1}(X,\Omega^m_X(logY))$ which when composed with the map res is zero. This implies that g lifts to a map $g : T_{s_0}S \to \mathrm{H}^{m+1}(X,\Omega^m_X)$ and hence gives an element in $\Omega^1_{U,o} \otimes \mathrm{H}^{m+1}(X,\Omega^m_X)$. This by section 3.0.3 is the infinitesimal invariant associated to the cycle $\mathcal{Z}'$.

We wish to understand how this map g depends on the cycle η. We now translate the above into a question in linear algebra.

Consider the following diagram of vector spaces over $\mathbb{C}$:

$$
\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & V & & \\
 & & \downarrow\beta & & \\
B & \to & W & \xrightarrow{f} & G \\
 & \cong\nwarrow p_\alpha & \downarrow & \searrow\alpha & \downarrow \\
 & & A & & H \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}
\tag{16}
$$

where as above, f is a linear map,

$$0 \to B \to W \xrightarrow{\alpha} H \to 0 \qquad\qquad 0 \to V \to W \to A \to 0$$

are short exact sequences.

The map $\alpha \circ \beta : V \to H$ is an isomorphism. Define

$$\bar{\alpha} := det(\alpha \circ \beta)^{-1} (\bigwedge^{\dim V-1} \alpha \circ \beta)^t : H \to V$$

Composing with β gives a splitting $\beta \circ \bar{\alpha} : H \to W$. This in turn gives a splitting of $\iota : A \to W$ as follows:

$$\begin{array}{ll} A \xrightarrow{\iota} & W \\ a \mapsto w - \beta(\bar{\alpha}^{-1}(\alpha(w))) \end{array}$$

Hence we get a map $g : A \to G$ by defining $g(a) := f \circ \iota(a)$.

In the situation of diagram (15), assume that Y is fixed and that X is varying in its parameter space. Then $d(\bar{\eta})$ depends linearly on the coordinates of $\eta \in K$ and the parameters defining X. From the linear algebra above, we have a map $g : T_{s_0}S \to \mathrm{H}^{m+1}(X, \Omega_X^m(logY))$ which is a rational function in the coefficients of the cycle η and the coordinates of the variety X in its parameter space.

Assumption 5.7. Suppose the denominator of the rational function g depends non-trivially on the coordinates of the cycle η and the parameters defining X.

Lemma 5.8. Let $\underline{r}_1 \in \mathbb{Q}^r$, $\underline{r}_2 \in \mathbb{Q}^s$ be r and s tuples of rational numbers and let λ be a transcendental number. Let $g(\underline{r}_1, \underline{r}_2, \lambda)$ be a rational function such that the denominator depends non-trivially on λ, $\underline{r}_1$ and $\underline{r}_2$. Then the $\mathbb{Q}$-vector space generated by g as $\underline{r}_1$ and $\underline{r}_2$ vary in $\mathbb{Q}^r$ and $\mathbb{Q}^s$ respectively is of infinite dimension over $\mathbb{Q}$.

Proof. (see [**33**]) □

invariants of the corresponding cycles defined by them on the family $\mathcal{X} \to S$ are linearly independent.

Proof. Let $\mathcal{Z}_i$ denote the cycles on $\mathcal{X}$ defined by the primitive cycles η_i in a neighbourhood of ξ in K. Let $L_\delta \subset \mathrm{H}^1(S, KS^{n,2n-1}(\mathcal{X}/S)) \otimes \mathcal{O}_T$ be the $\mathbb{Q}$-vector space generated by $\delta_1 \nu_{\mathcal{Z}_i}$ over $\mathbb{Q}$ and $L \subset \mathrm{H}^m(X, \Omega_X^{m+1}(logY))$ be the $\mathbb{Q}$-vector space generated by the g_i. From above there is an evaluation map

$$ev : L_\delta \to L$$

which maps $\delta_1 \nu_{\mathcal{Z}_i}$ to g_i for each i. Since g_i satisfy the assumption above, this means that L is infinite dimensional. This then implies that L_δ is infinite dimensional. This proves the statement of the proposition. □

Remark 5.10. Corollary above actually proves that the cycles $\{\mathcal{Z}_i\}$ are linearly independent in $\mathcal{X}$. The proof is the same as in the proof of Proposition 5.4.

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CORRIGENDUM

1. Preliminaries

Example 1. Let X be a smooth complex projective variety. Then for each n, the singular cohomology groups $V^n_{\mathbb{Z}} := \mathrm{H}^n(X;\mathbb{Z})$ come equipped with a pure Hodge structure of weight n. Here the complexification $V^n_{\mathbb{C}} = \mathrm{H}^n(X;\mathbb{C})$, is the cohomology with $\mathbb{C}$-coefficients. The filtration is such that $V^{p,q}$ is naturally isomorphic to $\mathrm{H}^q(X,\Omega^p_X)$ where the latter are the Hodge cohomology groups.

Definition 1. A *morphism of weight* $2n$ of pure Hodge structures $f : (V_{\mathbb{Z}}, F^\bullet) \to (V'_{\mathbb{Z}}, F'^{\,\bullet})$ is a homomorphism $f : V_{\mathbb{Z}} \to V'_{\mathbb{Z}}$ of abelian groups such that if $f_{\mathbb{C}} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$ is the induced $\mathbb{C}$-map, then $f_{\mathbb{C}}(\mathrm{F}^p V_{\mathbb{C}}) \subset \mathrm{F}^{p+n} V'_{\mathbb{C}}$ for all $p \in \mathbb{Z}$.

Example 2. Suppose $Y \hookrightarrow X$ is an inclusion of smooth projective varieties. Then for each i, the Gysin morphism $f_* : \mathrm{H}^i(Y;\mathbb{Z}) \to \mathrm{H}^{i+2d}(X;\mathbb{Z})$ where d is the codimension of Y in X, is a morphism of pure Hodge structures of weight $2d$.

Definition 2. The p-th *Intermediate Jacobian* of a pure Hodge structure H, $J^p(H)$ is

$$J^p(H) := \frac{H_{\mathbb{C}}}{F^p H_{\mathbb{C}} + H}$$

Lemma 1. The 0-th intermediate Jacobian of a pure Hodge structure H of weight -1 admits the following identification:

$$J^0(H_{\mathbb{Z}}) \otimes \mathbb{Q} \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H_{\mathbb{Z}} \otimes \mathbb{Q})$$

where MHS stands for the category of mixed Hodge structures.

Definition 3. Let X be a smooth projective variety over $\mathbb{C}$. The $2p$ *Deligne-Beilinson cohomology* (see [13]) is

$$\mathrm{H}^{2p}_{Db}(X) = \mathbb{H}^{2p}((2\pi i)^p\mathbb{Z} \to \Omega^{<p}_X)$$

Remark 1. We refer the reader to [13] for the case when X is not projective.

2. The method of degeneration

2.1. **The Setup and Notation.** Let X be a smooth projective variety of dimension $2d-1$ and let C be a codimension d subvariety of X. Let $f : \mathcal{X} \to S$ be a flat family where $\mathcal{X}$, S are smooth with X as the geometric generic fibre. Further suppose that S is a curve and the family has special fibre X_0 over $s_0 \in S$ containing only ordinary double points $\{p_i\}$ as singularities. Let $\mathcal{C} \to T$ be a smooth family with geometric generic fibre C such that there exists a diagram:

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\delta} & S \end{array} \tag{1}$$

Assume that the map $T \to S$ is a finite cover with simple ramification at $t_0 \in T$ where $\delta(t_0) = s_0$.

We consider the following two cases.

1. The special fibre C_0 of the family $\mathcal{C}$ passes through exactly one of the double points, denoted p_0, in the fibre X_0.
2. The special fibre C_0 of the family $\mathcal{C}$ misses the double points in the fibre X_0.

Let $\tilde{X}_0$ be the blow up of the double point p_0 and let E denote the exceptional fibre over the point p_0.

Let $\mathcal{X}_T := \mathcal{X} \times_S T$ be the pull back of $\mathcal{X}$ to a family over T. $\mathcal{X}_T$ then picks up ordinary double point singularities at the singular points of X_0. By the universal property of fibre products there is a lifting $\tilde{i} : \mathcal{C} \hookrightarrow \mathcal{X}_T$.

Let $\mathcal{Y} \to \mathcal{X}_T$ be the blowing up at the ordinary double point p_0. The special fibre of $\mathcal{Y} \to T$ at t_0 is the union of $\tilde{X}_0$ and a smooth quadric Q such that $\tilde{X}_0$ meets Q transversally along E. Let $\tilde{\mathcal{C}}$ be the strict transform under the blow up map. If C_0 passes through the double point p_0 then $\tilde{\mathcal{C}}$ intersects Q in a projective space $\mathbb{P}^{d-1} \subset E$.

We make the following assumption in order to simplify our arguments.

Assumption 1. For the generic fibre X in the family $\mathcal{X} \to S$, the cohomology group $\mathrm{H}^{2d}(X, \mathbb{Z}) \cong \mathbb{Z}$ so that the composite map (for the given embedding $X \hookrightarrow \mathbb{P}^N$)

$$\mathrm{CH}^d(X) \to \mathrm{H}^{2d}(X, \mathbb{Z}) \cong \mathrm{H}_{2d-2}(X, \mathbb{Z}) \cong \mathrm{H}_{2d-2}(\mathbb{P}^N, \mathbb{Z}) \cong \mathbb{Z}$$

is the degree map.

Let A be a relatively ample class on $\mathcal{X} \to S$. Let $\mathcal{H}$ be A^d and denote by H its restriction to a general fibre of the family $\mathcal{X} \to S$. Assume that the degree of H denoted by $deg(H)$ is odd. Then $\Xi = deg(H).\mathcal{C} - deg(C).\mathcal{H}$ is a cycle on $\mathcal{X}_T$ whose restriction to the generic fibre $\xi := deg(H).C - deg(C).H$ is a nullhomologous cycle. Let $\tilde{\mathcal{C}}$ be the strict transform of $\mathcal{C}$ under the map $\mathcal{Y} \to \mathcal{X}_T$ and let $\tilde{\Xi} := deg(H)\tilde{\mathcal{C}} - deg(C).\mathcal{H}$.

2.2. **Detecting nullhomologous cycles.** We shall now show that the cycle $\tilde{\Xi}$ can be detected in the cohomology of the total space $\mathcal{Y}$ in the case when C_0 passes through exactly one ordinary double point. Our method here is entirely topological and avoids any use of Hodge theory. We shall work locally over a disc Δ around the point $0 \in S$ in the base locus. We denote by $\tilde{\Delta}$ the component of the inverse image of Δ in T containing t_0. Having reduced the situation to such a neighbourhood, we note that the special fibre X_0 is a deformation retract of the family $\mathcal{X} \times_S \Delta \to \Delta$. Without loss of generality, we can assume that X_0 contains exactly one ordinary double point. We then have that in the blow up family $\mathcal{Y} \times_T \tilde{\Delta} \to \tilde{\Delta}$, $\tilde{X}_0 \cup Q$ is a deformation retract of $\mathcal{Y} \times_T \tilde{\Delta}$.

Theorem 1. We work with notation as above. Let $\mathcal{N} := (\mathcal{Y} \backslash Q) \times_T \tilde{\Delta}$.

1. Suppose that the special fibre C_0 passes through the ordinary double point p_0, then the image of $\tilde{\Xi} := deg(H).\tilde{\mathcal{C}} - (degC).\mathcal{H}$ is non-trivial 2-torsion in the cohomology of $\mathcal{N}$.

2. If C_0 does not pass through p_0 in X_0 then the class of $\tilde{\Xi}$ is trivial in the cohomology of $\mathcal{N}$.

Proof. (1) Consider the following diagram:

$$\begin{array}{ccccc} & & \mathrm{H}^{2d-1}(E,\mathbb{Z})=0 & & \\ & & \downarrow & & \\ \ldots \to \mathrm{H}^{2d-2}(Q,\mathbb{Z})) & \xrightarrow{i_{Q,*}} & \mathrm{H}^{2d}(\mathcal{Y}_{\hat{\Delta}},\mathbb{Z}) & \xrightarrow{j^*} & \mathrm{H}^{2d}(\mathcal{N};\mathbb{Z}) \to \ldots \\ & & \downarrow (i_Q^*, i^*) & & \\ & & \mathrm{H}^{2d}(Q;\mathbb{Z}) \oplus \mathrm{H}^{2d}(\tilde{X}_0,\mathbb{Z}) & & \\ & & \downarrow & & \\ & & \mathrm{H}^{2d}(E;\mathbb{Z}) & & \end{array}$$

where the horizontal sequence is the Gysin sequence and the vertical one is the Mayer-Vietoris sequence. The composite $i_Q^* \circ i_{Q,*}$ in the diagram

$$\mathrm{H}^{2d-2}(Q,\mathbb{Z}) \xrightarrow{i_{Q,*}} \mathrm{H}^{2d}(\mathcal{Y}_{\hat{\Delta}},\mathbb{Z}) \xrightarrow{i_Q^*} \mathrm{H}^{2d}(Q,\mathbb{Z})$$

is $\cup c_1(N_{Q/\mathcal{Y}})$, the cup product with the first Chern class of the normal bundle of Q in $\mathcal{Y}$. Now, $N_{Q/\mathcal{Y}} \cong \mathcal{O}_Q(-1)$ for the natural embedding $Q \hookrightarrow \mathbb{P}^{2d}$. Since $c_1(\mathcal{O}_Q(1))$ generates $\mathrm{H}^2(Q,\mathbb{Z})$, the first Chern class of this normal bundle can be identified with the class $-[E]$ where $[E]$ is the class of the exceptional fibre in $\tilde{X}_0$.

Furthermore,as Q occurs naturally as a smooth odd-dimensional quadric hypersurface of $\mathbb{P}^{2d}$, all its even cohomology groups are isomorphic to $\mathbb{Z}$.

Let $L \subset \tilde{C}$ be the exceptional fibre. This is contained in Q as a $\mathbb{P}^{d-1}$ and therefore $[L]$ generates $\mathrm{H}^{2d}(Q,\mathbb{Z})$. Thus the class of $\tilde{\Xi}$, denoted by $[\tilde{\Xi}]$, restricts to the class $deg(H).[L]$ in $\mathrm{H}^{2d}(Q,\mathbb{Z})$. Moreover its image under the isomorphism $\mathrm{H}^{2d}(\mathcal{Y}_{\hat{\Delta}}) \cong \mathrm{H}^{2d}(\tilde{X}_0 \cup Q)$ is the class $deg(H).[L]$. On the other hand, since the composition $i_Q^* \circ i_{Q,*} = \cup c_1(\mathcal{O}_Q(-1)$ one has by the projection formula,

$$\eta \cup i_Q^* c_1(\mathcal{O}_{\mathbb{P}^{2d}}(-1)) = i_Q^*(i_{Q,*}(\eta) \cup c_1(\mathcal{O}_{\mathbb{P}^{2d}}(-1)))$$

ordinary double point on each of the singular fibres over the points in the branch locus.

For any integer $1 \leq i \leq l$, one can now construct finitely many diagrams in the following manner:

We choose a small open disc $\Delta_i \subset S$ containing exactly one branch point of δ_i contained in S. Since the various branch loci are disjoint, Δ_i can be so chosen such that it does not contain branch points of δ_j for $j \neq i$. We then have the following diagrams:

$$\begin{array}{ccc} \mathcal{C}_i & \hookrightarrow & \mathcal{X}_i \\ \downarrow & & \downarrow \\ \tilde{\Delta}_i & \xrightarrow{\delta_i} & \Delta_i \end{array}$$

and for $j \neq i$,

$$\begin{array}{ccc} \mathcal{C}_{j,i} & \hookrightarrow & \mathcal{X}_i \\ \downarrow & & \downarrow \\ \Delta_i & \cong & \Delta_i \end{array}$$

Here the families $\mathcal{C}_{j,i}$, $\mathcal{C}_i$, $\mathcal{X}_i$ are the restrictions of the families in diagram (2) to the disc Δ_i. As earlier, we may assume without loss of generality that the singular fibres have exactly one ordinary double point.

Thus we have the following situation.

1. For $j \neq i$ the family $\mathcal{C}_{j,i}$ completely misses the singular locus of the special fibre of the family $\mathcal{X}_i \to \Delta_i$.
2. The map $\tilde{\Delta}_i \to \Delta_i$ is a double cover which is ramified at one point.
3. For the family of cycles $\mathcal{C}_i$, there exists a point $0 \in \Delta_i$ such that the special fibre C_0 at 0 passes through the ordinary double point of the (singular) fibre of $\mathcal{X}_i \to \Delta_i$.

On a general member X of the family $\mathcal{X}_i \to \Delta_i$, we then have finitely many codimension d cycles $\xi_j = deg(H).C_j - deg(C_j).H$ where C_j is a general member of the family $\mathcal{C}_{j,i}$ and H is a codimension d linear section in X.

2.4. **Relations between cycles.** We now wish to study the relations between the cycles $\tilde{\Xi}_i$ defined by ξ_i in X as it varies in the family $\mathcal{Y} \to T$.

Theorem 2. The cycles $\tilde{\Xi}_i$ are linearly independent modulo 2 in $\mathrm{CH}^d(\mathcal{Y})$, the Chow group of codimension d cycles.

Proof. Suppose there exists a relation

$$\sum_i n_i \tilde{\Xi}_i = 0$$

If $p : T \to S$ denotes the morphism between T and S, then one notes that for any i, $p^{-1}(\Delta_i) \cong \tilde{\Delta}_i$. For $i = i_0$, consider the restriction of the above sum to $\mathcal{Y}_{i_0} := \mathcal{Y}_{|_{p^{-1}(\Delta_i)}}$. Note that $\mathcal{Y}_{i_0}$ is isomorphic to $\mathcal{X}_i \times_{\Delta_i} \tilde{\Delta}_i$ blown up at the ordinary double point occuring in the special fibre of $\mathcal{X}_i \to \Delta_i$. Then one has a relation $\sum_i n_i[\tilde{\Xi}_i] = 0$ in $\mathrm{H}^{2d}(\mathcal{Y}_{i_0}, \mathbb{Z})$. Consider its image in the cohomology of $\mathcal{N}_{i_0}$ under the map $\mathrm{H}^{2d}(\mathcal{Y}_{i_0}, \mathbb{Z}) \to \mathrm{H}^{2d}(\mathcal{N}_{i_0}, \mathbb{Z})$. By Theorem 1 we know that $[\tilde{\Xi}_i]$ vanishes for $i \neq i_0$ since these do not intersect the exceptional divisors over the singularities in the special fibre of $\mathcal{X}_{i_0} \to \Delta_{i_0}$. This implies that $n_{i_0}.[\tilde{\Xi}_{i_0}] = 0$. Since $[\tilde{\Xi}_{i_0}]$ is a non-trivial 2-torsion class, this implies that n_{i_0} is divisible by 2. Similarly arguing, we see that 2 divides n_i for all i. Hence the cycles are linearly independent modulo 2. □

2.5. **Torsion in Chow groups.** We shall now make some remarks about the rank of $\mathrm{CH}^d(\mathcal{Y})$. We make note of the following useful lemma.

Lemma 2. If G is an abelian group such that its torsion subgroup G_{tor} is a subgroup of $(\mathbb{Q}/\mathbb{Z})^r$, then we have

$$rank_{\mathbb{Q}}(G \otimes \mathbb{Q}) + r \geq rank_{\mathbb{Z}/2\mathbb{Z}}(G \otimes \mathbb{Z}/2\mathbb{Z})$$

Proof. (see [10]) □

According to above lemma, we need to show that $\mathrm{CH}^d(\mathcal{X})_{tors}$, the torsion subgroup is a subgroup of $(\mathbb{Q}/\mathbb{Z})^r$ for some r. By results of Colliot-Thelene et al [9] one has the following commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}^i(X)(l) & \xrightarrow{\rho} & \mathrm{H}^{2i}(X;\mathbb{Z}_l) \\ \uparrow \alpha & \searrow \lambda_l^d & \uparrow \beta \\ \mathrm{H}^{i-1}(X,\mathcal{H}^i(\mathbb{Q}_l/\mathbb{Z}_l(i))) & \xrightarrow{\gamma} & \mathrm{H}^{2i-1}(X,\mathbb{Q}_l/\mathbb{Z}_l(i)) \end{array}$$

We note the following obvious corollary:

Corollary 1. For X smooth projective variety over $\mathbb{C}$, we have

$$\mathrm{CH}^2(X)_{tors} \subset (\mathbb{Q}/\mathbb{Z})^r$$

for some r.

Proof. The map γ is an inclusion in this case since $E_2^{-1,1}$ of the Bloch-Ogus spetral sequence vanishes. This implies that α is actually an isomorphism and therefore λ_l^2 is an inclusion. This in turn implies that the cycle class map ρ is an injection. Since $\mathrm{H}^4(X,\mathbb{Z}_l) \cong \mathrm{H}^4(X,\mathbb{Z}) \otimes \mathbb{Z}_l$, we get that the usual cycle class map into singular cohomology is injective on l-primary torsion cycles. Now $\mathrm{H}^4(X,\mathbb{Z}) \cong T \oplus \mathbb{Z}^m$ for some m where T is the torsion subgroup of the cohomology. Since $\mathbb{Q}/\mathbb{Z} \cong \bigcup_n \mathbb{Z}/n\mathbb{Z}$ one has $\mathrm{CH}^2(X)(l) \hookrightarrow (\mathbb{Q}/\mathbb{Z})^r$ for some r which is independent of l. The statement now follows by noting that the images of l and l' torsion cycles for $l \neq l'$ are distinct in $(\mathbb{Q}/\mathbb{Z})^r$, and then by taking union over all l. □

We now apply Lemma 2 to conclude that in the situation in the preceding section $rank_{\mathbb{Q}}(\mathrm{CH}^2(\mathcal{Y}) \otimes \mathbb{Q})$ is bounded from below.

Corollary 2. Suppose there exist countably infinite number of diagrams as in 1 satisfying the assumptions above. Further assume $d = 2$. Then the cycles $\{\Xi_i\}$ generate a subgroup whose rank when tensored with $\mathbb{Q}$ is infinite. In particular, the cycles of Clemens and Paranjape can be detected along with their relations after spreading out.

Proof. Let $G \subset \mathrm{CH}^2(\mathcal{Y})$ be the group generated by the cycles $\tilde{\Xi}_i$. Since $G \otimes \mathbb{Z}/2\mathbb{Z}$ has infinite rank we conclude from Lemmas 2 and 1 that G has infinite rank. □

3. The infinitesimal method

3.1. **Some invariants.** Let $f : \mathcal{X} \to S$ be a smooth family of projective varieties with S smooth defined over the field of complex numbers $\mathbb{C}$. We further assume that S is projective. Let $\mathrm{CH}^d(\mathcal{X}/S)_{hom} \subset \mathrm{CH}^d(\mathcal{X})$ be the Chow group of (relative) algebraic cycles which are fibrewise homologically trivial. We are interested in the various cohomological invariants associated to such a cycle, say $\mathcal{Z}$.

Consider the cycle class map

$$\mathrm{CH}^d(\mathcal{X}) \xrightarrow{cl_{dR}} \mathbb{H}^{2d}(\mathcal{X}; \Omega^\bullet_{\mathcal{X}}) \cong \mathrm{H}^{2d}(\mathcal{X}; \mathbb{C})$$

Here the isomorphism is given by the comparison theorem.

Let $[\mathcal{Z}] = cl_{dR}(\mathcal{Z})$ denote the cohomology class of the cycle $\mathcal{Z}$, which lies in the d-th filtered piece $F^d\mathbb{H}^{2d}(\mathcal{X}; \Omega^\bullet_{\mathcal{X}})$. Here F^* stands for the Hodge filtration.

The Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(S, \mathcal{H}^q_{\mathbb{C}}) \Rightarrow \mathrm{H}^{p+q}(\mathcal{X}, \mathbb{C})$$

Deligne (see [11]) has proved in this case that this sequence degenerates at E_2. Here $\mathcal{H}^q_{\mathbb{C}}$ is the local system which has stalk $\mathrm{H}^q(\mathcal{X}_s, \mathbb{C})$ at any point $s \in S$. One has an isomorphism $\mathrm{H}^p(S, \mathcal{H}^q_{\mathbb{C}}) \cong \mathbb{H}^p(S, \Omega^\bullet_S \otimes \mathcal{H}^q)$ where the complex in the right hand side is the de Rham complex associated to the local system $\mathcal{H}^q_{\mathbb{C}}$.

Since $\mathcal{Z}$ is fibrewise homologically trivial, $cl(\mathcal{Z})$ maps to zero in $\mathrm{H}^0(S, \mathcal{H}^{2d}_{\mathbb{C}})$. It then defines the cohomology class $\partial\nu_{\mathcal{Z}} \in \mathrm{H}^1(S, \mathcal{H}^{2d-1}_{\mathbb{C}})$ which we shall refer to as the cohomology class of the normal function associated to the cycle $\mathcal{Z}$. We shall be interested in an infinitesimal invariant associated to $\partial\nu_{\mathcal{Z}}$ which we shall denote by $\delta_1\nu_{\mathcal{Z}}$.

The Hodge filtration on the cohomology of $\mathcal{X}$ induces a filtration on each of the graded pieces in the Leray filtration.

Deligne (see [37]) has proved that the Hodge filtration is the same as the filtration induced by a filtration on the complex $\Omega_S^\bullet \otimes \mathcal{H}^q$ which we descibe below:

Consider the de Rham complex associated to the local system $\mathcal{H}_{\mathbb{C}}^{2d-1}$:

$$0 \to \mathcal{H}^{2d-1} \to \Omega_S^1 \otimes \mathcal{H}^{2d-1} \to \cdots \to \Omega_S^{dimS} \otimes \mathcal{H}^{2d-1} \to 0$$

Then the k-th piece in the filtration of the above complex is the complex $F^k(\mathcal{H}^{2d-1} \otimes \Omega^\bullet)$ defined by

$$0 \to \mathcal{F}^k\mathcal{H}^{2d-1} \to \Omega_S^1 \otimes \mathcal{F}^{k-1}\mathcal{H}^{2d-1} \to \Omega_S^2 \otimes \mathcal{F}^{k-2}\mathcal{H}^{2d-1} \to \ldots$$

$$\to \Omega_S^{dimS} \otimes \mathcal{F}^{k-dimS}\mathcal{H}^{2d-1} \to 0$$

The graded pieces of this are the complexes $KS^{k,2d-1}(\mathcal{X}/S)$:

$$0 \to \mathcal{H}^{k,2d-1-k} \to \mathcal{H}^{k-1,2d-1-k} \otimes \Omega_S^1 \to \ldots$$

The spectral sequence associated to this filtration has

$$E_1^{p,q} = \mathbb{H}^{p+q}(S, KS^{p,2d-1}(\mathcal{X}/S))$$

Lemma 3 (Deligne,([37])). The above spectral sequence degenerates at E_1.

We define $\delta_1\nu_Z$ as the image of $\partial\nu_Z$ under the projection map

$$\mathbb{H}^1(S, H_{\mathbb{C}}^{2d-1}) \to \mathbb{H}^1(S, KS^{d,2d-1})$$

We now state an obvious lemma.

Lemma 4. The non-triviality of the $\delta_1\nu_Z$ implies that $\mathcal{Z}$ has non-zero image under the cycle class map.

Zucker (see [37]) has proved analogous results when the base S is quasi-projective.

Remark 2. The conclusion of the above lemma follows from results in ([37]).

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