Exotic Smooth Structures on Products of Certain Smooth Manifolds with Spheres

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Ankur Sarkar

Ankur Sarkar

Dedicated to my parents and sisters

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Contents

Summary

The classification of smooth structures on manifolds is a central problem in differential topology, particularly for manifolds sharing the same underlying topological type. This thesis studies this problem for product manifolds of the form $M \times \mathbb{S}^k$, where $1 \leq k \leq 10$ and M is a closed, oriented, connected smooth 4-manifold, a closed simply connected smooth 5- or 6-manifold, or a closed, oriented, 3-connected smooth 8-manifold.

We first study smooth structures on manifolds up to concordance. For a smooth manifold N of dimension at least 5, Kirby–Siebenmann identified the set of concordance classes of smooth structures $\mathcal{C}(N)$ with the set of homotopy classes of maps from N to Top/O. From this correspondence, we show that the concordance inertia group $I_c(M \times \mathbb{S}^k)$ is determined by the stable top-cell attaching map of M. In particular, when the stable homotopy type of M is known, the group $I_c(M \times \mathbb{S}^k)$ can be computed explicitly. By analyzing the stable cell structures of the above-mentioned manifolds M and using known computations of the stable homotopy groups of spheres, we compute $I_c(M \times \mathbb{S}^k)$ for all $1 \leq k \leq 10$ and classify smooth structures on $M \times \mathbb{S}^k$ up to concordance for certain values of k.

The second part of the thesis addresses the classification of smooth structures up to diffeomorphism. Let $\mathcal{S}(N)$ denote the set of orientation-preserving diffeomorphism classes of smooth manifolds that are homeomorphic to a given smooth manifold N. The computation of $\mathcal{C}(N)$ plays a key role in this classification. The group of self-homeomorphisms $\mathrm{Homeo}(N)$ acts on $\mathcal{C}(N)$, and this action induces a one-to-one correspondence between $\mathcal{S}(N)$ and the orbit space $\mathcal{C}(N)/\pi_0(\mathrm{Homeo}(N))$. Equivalently, there is a bijection

$$\mathcal{C}(N)/\pi_0(\mathrm{Homeo}(N)) \;\leftrightarrow\; \mathcal{S}(N).$$

Building on the computations of $\mathcal{C}(M \times \mathbb{S}^k)$ and using surgery theory, we analyze this action and determine the inertia group for the manifolds $\mathbb{C}P^2 \times \mathbb{S}^k$ for $4 \leq k \leq 6$, $\mathbb{C}P^3 \times \mathbb{S}^k$ for $2 \leq k \leq 7$, and $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$, where $\widetilde{\mathbb{H}P}^2$ denotes a projective plane-like smooth 8-dimensional manifold. Finally, we obtain a complete diffeomorphism classification of all smooth manifolds homeomorphic to these product spaces, including the case of $\mathbb{C}P^3 \times \mathbb{S}^1$.

List of Symbols

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{H}$	Set of quaternionic numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{C}P^n$	Complex projective n-space
$\mathbb{H}P^n$	Quaternionic projective n-space
$\mathbb{S}^n$	Standard n-dimensional sphere
Θ_n	Group of homotopy n-sphere
$[X, Y]$	Set of homotopy classes of maps from X to Y
$\pi_n(X)$	n^{th}-homotopy group of X
$\mathcal{E}(X)$	Group of self-homotopy equivalence of X
$\Sigma^n X$	n-fold suspension of X
$\Sigma^n M(A)$	Moore spectrum
$X_{(p)}$	Localization of the space X at the prime p
HG	Eilenberg-Maclane spctrum for the abelian group G
$\{-,-\}$	the stable homotopy classes of maps between spectra
$\tau_{\leq m}X$	The m-th Postnikov section of the spectrum X
$\mathcal{P}_m(X)$	The m-th Postnikov section of the space X

$X_{(\frac{1}{p})}$	Localization of the space X away from prime p
$\widetilde{\mathbb{H}P}^2$	Projective plane-like smooth manifold of dimension 8

Chapter 1

Introduction

The classification of smooth manifolds up to diffeomorphism is one of the central problems in differential topology. A major tool for addressing this problem is surgery theory, which provides a general framework for classifying manifolds within a given homotopy type. This framework was effectively applied by Kervaire and Milnor [63] in their study of Θ_n, the group of homotopy n-spheres. However, for broader classes of manifolds, homotopy classification is often nearly as challenging as classification up to diffeomorphism.

Nevertheless, in higher dimensions, several important results have been obtained. For instance, simply connected 5-manifolds were classified by Smale [117] and Barden [8], while the classification of simply connected 6-manifolds was established by Wall [127], Jupp [55], and Zhubr [134]. In dimension 7, although a complete classification is still unknown, substantial progress has been made, including the classification of 2-connected 7-manifolds by Crowley and Nordström [35], and further results using rational homotopy theory to classify simply connected 7-manifolds up to finite ambiguity [36].

In dimension 8, Fang and Pan [41] classified a broad class of 2-connected 8-manifolds up to the action of $\Theta_8 \cong \mathbb{Z}/2$. Earlier, Wall's foundational work [126] established a classification of $(m-1)$-connected $2m$-manifolds for $m \geq 4$, up to the action of Θ_{2m}. More recently, the stabilizers of this Θ_{2m}-action for all $m \geq 4$ were determined by combining results from Senger–Zhang [112], Crowley–Nagy [34], and Crowley–Olbermann [112].

In this thesis, we focus on the diffeomorphism classification of all smooth manifolds homeomorphic to a manifold N, where N is the product of a certain closed, oriented, smooth manifold with a sphere. To address this problem, our first objective is to compute the group $\mathcal{C}(N)$ of concordance classes of smooth structures on N (see Definition 2.3.3), by using the Kirby–Siebenmann identification and analyzing the Θ_n-action on $\mathcal{C}(N)$. The second objective is to study the action of the group $\mathcal{E}(N)$ of self-homotopy equivalences of N on $\mathcal{C}(N)$ via the composition formula for normal invariants [32, Proposition 3.10], using techniques from surgery theory. Recall that the Θ_n-action on $\mathcal{C}(N)$ is given by sending the pair $([\Sigma], [(W, f)])$ to $[(W\#\Sigma, f)]$. The stabilizer of the concordance class of (N, Id) under this action is called the *concordance inertia group* of N, denoted by $I_c(N)$ (see Definition 2.4.11). There is also a natural action of the group Θ_n on the set $\mathcal{S}(N)$ of orientation-preserving diffeomorphism classes of smooth manifolds homeomorphic to N, given by $([\Sigma], [W]) \mapsto [W\#\Sigma]$. The stabilizer of the diffeomorphism class $[N]$ under this action is known as the *inertia group* of N, denoted by $I(N)$ (see Definition 2.4.1). Note that there is a surjective forgetful map $\mathcal{C}(N) \to \mathcal{S}(N)$, which sends $[(W, f)]$ to $[W]$.

The study of inertia groups has a long history. In the case of product manifolds, it was shown by Schultz [108] that $I(\mathbb{S}^p \times \mathbb{S}^q) = 0$ when $p+q \geq 5$. Kawakubo [61] further studied the inertia group of $\mathbb{S}^p \times \Sigma$, where Σ is an exotic sphere, and analyzed smooth structures on such products [62]. Later, Crowley [32] extended this line of study using a surgery-theoretic classification of manifolds homotopy equivalent to $\mathbb{S}^p \times \mathbb{S}^q$ for all $p, q \geq 2$ with $p + q \geq 5$. A natural generalization of these results is to consider the classification of all smooth manifolds homeomorphic to the product manifold $\mathbb{C}P^n \times \Sigma^k$, where Σ^k is an exotic k-sphere. This problem, which traces back to Browder, has been revisited in recent work by Masuda and Schultz [79] and by Belegradek, Kwasik and Schultz [18], using delicate arguments combining surgery theory and homotopy-theoretic techniques. In particular, [18] provides the diffeomorphism classification of all manifolds tangentially homotopy equivalent to $\mathbb{C}P^2 \times \mathbb{S}^7$, and it was shown in [79] that the inertia group of $\mathbb{C}P^2 \times W$ is Θ_7, where W is any closed, oriented 3-manifold.

There is no general method for computing inertia groups, and many problems remain open. Therefore, it is often useful to study more tractable subgroups, such as the concordance inertia group $I_c(N)$, which is a homotopy invariant of N (see [28]).

In this thesis, we compute the concordance inertia group $I_c(M \times \mathbb{S}^k)$, where M is a closed, oriented, smooth manifold of dimension 4, 5, 6, or 8, under certain mild assumptions (see Theorems 4.1.5, 4.2.2, 4.3.10, 4.4.1). We also determine the group $\mathcal{C}(M \times \mathbb{S}^k)$ of concordance classes of smooth structures for specific values of k (see Propositions 5.1.5, 5.2.1, Theorems 5.3.1, 5.4.2). These computations are carried out using the stable cell structure of M, together with known computations of the stable homotopy groups of spheres.

In particular, when $M = \mathbb{C}P^2$ or $\mathbb{C}P^3$, we analyze the normal invariants of self-homotopy equivalences of $M \times \mathbb{S}^k$, following the methods of [18] and [107], and study their action on $\mathcal{C}(M \times \mathbb{S}^k)$. As a consequence, we obtain the following:

Theorem A. *Let M be a closed, oriented, smooth manifold of dimension m.*

(i) If $M = \mathbb{C}P^2$, then

$$I(\mathbb{C}P^2 \times \mathbb{S}^k) = \begin{cases} \Theta_{4+k}, & \text{for } k = 4 \text{ or } 5; \\ \mathbb{Z}/2, & \text{for } k = 6. \end{cases}$$

(ii) If $M = \mathbb{C}P^3$, then

$$I(\mathbb{C}P^3 \times \mathbb{S}^k) = \begin{cases} 0, & \text{for } k = 2, 3 \text{ or } 6; \\ \mathbb{Z}/3, & \text{for } k = 4 \text{ or } 7; \\ \mathbb{Z}/62 & \text{for } k = 5. \end{cases}$$

(See Corollaries 6.1.3, 6.1.6 and Theorem 6.1.10 for part (i); Corollaries 6.2.9, 4.3.11, and Theorems 6.2.16, 6.2.22, 6.2.26 for (ii).)

Remark 1.0.1. *It is already known due to Brumfiel that $I(\mathbb{C}P^3 \times \mathbb{S}^1)$ is $\mathbb{Z}/7$ [28, Remark II.10].*

Theorem B. *Let M be a closed, oriented, smooth manifold of dimension m.*

(1) If $M = \mathbb{C}P^2$, then

  (a) $\mathcal{S}(\mathbb{C}P^2 \times \mathbb{S}^k) = \{[\mathbb{C}P^2 \times \mathbb{S}^k]\}$ *for* $k = 4, 5$.

  (b) $\mathcal{S}(\mathbb{C}P^2 \times \mathbb{S}^6) = \{[\mathbb{C}P^2 \times \mathbb{S}^6], [(\mathbb{C}P^2 \times \mathbb{S}^6)\#\Sigma_{\beta_1}], [(\mathbb{C}P^2 \times \mathbb{S}^6)\#\Sigma_{\beta_1}^{-1}]\}$, *where* $[\Sigma_{\beta_1}] \in \Theta_{10}$ *is the exotic sphere corresponding to the order* 3 *generator* β_1 *of* π_{10}^s.

(2) If $M = \mathbb{C}P^3$, then

  (a) $\mathcal{S}(\mathbb{C}P^3 \times \mathbb{S}^1) = \{[(\mathbb{C}P^3 \times \mathbb{S}^1)\#\Sigma^7] : [\Sigma^7] \in \mathbb{Z}/4 \subset \Theta_7\}$.

  (b) $\mathcal{S}(\mathbb{C}P^3 \times \mathbb{S}^2) = \{[\mathbb{C}P^3 \times \mathbb{S}^2], [(\mathbb{C}P^3 \times \mathbb{S}^2)\#\Sigma^8]\}$, *where* $[\Sigma^8]$ *is the exotic* 8*-sphere.*

  (c) $\mathcal{S}(\mathbb{C}P^3 \times \mathbb{S}^3) = \{[(\mathbb{C}P^3 \times \mathbb{S}^3)\#\Sigma^9] : [\Sigma^9] \in \Theta_9\}$.

  (d) $\mathcal{S}(\mathbb{C}P^3 \times \mathbb{S}^4) = \{[\mathbb{C}P^3 \times \mathbb{S}^4], [(\mathbb{C}P^3 \times \mathbb{S}^4)\#\Sigma_{\eta\mu}]\}$, *where* $[\Sigma_{\eta\mu}] \in \Theta_{10}$ *is the exotic* 10*-sphere of order* 2.

  (e) $\mathcal{S}(\mathbb{C}P^3 \times \mathbb{S}^5) = \{[(\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma^{11}] : [\Sigma^{11}] \in \mathbb{Z}/2^4 \subset \Theta_{11}\}$.

  (f) $\mathcal{S}(\mathbb{C}P^3 \times \mathbb{S}^6) = \{[\mathbb{C}P^3 \times \mathbb{S}^6],\ [\widetilde{N}]\}$, *where the manifold* $\widetilde{N}$ *is determined by normal invariant.*

  (g) $\mathcal{S}(\mathbb{C}P^3 \times \mathbb{S}^7) = \{[\mathbb{C}P^3 \times \mathbb{S}^7]\}$.

(See Theorems 6.1.2, 6.1.5, 6.1.11 for part (1); and Theorems 6.2.5, 6.2.8, 6.2.18, 6.2.23, 6.2.27, 6.2.29, and 6.2.32 for part (2).)

Furthermore, when M is a projective plane-like smooth manifold of dimension ≥ 8, we obtain the following:

Theorem C. *Let M be a smooth projective plane-like manifold of dimension $2n \geq 8$.*

(i) If $n = 4$, then

- *(a) $I(M \times \mathbb{S}^1) = bP_{10} \oplus \mathbb{Z}/2\{[\Sigma_{\nu^3}]\}$, where $[\Sigma_{\nu^3}]$ is the exotic 9-sphere associated with the generator $[\nu^3] \in \pi_9^s/Im(J)$.*
- *(b) $\mathcal{S}(M \times \mathbb{S}^1) = \{[M \times \mathbb{S}^1], [(M \times \mathbb{S}^1)\#\Sigma_\mu]\}$, where $[\Sigma_\mu]$ is the 9-dimensional exotic sphere corresponding to the element $[\mu] \in \pi_9^s/Im(J)$.*

(ii) If $n = 8$, then $I(M \times \mathbb{S}^1) = bP_{18} \oplus \mathbb{Z}/2\{[\Sigma_{\eta\eta^*}]\}$, where $[\Sigma_{\eta\eta^*}]$ is the exotic 17-sphere corresponding to the generator $[\eta\eta^*] \in \pi_{17}^s/Im(J)$.

(See Lemma 6.3.2 and Theorem 6.3.3 for parts (i)(a) and (i)(b), respectively; and Lemma 6.3.4 for part (ii).)

Organization of the Thesis

This thesis is organized into six chapters.

Chapter 2 introduces the necessary background material and collects foundational results that are used throughout the thesis.

Chapter 3 determines the stable homotopy types of the following classes of manifolds: closed, connected, oriented, smooth 4-manifolds; closed, simply connected, smooth 5-manifolds; and closed, 3-connected, smooth, 8-manifolds. It also analyzes the top cell attaching map of a closed, simply connected, smooth 6-dimensional manifold, based on its minimal cell structure.

Chapter 4 focuses on computing the concordance inertia group $I_c(M \times \mathbb{S}^k)$ for various values of k, using the stable decomposition of M obtained in Chapter 3.

Chapter 5 is devoted to determining the concordance smooth structures set $C(M \times \mathbb{S}^k)$, for some $1 \leq k \leq 10$, by combining the results from Chapter 4 with the decompositions established in Chapter 3.

Chapter 6 studies the normal invariants of self-homotopy equivalences of $M \times \mathbb{S}^k$, where $M = \mathbb{C}P^2$, $\mathbb{C}P^3$, or a projective plane-like smooth 8-manifold, and k ranges over selected values between 1 and 10. Using these computations, together with the surgery exact sequence and the results of Chapters 4 and 5, we establish the classification results stated in Theorems A, B, and C.

Chapter 2

Preliminaries

This chapter recalls the foundational tools from homotopy theory and surgery theory that will be used throughout the thesis.

2.1 Group of Homotopy Spheres

In this section, we recall the definition of the group of homotopy spheres and the basic results that will be required later.

Definition 2.1.1. *[86] A* cobordism *between two closed, oriented d-manifolds M and N is a $(d+1)$-dimensional manifold W whose boundary satisfies $\partial W = M \cup \overline{N}$, where $\overline{N}$ denotes the manifold with the same underlying topological and smooth structure as N, but equipped with the opposite orientation.*

If $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences, then W is a h-cobordism.

Theorem 2.1.2. *[86] Let W be a simply connected h-cobordism between two simply connected, closed manifolds M and N of dimension ≥ 5. Then W is (oriented) diffeomorphic to $M \times [0, 1]$.*

Definition 2.1.3 (Homotopy Sphere)**.** *A* homotopy d-sphere *is a closed, oriented, smooth d-manifold Σ^d that is homotopy equivalent to the standard sphere $\mathbb{S}^d$.*

The Generalized Poincaré Conjecture asserts that: Every closed, topological manifold that is homotopy equivalent to the d-sphere $\mathbb{S}^d$ is homeomorphic to $\mathbb{S}^d$.

The Generalized Poincaré Conjecture has been established in all dimensions. In dimensions one and two, the result is classical: any 1-manifold homotopy equivalent to $\mathbb{S}^1$ is homeomorphic to it, and the case $n = 2$ follows from the classification of compact surfaces. The three-dimensional case corresponds to the original Poincaré Conjecture and was resolved by Perelman using Ricci flow with surgery [100, 88]. In dimension four, Freedman proved that any closed topological 4-manifold homotopy equivalent to $\mathbb{S}^4$ is indeed homeomorphic to it [43]. For dimensions $n \geq 5$, Smale established the result under additional smoothness assumptions using handlebody theory and the h- and s-cobordism theorems [116], while the general topological case, without any smoothness assumption, was later proved by Newman [97, Theorem 7] and Connell [31, Corollary 1]. Therefore, a homotopy d-sphere refers to a closed, oriented, smooth manifold that is homeomorphic to $\mathbb{S}^d$.

One may ask whether the Generalized Poincaré Conjecture holds in the smooth category i.e.,

Is every homotopy d-sphere diffeomorphic to the standard d-sphere?

This turns out to be true for $d \leq 3$ while the case $d = 4$ remains unsolved. In 1956, Milnor provided a counterexample in dimension 7 by constructing a homotopy 7-sphere which is not diffeomorphic to standard unit sphere $\mathbb{S}^7$ [84]. This example prompts the following definition of an *exotic sphere*.

Definition 2.1.4 (Exotic Sphere)**.** *A homotopy d-sphere Σ^d is said to be* exotic *if it is not diffeomorphic to standard $\mathbb{S}^d$.*

Let

$$\bar{\Theta}_d := \frac{\{\Sigma^d \mid \Sigma^d \text{ is a homotopy } d\text{-sphere}\}}{\{\text{(oriented) diffeomorphism}\}}.$$

This raises the following natural questions:

(i) How many elements does $\bar{\Theta}_d$ contain?

(ii) For which d is $\bar{\Theta}_d$ trivial?

Kervaire and Milnor [63] addressed these questions by relating $\bar{\Theta}_d$ to

$$\Theta_d := \frac{\{\Sigma^d \mid \Sigma^d \text{ is a homotopy } d\text{-sphere }\}}{\{\text{ (oriented) h-cobordism}\}}.$$

and computing Θ_d in many cases.

Theorem 2.1.5 (Properties of Θ_d)**.**

(i) The set Θ_d forms an abelian group under the connected sum operation. The identity element is the class of the standard unit sphere $\mathbb{S}^d$, and the inverse of an element $[M^d]$ is $[\overline{M^d}]$, where $\overline{M^d}$ denotes the manifold M^d with the opposite orientation [63].

(ii) The group Θ_d is finite for all $d \geq 1$ [63, 124, 88].

Since two (oriented) diffeomorphic manifolds are h-cobordant, there is a well-defined surjective map

$$\mathcal{X} : \Theta_d \to \bar{\Theta}_d.$$

Let M and N be closed manifolds homeomorphic to $\mathbb{S}^d$ with $d \geq 5$. Then, by Theorem 2.1.2, M is (oriented) diffeomorphic to N. Thus, for $d \geq 5$, the map $\mathcal{X} : \Theta_d \to \bar{\Theta}_d$ is a bijection.

It also follows from the classification of manifolds in dimensions $d \leq 3$ that two homotopy d-spheres are h-cobordant if and only if they are oriented diffeomorphic. Hence, $\mathcal{X} : \Theta_d \to \bar{\Theta}_d$ is a bijection for $d \leq 3$ as well. Thus, $\bar{\Theta}_d$ is a finite abelian group for all $d \neq 4$. Using this bijection along with the computation of Θ_d by Kervaire and Milnor, we have the following:

Theorem 2.1.6. *[63] The values of $\bar{\Theta}_d$ in the low-dimensional cases are as follows.*

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\bar{\Theta}_d$	0	0	0	?	0	0	$\mathbb{Z}/28$	$\mathbb{Z}/2$	$3\mathbb{Z}/2$	$\mathbb{Z}/6$	$\mathbb{Z}/992$	0	$\mathbb{Z}/3$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus G$	$\mathbb{Z}/2$	$4\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/8$	$H \oplus \mathbb{Z}/2$	$\mathbb{Z}/24$

where $G = \mathbb{Z}/8128$ *and* $H = \mathbb{Z}/130816$.

Note 2.1.7. *Kervaire and Milnor showed that* $\Theta_4 = 0$, *while* $\bar{\Theta}_4$ *remains unknown.*

From now on, we denote by Θ_d the group of smooth homotopy d-spheres up to orientation-preserving diffeomorphism, for $d \neq 4$.

In the study of homotopy spheres, a natural and well-understood subclass consists of those that bound parallelizable manifolds. These spheres form a subgroup of Θ_d, denoted bP_{d+1}, which is particularly amenable to explicit computation and often serves as a tractable component in the analysis of Θ_d. A homotopy d-sphere Σ^d lies in bP_{d+1} if and only if it bounds a parallelizable $(d+1)$-manifold W, that is, $\partial W = \Sigma^d$.

Theorem 2.1.8. *[63] The group* bP_{d+1} *is a finite cyclic group for all* d. *In particular,*

(i) $bP_{2k+1} = 0$ *for all* $k \geq 0$.

(ii) bP_{4k+2} *is either* 0 *or* $\mathbb{Z}/2$ *for all* $k \geq 0$. *Moreover, if* $4k + 2 \neq 2^j - 2$, *then* $bP_{4k+2} = \mathbb{Z}/2$.

(iii) bP_2, bP_6, bP_{14} *are all zero.*

(iv) For each $k \geq 2$, *the group* bP_{4k} *is cyclic of order*

$$\frac{3-(-1)^k}{2}\, 2^{2k-2}\,(2^{2k-1}-1)\, \mathrm{num}\left(\frac{B_k}{4k}\right),$$

where B_k *denotes the* k*-th Bernoulli number and* $\mathrm{num}(\cdot)$ *denotes the numerator.*

Remark 2.1.9. *The generator of* bP_{4k} *for* $k \geq 2$ *is known as a Milnor sphere; it bounds a parallelizable manifold with signature* ± 8.

Theorem 2.1.10. *The following computations of bP_n are known.*

(i) $bP_{30} = 0$ *[22, 78, 48].*

(ii) $bP_{62} = 0$ *[10, 129].*

(iii) $bP_{8k+2} = \mathbb{Z}/2$ *[24, Corollary 1.2].*

(iv) $bP_{4k+2} = \mathbb{Z}/2$ *if* $4k+1 \neq 2^i - 3$ *[22, Corollary 2].*

(v) $bP_{2^j-2} = \mathbb{Z}/2$ *for* $j \geq 8$ *[48].*

(vi) $bP_{126} = 0$ *[73].*

We now recall the following description of the group Θ_d in terms of the stable homotopy groups of spheres π_d^s and the groups bP_{d+1}.

Theorem 2.1.11. *Let* $d \geq 5$*.*

(a) For $d \not\equiv 2 \pmod 4$*, there is an exact sequence*

$$0 \longrightarrow bP_{d+1} \longrightarrow \Theta_d \longrightarrow \pi_d^s/Im(J) \longrightarrow 0,$$

where $\pi_n^s/Im(J)$ *denotes the cokernel of the* J *is the classical* J*-homomorphism.*

(b) There is an exact sequence

$$0 \to \Theta_{4k+2} \to \pi_{4k+2}^s/Im(J) \xrightarrow{\Phi} \mathbb{Z}/2 \to bP_{4k+2} \to 0,$$

where Φ *is the Kervaire invariant.*

For all values of $4k+2$ *except 6, 14, 30, 62, and 126, the group* $\bar{\Theta}_{4k+2} \cong \pi_{4k+2}^s/Im(J)$*. In these exceptional dimensions, there is a short exact sequence*

$$0 \to \Theta_{4k+2} \to \pi_{4k+2}^s/Im(J) \xrightarrow{\Phi} \mathbb{Z}/2 \to 0.$$

(c) $\Theta_{4k+3} \cong bP_{4k+4} \oplus \pi_{4k+3}^s/Im(J)$ *[25, Theorem 1.3].*

(d) $\Theta_{4k+1} \cong bP_{4k+2} \oplus \pi^s_{4k+1}/Im(J)$ *[27, Theorem 1.1].*

The following theorem states the dimensions in which $\mathbb{S}^d$ admits a unique smooth structure.

Theorem 2.1.12.

- *For* $d \leq 140$*, the group* $\bar{\Theta}_d$ *is trivial exactly when* $d = 1, 2, 3, 5, 6, 12, 56, 61$*, and potentially also when* $d = 4$ *[17].*

- *The only odd-dimensional spheres that are known to admit a unique smooth structure are those of dimensions 1, 3, 5, and 61 [129].*

Conjecture 2.1.13. *[129] For dimensions greater than* 4*, the only spheres known to admit a unique smooth structure are* $\mathbb{S}^5$*,* $\mathbb{S}^6$*,* $\mathbb{S}^{12}$*,* $\mathbb{S}^{56}$*, and* $\mathbb{S}^{61}$*.*

2.2 Bundle Theory in Classification of High-Dimensional Manifolds

Bundle theory plays a crucial role in the classification of high-dimensional manifolds by providing the framework to study tangent bundles and their reductions. Classifying spaces associated with structure groups help encode smooth, PL, or topological structures through homotopy-theoretic data.

Recall that

Definition 2.2.1. *Let* G *be a topological group. A* universal principal G-bundle *is a principal* G*-bundle* $G \to EG \xrightarrow{\pi} BG$ *satisfying the following conditions:*

- *The total space* EG *is contractible.*

- *The base space* BG*, known as the* classifying space *of* G*, is a CW complex.*

- *For any principal* G*-bundle* $p : E \to B$ *over a CW complex* B*, there exists a continuous map* $f : B \to BG$*, unique up to homotopy, such that the pullback bundle* $f^*(EG)$ *is isomorphic to* E*.*

Definition 2.2.2.

- *A* real vector bundle of rank n *over a topological space* B *is a topological space* E *together with a continuous surjection* $p : E \to B$ *such that:*
  1. *For each* $b \in B$*, the fiber* $E_b = p^{-1}(b)$ *is a real vector space of dimension* n*, and*
  2. *There exists an open cover* $\{U_\alpha\}$ *of* B *and homeomorphisms (called local trivializations)*

     $$\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$$

     such that $p = Pr_1 \circ \varphi_\alpha$*, and for each* $x \in U_\alpha$*, the restriction*

     $$\varphi_\alpha|_{E_x} : E_x \to \{x\} \times \mathbb{R}^n$$

     is a linear isomorphism.
- *Two vector bundles* $p_i : E_i \to B$*, for* $i = 1, 2$ *over a base space* B*, are said to be* isomorphic *if there exists a map* $H : E_1 \to E_2$ *such that* $p_2 \circ H = p_1$*, and for each* $b \in B$*, the restriction* $H|_{p_1^{-1}(b)} : p_1^{-1}(b) \to p_2^{-1}(b)$ *is a linear isomorphism.*

Let $O(n)$ be the orthogonal group of real $n \times n$ matrices. The space $BO(n)$ is the corresponding classifying space, equipped with a universal rank n real vector bundle.

Proposition 2.2.3. *[82, 119] Let* B *be a paracompact, Hausdorff space. Then the isomorphism classes of rank* n *real vector bundles over* B *are in one-to-one correspondence with the homotopy classes of maps* $B \to BO(n)$*.*

In particular, if M *is an* n*-dimensional smooth manifold, then its tangent bundle is classified by a map* $M \to BO(n)$*.*

Given a map $f : B \to BO(n)$, the corresponding vector bundle over B is obtained as the pullback of the universal bundle over $BO(n)$ along f. This correspondence provides a complete topological classification of rank n real vector bundles.

Definition 2.2.4.

- *Two vector bundles $p : E \to B$ and $q : F \to B$ are* stably isomorphic *if there exist trivial bundles ε^k and ε^ℓ such that*

$$E \oplus \varepsilon^k \cong F \oplus \varepsilon^\ell.$$

- *A stable vector bundle over a space B is an equivalence class of vector bundles under the relation of stable isomorphism.*

Taking Whitney sum of the universal rank n real vector bundle with a trivial line bundle yields the canonical inclusion

$$BO(n) \hookrightarrow BO(n+1).$$

The *stable classifying space* is defined as the colimit:

$$BO = \varinjlim_{n} BO(n).$$

The space BO is a homotopy associative, homotopy commutative H-space with homotopy inverse [4, 77]. Hence $[X, BO]$ is an abelian group.

Theorem 2.2.5. *The isomorphism classes of stable vector bundles over a paracompact, Hausdorff space X are in one-to-one correspondence with $[X, BO]$.*

Theorem 2.2.6. *[3] $[X, BO] \cong \widetilde{KO}^0(X)$, where $KO(X)$ denotes the Grothendieck group of real vector bundles over X.*

Using the fact that $\widetilde{KO}^0(\Sigma^n X) \cong \widetilde{KO}^{-n}(X)$ and Bott periodicity $\widetilde{KO}^{n+8}(X) \cong \widetilde{KO}^n(X)$, we get

Theorem 2.2.7. *[45] The homotopy group of BO in degree n is as follows:*

$$\pi_n(BO) = \begin{cases} \mathbb{Z} & \text{if } n = 8k \text{ or } n = 8k+4, \\ \mathbb{Z}/2 & \text{if } n = 8k+1 \text{ or } n = 8k+2, \\ 0 & \text{otherwise.} \end{cases}$$

Now we define the notion of tangent bundle over a topological manifold.

Definition 2.2.8 (Microbundle [81])**.** *A* microbundle *is a diagram $B \xrightarrow{i} E \xrightarrow{p} B$ with the following data:*

(i) B and E are topological spaces.

(ii) $p \circ i = Id_B$.

(iii) For each $b \in B$, there exist open neighbourhoods U containing b and V containing $i(b)$, together with a homeomorphism $h : V \to U \times \mathbb{R}^n$ such that the following diagram commutes:

$$\begin{tikzcd} & V \arrow[dr, "p|_V"] \arrow[dd, "h"] & \\ U \arrow[ur, "i|_U"] \arrow[dr, "Id_U\times\{0\}"'] & & U \\ & U \times \mathbb{R}^n. \arrow[ur, "Pr_1"'] & \end{tikzcd}$$

Here $Pr_1 : U \times \mathbb{R}^n \to U$ denotes the projection map onto the first factor.

Example 2.2.9. *(1) For any topological space B and $n \geq 0$, the diagram*

$$B \xrightarrow{Id_B\times\{0\}} B \times \mathbb{R}^n \xrightarrow{Pr_1} B$$

forms a microbundle, known as trivial microbundle. It is denoted by ϵ^n_B.

(2) Let M be a topological manifold, and let $\Delta : M \to M \times M$ be the diagonal map. Then the diagram $M \xrightarrow{\Delta} M \times M \xrightarrow{Pr_1} M$ defines a microbundle over M, called the tangent microbundle and denoted by t_M.

Definition 2.2.10 (Equivalence of Microbundles)**.** *Let* $B \xrightarrow{i_l} E_l \xrightarrow{p_l} B$, *for* $l = 1, 2$, *be two microbundles over* B. *We say that these microbundles are* isomorphic *if there exist neighborhoods* $V_1 \subset E_1$ *of* $i_1(B)$ *and* $V_2 \subset E_2$ *of* $i_2(B)$, *along with a homeomorphism* $\phi : V_1 \to V_2$, *such that the following diagram commutes:*

$$\begin{tikzcd}
 & V_1 \arrow[dd, "\phi"] \arrow[dr, "p_1"] & \\
B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\
 & V_2. \arrow[ur, "p_2"'] &
\end{tikzcd}$$

The following theorem relates microbundle theory to the classical theory of vector bundles.

Theorem 2.2.11. *[81, Theorem 2.2] Let* M *be a smooth paracompact manifold, and let* $p : TM \to M$ *its tangent bundle with zero section* $i : M \to TM$. *Then the microbundle given by the diagram* $M \xrightarrow{i} TM \xrightarrow{p} M$ *is isomorphic to the tangent microbundle* t_M.

Recall a Top_n-bundle (or topological $\mathbb{R}^n$-bundle) over a topological space B is a fiber bundle $E \to B$ whose fiber is homeomorphic to $\mathbb{R}^n$.

Theorem 2.2.12. *[66] Let* B *be a paracompact topological space. Then there is a natural bijection between isomorphism classes of* Top_n*-bundles over* B *and equivalence classes of rank* n*-microbundles over* B.

We note that the structure group of Top_n-bundle is the group of homeomorphisms of $\mathbb{R}^n$ fixing the origin. We denote this group by $Top(n)$. The classifying space of topological $\mathbb{R}^n$-bundles is denoted by $BTop(n)$, and there is a universal topological $\mathbb{R}^n$-bundle with base space $BTop(n)$. Then combining Theorem 2.2.12 with standard classifyng space theory we get the following.

Theorem 2.2.13. *The equivalence classes of rank* n*-microbundles over a paracompact topological space* B *is in one to one correspondence with the homotopy classes of maps* $[B, BTop(n)]$.

Definition 2.2.14. *Let B be a topological space.*

(i) Let $\chi = (B \xrightarrow{i} E \xrightarrow{p} B)$ and $\chi' = (B \xrightarrow{i'} E' \xrightarrow{p'} B)$ be two microbundles over B. Then the whitney sum of $\chi \oplus \chi'$ is defined as the diagram

$$B \xrightarrow{\tilde{i}} E(\chi \oplus \chi') \xrightarrow{\tilde{p}} B,$$

where $E(\chi \oplus \chi') = \{(e, e') \in E \times E' : p(e) = p'(e')\}$, $\tilde{i}(b) = (i(b), i'(b))$ and $\tilde{p}(e, e') = p(e)$.

(ii) Two microbundles χ and χ' over B are stably equivalent *if there exist integers $k, l \geq 0$ such that $\chi \oplus \epsilon^k_B \cong \chi' \oplus \epsilon^l_B$.*

Theorem 2.2.15. *Let X be a paracompact Hausdorff space. Then stable isomorphism classes of topological microbundles over X correspond bijectively to homotopy classes of maps $[X, BTop]$, where*

$$BTop = \varinjlim_n BTop(n)$$

is the colimit taken over the standard inclusions $BTop(n) \hookrightarrow BTop(n+1)$, induced by taking the Whitney sum of the universal topological $\mathbb{R}^n$-bundle with a trivial rank-one microbundle.

There is an analogous concept of a PL microbundle, along with a notion of equivalence between two PL microbundles over the same base space. In Definitions 2.2.8 and 2.2.10, we need to take both E and B to be locally finite simplicial complexes, the maps i and p to be PL-maps, and h and ϕ to be PL-homeomorphisms.

A simplicial complex M is called a *PL manifold* if each point $p \in M$ has a neighborhood that is PL-homeomorphic to $\mathbb{R}^n$. Then as in the topological setting, the PL tangent microbundle of M is given by the diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{Pr_1} M.$$

Definition 2.2.16. *A* PL $\mathbb{R}^n$-bundle is a topological $\mathbb{R}^n$-bundle $p : E \to B$ *such that:*

- *both E and B are simplicial complexes;*
- *the projection map p is piecewise linear;*
- *the zero section $s : B \to E$ is piecewise linear;*
- *for each simplex $\Delta \subseteq B$, there exists a PL-homeomorphism $\varphi : p^{-1}(\Delta) \to \Delta \times \mathbb{R}^n$ such that $\varphi(s(\Delta)) = \Delta \times \{0\}$.*

Theorem 2.2.17. *[69] Let B be a locally finite simplicial complex. Then there is a bijection between isomorphism classes of PL $\mathbb{R}^n$-bundles over B and equivalence classes of rank n PL microbundles over B.*

Therefore, to study rank n PL microbundles, it suffices to consider PL $\mathbb{R}^n$-bundles. We now describe a classifying space for PL $\mathbb{R}^n$-bundles.

Definition 2.2.18. *For each $n \in \mathbb{N}_0$, the simplicial group $(PL_n)_\bullet$ is defined as a contravariant functor from the simplicial category to the category of groups as follows:*

(a) For each $k \geq 0$, the group $(PL_n)_k$ assigned to the k-simplex consists of PL homeomorphisms

$$f : \Delta^k \times \mathbb{R}^n \to \Delta^k \times \mathbb{R}^n$$

that preserve the zero section and commute with the projection onto Δ^k.

(b) Given a morphism $\lambda : \Delta^l \to \Delta^k$ in the simplicial category, the induced map $\lambda^* : (PL_n)_k \to (PL_n)_l$ is defined by pullback. For $f \in (PL_n)_k$, the element $\lambda^*(f) \in (PL_n)_l$ is such that the following diagram commutes:

$$\begin{tikzcd}
\Delta^l \times \mathbb{R}^n \arrow[r, "\lambda^*(f)"] \arrow[d, "\lambda \times Id_{\mathbb{R}^n}"'] & \Delta^l \times \mathbb{R}^n \arrow[d, "\lambda \times Id_{\mathbb{R}^n}"] \\
\Delta^k \times \mathbb{R}^n \arrow[r, "f"'] & \Delta^k \times \mathbb{R}^n
\end{tikzcd}$$

The geometric realization $PL_n = |(PL_n)_\bullet|$ is a topological group [85]. Using Milnor's join construction [83], the classifying space $BPL(n)$ is defined to classify PL $\mathbb{R}^n$-bundles over CW complexes.

Theorem 2.2.19. *There exists a universal rank n PL microbundle γ^n over $BPL(n)$ such that for any rank n PL microbundle over ξ over a locally finite simplicial complex B there exits a unique homotopy class of maps $f : B \to BPL(n)$ such that $f^*(\gamma^n) \cong \xi$.*

In other words, the equivalence classes of rank n PL microbundles over a locally finite simplicial complex B are in one to one correspondence with the homotopy classes of maps $[B, BPL(n)]$.

Now, as in Definition 2.2.14, one can define the Whitney sum of two PL microbundles over the same base space, along with the notion of stable equivalence. Then we have

Theorem 2.2.20. *Let B be a locally finite simplicial complex. Then stable isomorphism classes of PL microbundles over B are in one-to-one correspondence with homotopy classes of maps $[B, BPL]$, where*

$$BPL = \varinjlim_n BPL(n).$$

is the colimit taken over the standard inclusions $BPL(n) \hookrightarrow BPL(n+1)$ induced by taking the Whitney sum of the universal rank n PL microbundle with a trivial PL microbundle of rank one.

Theorem 2.2.21.

(i) $\pi_4(BPL) = \mathbb{Z}$ *[131].*

(ii) $\pi_8(BPL) = \mathbb{Z} \oplus \mathbb{Z}/4$ *[131].*

(ii) $\pi_{4n}(BPL) \cong \mathbb{Z} \oplus \pi^s_{4n-1}/\mathrm{Im}(J)'$ *for all* $n > 2$, *where* $\mathrm{Im}(J)' \subset \mathrm{Im}\left(J : \pi_{4n}(BO) \to \pi^s_{4n-1}\right)$ *is the subgroup of elements of odd order [25, Theorem 1.4].*

Definition 2.2.22 (Spherical Fibration and Fiber homotopy equivalence)**.** *Let B be a CW complex. A $(n-1)$-*spherical fibration *is a hurewicz fibration $p : E \to B$ whose fibers are homotopy equivalent to $\mathbb{S}^{n-1}$.*

Two $(n-1)$-spherical fibrations $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are fiber homotopy equivalent *if there is a homotopy equivalence $f : E_1 \to E_2$ and E_2 such that $p_2 \circ f \simeq p_1$.*

Let $G(n)$ be the topological monoid of self-homotopy equivalences of $\mathbb{S}^{n-1}$ and $BG(n)$ be the corresponding classifying space.

Theorem 2.2.23. *Let X be a CW complex. Then the set of fiber homotopy equivalence classes of $(n-1)$-spherical fibrations over X is in one-to-one correspondence with the set $[X, BG(n)]$.*

To understand stable spherical fibrations, we first define the direct sum of two spherical fibrations.

Definition 2.2.24.

(a) Let $\xi_1 : E_1 \to B_1$ be a $(k-1)$-spherical fibration and $\xi_2 : E_2 \to B_2$ an $(l-1)$-spherical fibration. One can construct a new $(k+l-1)$-spherical fibration $\xi_1 * \xi_2 : E_1 * E_2 \to B_1 \times B_2$ by forming the fiberwise join *of these two fibrations. The total space of this fibration is obtained from the product $E_1 \times E_2 \times [0,1]$ by identifying points as follows: $(e_1, e_2, 0)$ is identified with $(e_1, e_2', 0)$ whenever $e_1 \in E_1$ is fixed and $\xi_2(e_2) = \xi_2(e_2')$, and $(e_1, e_2, 1)$ is identified with $(e_1', e_2, 1)$ whenever $e_2 \in E_2$ is fixed and $\xi_1(e_1) = \xi_1(e_1')$.*

If both fibrations are over the same base B (i.e., $B_1 = B_2 = B$), then the fiberwise join can be pulled back along the diagonal map $\Delta : B \to B \times B$, and the resulting fibration over B is denoted again by $\xi_1 * \xi_2$ by abuse of notation.

Given a $(k-1)$-spherical fibration $\eta : B \to BG(k)$, its fiberwise join with the trivial $(l-1)$-spherical fibration yields a $(k+l-1)$-spherical fibration classified by the map $\eta * \epsilon^l : B \to BG(k+l)$.

(b) *A stable spherical fibration over B is an equivalence class of spherical fibrations, where two spherical fibrations ξ_1 and ξ_2 are equivalent if $\xi_1 * \epsilon^k$ is fiber homotopy equivalent to $\xi_2 * \epsilon^l$ for some integers $k, l \geq 0$.*

Theorem 2.2.25. *Let X be a CW complex. Then the set $[X, BG]$ is in bijection with the set of stable fiber homotopy equivalence classes of spherical fibrations over X, where*

$$BG = \varinjlim_{n} BG(n),$$

is the colimit taken over the standard inclusions $BG(n) \hookrightarrow BG(n+1)$, each induced by taking the fiberwise join of the universal $\mathbb{S}^{n-1}$-spherical fibration with the trivial $\mathbb{S}^0$-spherical fibration.

Theorem 2.2.26. *[28] $[X, G] \cong \pi_s^0(X)$, where $\pi_s^0(X) \cong \lim_{n\to\infty} [\Sigma^n X, \mathbb{S}^n]$ is the 0-th stable cohomotopy group of X.*

Theorem 2.2.27. *[77, Corrollary 3.8] $\pi_{i+1}(BG) \cong \pi_i(G) \cong \pi_i^s$, where $\pi_i^s = \pi_{i+k}(\mathbb{S}^k)$ denotes the i-th stable homotopy group spheres for $k \geq i+2$.*

We present a list of stable homotopy groups in low dimensions along with their generators.

Theorem 2.2.28. *[125, Page 189]*

i	*1*	*2*	*3*	*4*	*5*	*6*
π_i^s	$\mathbb{Z}/2\{\eta\}$	$\mathbb{Z}/2\{\eta^2\}$	$\mathbb{Z}/8\{\nu\} \oplus \mathbb{Z}/3\{\alpha_1\}$	0	0	$\mathbb{Z}/2\{\nu^2\}$

i	*7*	*8*	*9*
π_i^s	$\mathbb{Z}/16\{\sigma\} \oplus \mathbb{Z}/3\{\alpha_2\} \oplus \mathbb{Z}/5\{\alpha_{1,5}\}$	$\mathbb{Z}/2\{\bar{\nu}\} \oplus \mathbb{Z}/2\{\epsilon\}$	$\mathbb{Z}/2\{\nu^3\} \oplus \mathbb{Z}/2\{\mu\} \oplus \mathbb{Z}/2\{\eta\epsilon\}$

i	*10*	*11*	*12*	*13*
π_i^s	$\mathbb{Z}/2\{\eta\mu\} \oplus \mathbb{Z}/3\{\beta_1\}$	$\mathbb{Z}/8\{\zeta\} \oplus \mathbb{Z}/9\{\alpha_3'\} \oplus \mathbb{Z}/7\{\alpha_{1,7}\}$	0	$\mathbb{Z}/3\{\alpha_1\beta_1\}$

i	*14*	*15*	*16*
π_i^s	$\mathbb{Z}/2\{\sigma^2\} \oplus \mathbb{Z}/2\{\kappa\}$	$\mathbb{Z}/32\{\rho\} \oplus \mathbb{Z}/2\{\eta\kappa\} \oplus \mathbb{Z}/3\{\alpha_4\} \oplus \mathbb{Z}/5\{\alpha_{2,5}\}$	$\mathbb{Z}/2\{\eta^*\} \oplus \mathbb{Z}/2\{\eta\rho\}$

i	*17*	*18*
π_i^s	$\mathbb{Z}/2\{\eta\eta^*\} \oplus \mathbb{Z}/2\{\nu\kappa\} \oplus \mathbb{Z}/2\{\eta^2\rho\} \oplus \mathbb{Z}/2\{\bar{\mu}\}$	$\mathbb{Z}/8\{\nu^*\} \oplus \mathbb{Z}/2\{\eta\bar{\mu}\}$

i	*19*
π_i^s	$\mathbb{Z}/8\{\bar{\zeta}\} \oplus \mathbb{Z}/2\{\bar{\sigma}\} \oplus \mathbb{Z}/3\{\alpha_5\} \oplus \mathbb{Z}/11\{\alpha_{1,11}\}$

Given a vector bundle over a smooth manifold, it can be regarded as a PL microbundle as the smooth manifold is equipped with a PL triangulation. Moreover, every PL microbundle can be viewed as a topological microbundle by forgetting the PL structure. These observations induce the following forgetful maps between classifying spaces:

$$g_n : BO(n) \to BPL(n), \quad h_n : BPL(n) \to BTop(n).$$

Passing to the colimit under stabilization yields the maps between stable classifying spaces:

$$g : BO \to BPL, \quad h : BPL \to BTop.$$

Let $PL/O, Top/PL$ and Top/O be the homotopy fibers of the maps $g : BO \to BPL,\ h : BPL \to BTop$ and $h \circ g : BO \to BTop$ respectively.

Any topological microbundle of rank n gives rise to a spherical fibration with fiber $\mathbb{S}^{n-1}$, constructed by taking the boundary of a fiberwise neighborhood around the zero section. This defines a map

$$l_n : BTop(n) \to BG(n).$$

By passing to the colimit using the standard stabilization maps, which are defined by taking direct sums with trivial microbundles and fiberwise joins with trivial spherical fibrations, one obtains a canonical map

$$l : BTop \longrightarrow BG,$$

from the classifying space of stable topological microbundles to the classifying space of stable spherical fibrations.
Similarly, there are canonical maps

$$l \circ h : BPL \to BG, \;\; J = l \circ g : BO \to BG.$$

Let G/Top, G/PL, and G/O denote the homotopy fibers of the maps

$$l : BTop \to BG, \quad l \circ h : BPL \to BG, \quad \text{and} \quad J = l \circ g : BO \to BG,$$

respectively.

According to [19], the classifying spaces BO, BPL, $BTop$, and BG admit structures of infinite loop spaces. Since the maps between these classifying spaces are infinite loop maps [5], their homotopy fibres inherit infinite loop space structures. Consequently, the homotopy fibres PL/O, Top/PL, Top/O, G/Top, G/PL, and G/O, taken in the category of infinite loop spaces (equivalently, spectra), admit natural infinite loop space structures.

For an infinite loop space X, we write x for a connective spectrum whose zeroth space satisfies $\Omega^\infty x \simeq X$.

Theorem 2.2.29. *The homotopy groups of PL/O is given as follows.*

(i) $\pi_n(PL/O) \cong \Theta_n$ *for* $n \geq 5$ *[118].*

(ii) $\pi_n(PL/O) \cong 0$ *for* $n \leq 3$ *[115, 95].*

(iii) $\pi_4(PL/O) \cong 0$ *[29].*

Theorem 2.2.30. *[53, 13] Let PL/O be the infinite loop space $\Omega^\infty pl/o$, where pl/o is the corresponding spectrum. Then*

(i) $\tau_{\leq 8}(pl/o) \simeq \Sigma^7 H\mathbb{Z}/28 \vee \Sigma^8 H\mathbb{Z}/2$.

(ii) $\tau_{\leq 9}(pl/o) \simeq \Sigma^8 \mathcal{F} \vee \Sigma^7 \mathcal{F}_2 \vee \Sigma^7 H\mathbb{Z}/7 \vee \Sigma^9 H\mathbb{Z}/2$, *where $\mathcal{F}$ and $\mathcal{F}_2$ denote the fiber of the maps $Sq^2 : H\mathbb{Z}/2 \to \Sigma^2 H\mathbb{Z}/2$ and $Sq^2 \circ d : H\mathbb{Z}/4 \to \Sigma^3 H\mathbb{Z}/2$ respectively with $d : H\mathbb{Z}/4 \to H\mathbb{Z}/2$ is the mod 2 reduction map.*

(iii) $\tau_{\leq 10}(pl/o) \simeq \Sigma^8 \mathcal{F} \vee \Sigma^7 \mathcal{E} \vee \Sigma^7 H\mathbb{Z}/7 \vee \Sigma^9 H\mathbb{Z}/2 \vee \Sigma^{10} H\mathbb{Z}/3$, *where $\mathcal{E}$ is the fiber of the map $\mathcal{F}_2 \to \Sigma^4 H\mathbb{Z}/2$ which exists since the composition $\Sigma^{-1} H\mathbb{Z}/4 \xrightarrow{Sq^2 \circ d} \Sigma^2 H\mathbb{Z}/2 \xrightarrow{Sq^2} \Sigma^4 H\mathbb{Z}/2$ is nullhomotopic.*

Here $\tau_{\leq n}(X)$ denotes the m-th Postnikov section of the spectrum X.

Theorem 2.2.31. *[64, 106] The homotopy groups of Top/PL are:*

$$\pi_l(Top/PL) = \begin{cases} 0 & \text{if } l \neq 3, \\ \mathbb{Z}/2 & \text{if } l = 3. \end{cases}$$

Thus $Top/PL \simeq K(\mathbb{Z}/2, 3)$.

Theorem 2.2.32. *[64, Page 246] The homotopy groups of Top/O are given as follows:*

$$\pi_l(Top/O) \cong \begin{cases} \Theta_l & \text{if } l \geq 5, \\ \pi_l(K(\mathbb{Z}/2, 3)) & \text{if } l \leq 6. \end{cases}$$

The following table lists the homotopy groups of Top/O up to dimension 20.

i	*1*	*2*	*3*	*4*	*5*	*6*
$\pi_i(Top/O)$	0	0	$\mathbb{Z}/2$	0	0	0

i	*7*	*8*	*9*
$\pi_i(Top/O)$	$\mathbb{Z}/28 \cong bP_8$	$\mathbb{Z}/2\{[\Sigma_\epsilon]\}$	$bP_{10} \oplus \mathbb{Z}/2\{[\Sigma_{\eta\circ\epsilon}]\} \oplus \mathbb{Z}/2\{[\Sigma_\mu]\}$

i	*10*	*11*	*12*
$\pi_i(Top/O)$	$\mathbb{Z}/2\{[\Sigma_{\eta\circ\mu}]\} \oplus \mathbb{Z}/3\{[\Sigma_{\beta_1}]\}$	$bP_{12} \cong \mathbb{Z}/992$	0

i	*13*	*14*
$\pi_i(Top/O)$	$\mathbb{Z}/3\{[\Sigma_{\alpha_1\circ\beta_1}]\}$	$\mathbb{Z}/2\{[\Sigma_\kappa]\}$ *or* $\mathbb{Z}/2\{[\Sigma_{\kappa+\sigma^2}]\}$

i	*15*	*16*
$\pi_i(Top/O)$	$\mathbb{Z}/2\{[\Sigma_{\eta\circ\kappa}]\} \oplus bP_{16} \cong \mathbb{Z}/2\{[\Sigma_{\eta\circ\kappa}]\} \oplus \mathbb{Z}/8128$	$\mathbb{Z}/2\{[\Sigma_{\eta^*}]\}$

i	*17*
$\pi_i(Top/O)$	$bP_{18} \oplus \mathbb{Z}/2\{[\Sigma_{\eta\eta^*}]\} \oplus \mathbb{Z}/2\{[\Sigma_{\nu\kappa}]\} \oplus \mathbb{Z}/2\{[\Sigma_{\bar{\mu}}]\} \cong \mathbb{Z}/2 \oplus \oplus\mathbb{Z}/2\{[\Sigma_{\eta\eta^*}]\} \oplus \mathbb{Z}/2\{[\Sigma_{\nu\kappa}]\} \oplus \mathbb{Z}/2\{[\Sigma_{\bar{\mu}}]\}$

i	*18*	*19*	*20*
$\pi_i(Top/O)$	$\mathbb{Z}/2\{[\Sigma_{\nu^*}]\} \oplus \mathbb{Z}/2\{[\Sigma_{\eta\bar{\mu}}]\}$	$bP_{20} \oplus \mathbb{Z}/2\{[\bar{\sigma}]\}$	$\mathbb{Z}/24\{[\Sigma_{\bar{\kappa}}]\}$

Theorem 2.2.33. *[70] The 8-th postnikov section of the spectrum top/o is given by*

$$\tau_{\leq 8}(top/o) \simeq \Sigma^3 H\mathbb{Z}/2 \vee \Sigma^7 H\mathbb{Z}/4 \vee \Sigma^8 H\mathbb{Z}/2.$$

Theorem 2.2.34. *[121, 77] The decomposition of G/Top as an infinite loop space is given by:*

(i) $G/Top_{(2)} \simeq \prod_{n\geq 1}\left(K(\mathbb{Z}_{(2)}, 4n) \times K(\mathbb{Z}/2, 4n-2)\right)$.

(ii) $G/Top_{(\frac{1}{2})} \simeq BO_{(\frac{1}{2})}$.

Theorem 2.2.35. *[121, 77]*

$$\pi_l(G/PL_{(2)}) = \begin{cases} \mathbb{Z}_{(2)} & \text{if } l \equiv 0 \pmod 4, \\ \mathbb{Z}/2 & \text{if } \ l \equiv 2 \pmod 4, \\ 0 & \text{if } l \equiv 1 \pmod 2). \end{cases}$$

Theorem 2.2.36. *[121, 77] The space G/PL admits the following decompositions as H-spaces:*

(a)

$$G/PL_{(2)} \simeq E \times \prod_{n>1} \big(K(\mathbb{Z}_{(2)}, 4n) \times K(\mathbb{Z}/2, 4n-2)\big)$$

where E denotes the fiber of the fibration $\beta \circ Sq^2 : K(\mathbb{Z}/2, 2) \to K(\mathbb{Z}_{(2)}, 5)$ and $\beta : K(\mathbb{Z}/2, 4) \to K(\mathbb{Z}_{(2)}, 5)$ is the Bockstein operator.

(b) $G/PL_{(p)} \simeq BO_{(p)}$ for all odd prime p.

Theorem 2.2.37. *[121, 103, 87, 74] The first twenty homotopy groups of G/O are given in the following table.*

$k =$	1	2	3	4	5	6	7	8	9	10
$\pi_k(G/O)$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	$2\mathbb{Z}/2$	$\mathbb{Z}/6$
$k =$	11	12	13	14	15	16	17	18	19	20
$\pi_k(G/O)$	0	$\mathbb{Z}$	$\mathbb{Z}/3$	$2\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$3\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/8$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/24$

Theorem 2.2.38. *[77, Theorem 5.18] For each prime p,*

$$G/O_{(p)} \simeq BSO_{(p)} \times \mathrm{Cok}J_p,$$

where J_p is the homotopy fiber of $\psi^k - 1 : BSO_{(p)} \to BSO_{(p)}$, where k is a primitive root of unity mod p^2 and ψ^k is the Adams operation in K-theory. The space $\mathrm{cok}J_p$ is defined in [77, Definition 5.16].

Remark 2.2.39. *[74]*

(i) For each prime p, the homotopy groups $\pi_k(\mathrm{Cok}J_p)$ are finite p-primary groups.

(ii) If $k \not\equiv 1, 2 \pmod 8$, then $\pi_k(\mathrm{Cok}J_2) \cong (\pi_k^s/\operatorname{Im}(J))_{(2)}$.

(iii) If $k \equiv 1, 2 \pmod 8$, then $\pi_k(\mathrm{Cok}J_2)$ is isomorphic to an index two summand of $(\pi_k^s/\operatorname{Im}(J))_{(2)}$.

(iv) For any odd prime p, one has $\pi_k(\mathrm{Cok}J_p) \cong (\pi_k^s/\operatorname{Im}(J))_{(p)}$.

Note 2.2.40. *The homotopy equivalence in Theorem 2.2.38 fails to be an H-space equivalence after localization at the prime 2 [76, Page 236], but it does become an H-space equivalence when localized at any odd prime [80].*

The natural maps $J = l \circ g : BO \to BG$ and $l : BTop \to BG$ fit into a commutative diagram in which both the rows and columns are homotopy fiber sequences:

$$\begin{tikzcd}
\Omega(G/Top) \arrow[r, dashed, "\omega"] \arrow[d] & Top/O \arrow[r, dashed, "\psi"] \arrow[d, "\cong"'] & G/O \arrow[r, dashed, "\phi"] \arrow[d, "i"'] & G/Top \arrow[d] \\
Top \arrow[r] & Top/O \arrow[r] & BO \arrow[r, "h\circ g"] \arrow[d, "J"'] & BTop \arrow[d, "l"] \\
 & & BG \arrow[r, "\cong"'] & BG
\end{tikzcd}$$

Similarly, there are fibrations of the form

$$\cdots \to \Omega(G/PL) \to PL/O \to G/O \to G/PL,$$

$$\cdots \to \Omega(G/Top) \to Top/PL \to G/PL \to G/Top.$$

2.3 Structure Set

Definition 2.3.1. *Let M be a topological n-dimensional manifold.*

- *A* smooth atlas *on a topological manifold M is an atlas such that all the transition maps are smooth.*

- *A* smoothing *(or smooth structure) on a topological manifold M is a maximal smooth atlas on M. Equivalently, a smoothing can be described by a pair (N, f), where N is a smooth n-manifold and $f : N \to M$ is a homeomorphism.*

Definition 2.3.2. *(Smooth Structure Set) Two smoothings* (N_1, f_1) *and* (N_2, f_2) *on a topological manifold* M *are said to be equivalent if there exists an orientation-preserving diffeomorphism* $\Psi : N_1 \to N_2$ *such that* $f_2 \circ \Psi \simeq f_1$*. The set of equivalence classes of smoothings on* M *is denoted by* $\mathcal{S}^{Diff}(M)$*.*

We note that there is a natural forgetful map

$$\mathcal{L} : \mathcal{S}^{Diff}(M) \twoheadrightarrow \mathcal{S}(M) \qquad [(N, f)] \mapsto [N]. \tag{2.1}$$

We first note that the group Θ_n acts on the smooth structure set $\mathcal{S}^{Diff}(M)$ via the following operation:

$$\begin{gathered} \Theta_n \times \mathcal{S}^{Diff}(M) \twoheadrightarrow \mathcal{S}^{Diff}(M) \\ ([\Sigma^n], [(N, f)]) \mapsto [(N \# \Sigma^n, h_{\Sigma^n} \circ (f \# Id_{\Sigma^n}))], \end{gathered} \tag{2.2}$$

where $h_{\Sigma^n} : M \# \Sigma^n \to M$ is the canonical homeomorphism associated to $\Sigma^n \in \Theta_n$.

There is also a natural action of Θ_n on $\mathcal{S}(M)$, defined by:

$$\begin{gathered} \Theta_n \times \mathcal{S}(M) \twoheadrightarrow \mathcal{S}(M) \\ ([\Sigma^n], [N]) \mapsto [N \# \Sigma^n]. \end{gathered} \tag{2.3}$$

While $\mathcal{S}(M)$ classifies smooth manifolds homeomorphic to M up to diffeomorphism, it is often difficult to analyze directly due to the absence of a convenient algebraic structure. To obtain a more refined and computable invariant, one considers the concordance structure set, which admits effective techniques from stable homotopy theory.

Definition 2.3.3. *(Concordance Structure Set) Let* M *be an* n*-dimensional topological manifold. We say that two smoothings* (N_1, f_1) *and* (N_2, f_2) *on* M *are* concordant *if there exist an oriented diffeomorphism* $\Phi : N_1 \to N_2$ *and a homeomorphism* $H : N_1 \times [0, 1] \to M \times [0, 1]$ *such that* $H|_{N_1 \times \{0\}} = f_1$ *and* $H|_{N_1 \times \{1\}} = f_2 \circ \Phi$*. The set of all such concordance classes is denoted by* $\mathcal{C}(M)$ *and the concordance class of* (N, f)

in $\mathcal{C}(M)$ *is written as* $[(N, f)]$.

If M *is a smooth manifold, then the base point of* $\mathcal{C}(M)$ *is the concordance class* $[(M, Id_M)]$ *of the identity map* $Id_M : M \to M$.

Example 2.3.4. *(i)* $\mathcal{C}(\mathbb{S}^n) = \Theta_n, n \geq 5$ *[63].*

(ii) $\mathcal{C}(\mathbb{S}^p \times \mathbb{S}^q) = \mathcal{C}(\mathbb{S}^p) \oplus \mathcal{C}(\mathbb{S}^q) \oplus \mathcal{C}(\mathbb{S}^{p+q})$, *if* $p, q \geq 5$ *[53, Page-11].*

Note that there is a well-defined surjective map

$$\mathcal{C}(M) \twoheadrightarrow \mathcal{S}^{Diff}(M) \qquad [(N, f)] \mapsto [(N, f)]. \tag{2.4}$$

To relate the smooth structure set $\mathcal{C}(M)$ to $\mathcal{S}(M)$, one must account for the action of the group of self-homeomorphism, denoted by Homeo(M) on $\mathcal{C}(M)$. The action of Homeo(M) on $\mathcal{C}(M)$ is given by:

$$\begin{aligned} \text{Homeo}(M) \times \mathcal{C}(M) &\twoheadrightarrow \mathcal{C}(M) \\ ([g], [(N, f)]) &\mapsto [(N, g \circ f)]. \end{aligned} \tag{2.5}$$

Combining (2.1), (2.4) and (2.5), we obtain a natural surjective map

$$\mathcal{B} : \mathcal{C}(M)/\text{Homeo}(M) \twoheadrightarrow \mathcal{S}(M).$$

To show that $\mathcal{B}$ is bijective, it suffices to prove that any two elements $[(M, f_1)], [(M, f_2)] \in \mathcal{C}(M)$ lie in the same orbit under the action of Homeo(M). Observe that

$$f_1 = (f_1 \circ f_2^{-1}) \circ f_2,$$

where $f_1 \circ f_2^{-1} \in \text{Homeo}(M)$. Hence, $[(M, f_1)] = [(M, f_2)]$ in $\mathcal{C}(M)/\text{Homeo}(M)$.

Hence

$$\mathcal{B} : \mathcal{C}(M)/\text{Homeo}(M) \xrightarrow{\cong} \mathcal{S}(M).$$

Theorem 2.3.5 (Fundamental Theorem of Smoothing [64])**.** *Let M be a topological manifold of dimension $n \geq 5$ and let $\tau_M : M \to BTop$ denotes the classifying map of the stable tangent microbundle. Then M admits a smooth structure if and only if there exits a lift $M \to BO$ such that following diagram commutes:*

$$\begin{tikzcd}
 & BO \arrow[d, "h\circ g"] \\
M \arrow[ur, dashed] \arrow[r, "\tau_M"'] & BTop.
\end{tikzcd}$$

Moreover, if M admits a smooth structure, then is a bijection between $\mathcal{C}(M)$ and $[Lift_{\tau_M}(h \circ g)]$, where $[Lift_{\tau_M}(h \circ g)]$ denotes the set of homotopy classes of lifttings of the map τ_M over $h \circ g$. Since the map $h \circ g : BO \to BTop$ is a principal Top/O-fibration, $[Lift_{\tau_M}(h \circ g)]$ is identified with abelian group $[M, Top/O]$. Therefore, for M is a smooth manifold, there is a bijection between $\mathcal{C}(M)$ and $[M, Top/O]$.

As an immediate consequence of this identification:

Remark 2.3.6. *The set $\mathcal{C}(M)$ admits an abelian group structure for any smooth manifold M of dimension ≥ 5.*

2.4 Inertia Groups and Their Subgroups

In the classification of smooth structures, a fundamental question arises: does forming the connected sum of a manifold M with an exotic sphere yield a genuinely new smooth structure? This leads to the definition of the inertia group. To investigate this question under weaker equivalence relations, one introduces the homotopy inertia group and the concordance inertia group.

Definition 2.4.1 (Inertia Group)**.** *Let M be a closed, smooth manifold of dimension n. The* inertia group *$I(M)$ is the subgroup of Θ_n consisting of those homotopy n-spheres $[\Sigma^n]$ such that there exists an orientation-preserving diffeomorphism between $M\#\Sigma^n$ and M.*

The inertia group $I(M)$ can also be characterized as the stabilizer of the class $[M] \in \mathcal{S}(M)$ under the action described in equation (2.3).

Example 2.4.2.

1. *$I(\mathbb{S}^n) = 0$ for all $n \in \mathbb{N}$.*

2. *$I(\mathbb{C}P^n) = 0$ for all $n \leq 8$ [60, Theorem 1].*

3. *By [11, Theorem 3.1], the group $I(\mathbb{C}P^9)$ is either $\mathbb{Z}/2$ or $\mathbb{Z}/4$. Furthermore, $I(\mathbb{C}P^9) \subset \mathbb{Z}/8 \subset \Theta_{18}$ by [11, Proposition 4.2].*

4. *$I(\mathbb{H}P^2) = \Theta_8$ [68], [57, Theorem 1.1] and [34, Theorem 1.3].*

5. *$I(\mathbb{H}P^4) = \Theta_{16}$ [12, Corollary 5.8].*

6. *$I(\mathbb{R}P^8) = \mathbb{Z}/2$ [13, Corollary 4.6].*

7. *For any $k \geq 0$, the inertia group of the lens space $L^9(m) = \mathbb{S}^9/\mathbb{Z}/m$, with the action $(z_1, z_2, z_3, z_4) \mapsto (\alpha z_1, \alpha z_2, \alpha z_3, \alpha z_4)$, where $\alpha = e^{2\pi i/m}$, is given by*

$$I(L^9(m)) = \begin{cases} 0 & \text{if } m = 2k+1, \\ \mathbb{Z}/2 & \text{if } m = 4k+2, \\ \mathbb{Z}/2 \oplus bP_{10} & \text{if } m = 4k, \end{cases}$$

as shown in [13, Theorem 4.10].

Note 2.4.3. *Kawakubo showed that $I(\mathbb{S}^p \times \mathbb{S}^q) = 0$ for $p + q \geq 5$ [61, Corollary 3], while $I(\mathbb{S}^3 \times \Sigma^{10}) = \Theta_{13}$ [61, Corollary 2], where Σ^{10} denotes the generator of $(\Theta_{10})_{(3)}$. It follows that the inertia group $I(M)$ is not a homotopy invariant.*

Theorem 2.4.4.

(a) For each $n \in \mathbb{N}$, there exists a simply connected, closed manifold M of dimension n such that $I(M) = \Theta_n$ [132, Theorem 2.10].

(b) *For $n = 3, 7$, there exists a $(n-1)$-connected, closed, smooth manifold of dimension $2n+1$ whose inertia group $I(M)$ is a proper nontrivial subgroup of Θ_{2n+1} [130].*

Theorem 2.4.5. *[109, Theorem A] If $n \geq 5$ and M^n is a product of standard spheres, then its inertia group $I(M)$ vanishes.*

Example 2.4.2 illustrates that the inertia group of a manifold can be trivial, equal to the entire group Θ_n, or a proper nontrivial subgroup of Θ_n. On the other hand, various limitations on the size of the inertia group have been investigated by Wall, Browder, Kosinski, and Novikov. In fact, there is no general framework for computing the inertia group, and each case typically requires separate analysis. This has led to a focus on certain more computable subgroups, notably the *homotopy inertia group* and the *concordance inertia group*.

Let M be a closed, smooth manifold of dimension n, and let $\sigma : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be an orientation-preserving diffeomorphism representing the homotopy sphere $[\Sigma] \in \Theta_n$. Since $M\#\Sigma$ can be described as $(M \setminus \text{int}(\mathbb{D}^n)) \cup_\sigma \mathbb{D}^n$, we define a homeomorphism $h_\Sigma : M\#\Sigma \to M$ by setting $h_\Sigma|_{\mathbb{D}^n} = \text{cone}(\sigma)$ and $h_\Sigma|_{M\setminus\text{int}(\mathbb{D}^n)} = Id$. We refer to this map h_Σ as the *canonical homeomorphism* corresponding to the homotopy sphere $[\Sigma]$.

Definition 2.4.6 (Homotopy Inertia Group)**.** *The* homotopy inertia group *of a closed, smooth n-manifold M, denoted $I_h(M)$, is the subgroup of Θ_n containing those $[\Sigma] \in \Theta_n$ for which there exists a orientation preserving diffeomorphism $M\#\Sigma \to M$ that is homotopic to the canonical homeomorphism $h_\Sigma : M\#\Sigma \to M$.*

Equivalently, with respect to the action defined in (2.2), the homotopy inertia group $I_h(M)$ is the stabilizer of the element $[(M, Id)] \in \mathcal{S}^{Diff}(M)$.

Example 2.4.7.

1. $I_h(\mathbb{R}P^7) = 0$ *[58].*

2. $I_h(\mathbb{R}P^{10}) = \Theta_{10}$ *[13, Theorem 4.16].*

3. *$I_h(M) = 0$, where M is homotopy equivalent to $\mathbb{H}P^2$ [57].*

4. *$I_h(M)$ is trivial for any closed, 3-connected, 8-dimensional smooth manifold M [59, Corollary 3.4].*

5. *$I_h(\mathbb{C}P^2 \times \mathbb{S}^3) = 2\Theta_7$ [21, 107].*

6. *$2bP_{4k+4} \subseteq I_h(\mathbb{C}P^{2k} \times \mathbb{S}^3)$ for all $k \geq 1$ [21, 107].*

7. *$I_h(\mathbb{C}P^3 \times \mathbb{S}^1) = \mathbb{Z}/7$ [28, Page 406].*

8. *$I_h(M) = 0$ for any closed, smooth manifold M of dimension 8 [13, Theorem 4.5].*

Note 2.4.8. *Brumfiel [28, Remark II.12] showed that $I_h(P_j^6 \times \mathbb{S}^1) = 0$ when $j \equiv 1 \pmod 7$, where P_j^6 is a smooth manifold homotopy equivalent to $\mathbb{C}P^3$ with first Pontrjagin class $p_1(P_j^6) = (4 + 24j)z^2$ for some generator $z \in H^2(P_j^6; \mathbb{Z})$. On the other hand, $I_h(\mathbb{C}P^3 \times \mathbb{S}^1) = \mathbb{Z}/7$. This example shows that the homotopy inertia group is not a homotopy invariant. It also shows that the homotopy inertia group is not an h-cobordism invariant in dimension 7.*

In contrast, the situation improves in higher dimensions.

Proposition 2.4.9. *([42, Page 438], [107, Theorem 2.1]) Let M and N be closed, smooth manifolds of dimension $n \geq 8$ that are h-cobordant. Then $I_h(M) = I_h(N)$.*

The next result shows that Theorem 2.4.4(a) does not remain valid if the inertia group $I(M)$ is replaced by the homotopy inertia group $I_h(M)$.

Theorem 2.4.10. *[18, Theorem 2.2] Let M be a closed, oriented, smooth manifold of dimension $4k - 1 \geq 7$. Then the subgroup $I_h(M) \cap bP_{4k} \subseteq bP_{4k}$ has index at least 2. Hence $I_h(M) \neq \Theta_{4k-1}$.*

This theorem also provides examples where $I_h(M) \neq I(M)$.

Although $I_h(M)$ offers a refinement of the inertia group, one often considers the concordance inertia group $I_c(M)$, which is more accessible from a computational standpoint.

Definition 2.4.11 (Concordance Inertia Group)**.** *The* concordance inertia group $I_c(M)$ *is the subgroup of* $I(M)$ *made up of those homotopy* n*-spheres* $[\Sigma]$ *such that the pairs* (M, Id_M) *and* $(M\#\Sigma, h_\Sigma)$ *are concordant.*

Note from (2.5) that the isotropy subgroup $(\Theta_n)_{[(M,Id)]}$ is precisely the concordance inertia group $I_c(M)$ of M.

From Definitions 2.4.1, 2.4.6, and 2.4.11, we immediately observe:

(a) $I_c(M) \subseteq I_h(M) \subseteq I(M)$.

(b) $I_c(\mathbb{S}^n) = I_h(\mathbb{S}^n) = I(\mathbb{S}^n) = 0$.

The following examples illustrate known computations of inertia groups.

Example 2.4.12.

1. $I_c(\mathbb{C}P^n) = I_h(\mathbb{C}P^n) = I(\mathbb{C}P^n)$ *[56, Remark 4.3].*

2. $I_c(\mathbb{H}P^n) = I_h(\mathbb{H}P^n)$ *for all* $n \geq 1$ *[59, Corollary 3.2].*

3. $I_h(M) = I_c(M) = 0$ *for any* $(n-1)$*-connected, closed, smooth, oriented* $2n$*-dimensional manifold where* $n \geq 3$ *[112, Theorem 1.9].*

Example 2.4.13.

1. $I_c(M)$ *is trivial for any closed, oriented, smooth* 7*-dimensional manifold* M *[58, Remark 2.9].*

2. $I_c(M) = 0$*, where* M *is a closed, smooth* 8*-manifold [13, Theorem 4.1].*

3. *Let* M *be a closed, oriented, non-spin, smooth manifold of dimension* 9*. Then* $I_c(M)$ *is either* $\mathbb{Z}/2$ *or* $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ *[13, Theorem 4.2].*

Remark 2.4.14. *[96, Remark 1.5] Let* M *be a connected, smooth manifold of dimension* $n \geq 5$*. Then:*

(a) If M *is non-compact, then* $I_c(M) = \Theta_n$*.*

(b) If M is non-orientable, then $I_c(M)$ contains all elements of Θ_n of order dividing 2.

Therefore, the most interesting nontrivial case arises when M is compact and orientable.

Let M be a closed, oriented, smooth manifold of dimension $n \geq 5$. Up to homotopy, there exists a well-defined degree one map $f_M : M \to \mathbb{S}^n$. Using the description of $\mathcal{C}(M)$ provided by Theorem 2.3.5, this map induces a group homomorphism

$$\begin{aligned}(f_M)^* : \Theta_n &\to \mathcal{C}(M)\\ [(\Sigma, h_\Sigma)] &\mapsto [(M\#\Sigma, h_\Sigma)].\end{aligned}$$

Combining this equation with Definition 2.4.11, we obtain the following description of the concordance inertia group.

Proposition 2.4.15. *Let M be a closed, oriented, smooth manifold of dimension $n \geq 5$. Then the concordance inertia group $I_c(M)$ is equal to the kernel of the group homomorphism $f_M^* : \Theta_n \to \mathcal{C}(M)$, induced from the degree one collapse map $f_M : M \to \mathbb{S}^n$.*

This description of $I_c(M)$ implies that it depends on the homotopy type of M.

Proposition 2.4.16. *If two closed, oriented, smooth manifolds M and N of the same dimension are homotopy equivalent, then $I_c(M) = I_c(N)$. In particular, the concordance inertia group is a homotopy invariant.*

Since $I_c(M) \subseteq I_h(M)$, Theorem 2.4.10 implies that Theorem 2.4.4(a) also fails to hold when stated for $I_c(M)$ in place of $I(M)$.

2.5 Preliminaries of Surgery theory

The surgery exact sequence is one of the main tools for solving classification problem for manifolds of dimension ≥ 5. In this section, we introduce each term of the surgery exact sequence in detail and explain the maps in the surgery exact sequence. Most of the material in this section can be found in [128, 74, 102, 30].

Questions Addressed by Surgery Theory

- Checks whether two homotopy equivalent (or homeomorphic) manifolds are diffeomorphic.
- Determines whether a given complex X satisfying Poincaré duality admits a homotopy equivalence to a smooth manifold.

Definition 2.5.1 (Poincaré Complex)**.** *A connected finite CW-complex X of dimension n is an n-*dimensional Poincaré complex *if there exists an* orientation homomorphism $w = w_1(X)\colon \pi_1(X) \to \mathbb{Z}/2$ *and a* fundamental class $[X] \in H_n\big(X;\mathbb{Z}^w\big)$ *(with $\mathbb{Z}^w$ the local orientation coefficient system twisted by w) such that the cap product maps*

$$[X] \cap -\colon H^k\big(X;\mathbb{Z}[\pi_1(X)]\big) \to H_{n-k}\big(X;\mathbb{Z}^w[\pi_1(X)]\big)$$

are isomorphisms for all $k \geq 0$.

Theorem 2.5.2. *A closed, connected n-dimensional manifold is a finite n-dimensional Poincaré Complex.*

Thus a CW complex is homotopy equivalent to a closed n-manifold if and only if it is homotopy equivalent to a finite n-dimensional Poincaré Complex.

Definition 2.5.3 (Geometric Structure Set)**.** *Let X be an n-dimensional Poincaré complex. Then the* geometric structure set $\mathcal{S}_h^{Diff}(X)$ *of X is the equivalence classes of the pair (N, f), where N is a closed n-dimensional manifold and $f : N \to X$ is a homotopy equivalence. We say (N_1, f_1) and (N_2, f_2) are equivalent if there exists an h-cobordism W between N_1 and N_2 and a homotopy equivalence $F : W \to X \times [0,1]$ such that $F|_{\partial_1(W)} = f_1$ and $F|_{\partial_2(W)} = f_2$.*

Note 2.5.4. *If X is a compact manifold of dimension $n \geq 5$, then we get $\mathcal{S}^{Diff}(X)$ from $\mathcal{S}_h^{Diff}(X)$.*

In the context of the manifold structure set, surgery theory determines whether $\mathcal{S}_h^{Diff}(X)$ is non-empty and whether two elements (N_1, f_1) and (N_2, f_2) in $\mathcal{S}_h^{Diff}(X)$ are equivalent.

Surgery programme: Let M and X be two closed manifolds.

- Construct a homotopy equivalence $f : M \to X$.
- Build a Cobordism (W, M, X) and $F : W \to X \times [0, 1]$ such that $F|_M = f$ and $F|_N = Id_N$.
- Modify F and W so that F becomes a homotopy equivalence.

Now if M and X are simply connected and are of dimension at least 5, then by Smale's h-cobordism theorem M is diffeomorphic to X.

Motovation of Surgery steps: Let M be a closed n-dimensional manifold, X a CW complex, and $f : M \to X$ be a k-connected map.

Recall that

$$\pi_{k+1}(f) := \pi_{k+1}(M_f, M),$$

where M_f denotes the mapping cylinder of $f\colon M \to X$. Since M_f is homotopy equivalent to X, each element of $\pi_{k+1}(f)$ can be represented by a pair of maps $\mathbb{S}^k \to M$ and $\mathbb{D}^{k+1} \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{S}^k \arrow[r] \arrow[d, hook] & M \arrow[d, "f"] \\
\mathbb{D}^{k+1} \arrow[r] & X.
\end{tikzcd}$$

We know that $\pi_{k+1}(f)$ fits into the following short exact sequence

$$\cdots \to \pi_{k+1}(M) \to \pi_{k+1}(X) \to \pi_{k+1}(f) \to \pi_k(M) \to \pi_k(X) \to \cdots .$$

This short exact sequence, together with Whitehead's theorem, implies that eliminating $\pi_{k+1}(f)$ requires attaching a $(k + 1)$-cell to M. However, merely killing the homotopy group does not necessarily yield a manifold. In essence, surgery can be viewed as the process of attaching cells while preserving the manifold structure.

Let $\varphi_0 : \mathbb{S}^k \to M$ extend to an embedding $\varphi : \mathbb{S}^k \times \mathbb{D}^{n-k} \hookrightarrow M$. Then removing the interior of the image of φ and attaching $\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1}$ is called *surgery.* We call the resulting manifold

$$M' = \big(M - int(\varphi(\mathbb{S}^k \times \mathbb{D}^{n-k}))\big) \cup_{\mathbb{S}^k \times \mathbb{S}^{n-k-1}} \big(\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1}\big)$$

the *effect of the surgery.*

The *trace of surgery* W is the cobordism between M and M', given by

$$W := M \times [0,1] \cup_\varphi (\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}).$$

Thus, performing surgery to eliminate an element of $\pi_{k+1}(f)$ requires determining whether $\varphi_0 : \mathbb{S}^k \to M$ can be changed to an embedding up to homotopy and, furthermore, whether it extends to an embedding $\varphi : \mathbb{S}^k \times \mathbb{D}^{n-k} \hookrightarrow M$.

Definition 2.5.5.

(i) Normal bundle of an immersion $g : Y^n \to Z^m$ *is the quotient bundle* $\nu_g := g^*(TZ)/TY$. *In terms of classifying space, this is denoted by the map* $\nu_g : Y \to BO(m-n)$.

(ii) A framing of an immersion $g : Y^n \to Z^m$ *is a framing* $b : \nu_g \cong \epsilon^{m-n}$ *of the normal bundle* $\nu_g : Y \to BO(n-m)$.

(iii) A normal bundle of an n-dimensional manifold M *is given by* $\nu_M = \nu_g : M \to BO(k)$ *for any embedding* $g : M \hookrightarrow \mathbb{R}^{n+k}$ *with* $k \geq 1$.

Using the Tubular Neighbourhood Theorem, we can show that

Proposition 2.5.6. *The framings* b *of an embedding* $g : Y^n \hookrightarrow Z^m$ *correspond bijectively to extensions of* g *to an embedding* $\tilde{g} : Y^n \times \mathbb{D}^{m-n} \hookrightarrow Z^m$.

From the above proposition, we obtain

Theorem 2.5.7. *Let M be an n-dimensional manifold. Then a framed embedding $\varphi_0 : \mathbb{S}^r \hookrightarrow M$ determines an embedding $\varphi : \mathbb{S}^r \times \mathbb{D}^{n-r} \hookrightarrow M$ such that $\varphi|_{\mathbb{S}^r \times \{0\}} = \varphi_0$.*

2.5.1 Motivation for normal map

According to Theorem 2.5.7, we first consider representing each element of $\pi_{k+1}(f)$ by a framed embedding $\varphi_0 : \mathbb{S}^k \hookrightarrow M$.

Lemma 2.5.8. *Let $\varphi_0 : \mathbb{S}^k \hookrightarrow M$ be an embedding.*

- *If ν_{φ_0} is trivial, then the pullback bundle $\varphi_0^*(TM)$ is trivial.*
- *If $\varphi_0^*(TM)$ is stably trivial, then ν_{φ_0} is stably trivial.*
- *If $2r < n$ and ν_{φ_0} is stably trivial, then ν_{φ_0} is trivial.*

Therefore, the framing of an embedding is related to the tangent bundle.

Lemma 2.5.9. *Let ξ be a vector bundle over X and $TM \cong f^*(\xi)$. Then if $w \in \pi_{k+1}(f)$ is represented by an embedding $\varphi_0 : \mathbb{S}^k \hookrightarrow M$, then $\varphi_0^*(TM)$ is trivial and hence ν_{φ_0} is stably trivial.*

Observe that the condition $TM \cong f^*(\xi)$ ensures that $\varphi_0^*(TM)$ is trivial, while its stable triviality is sufficient by Lemma 2.5.8. The following definition imposes a stronger condition on f to guarantee that the embedding $\varphi_0 : \mathbb{S}^k \hookrightarrow M$ admits a stable framing.

Definition 2.5.10 (Normal Map with Respect to the Tangent Bundle)**.** *Let X be a finite n-dimensional Poincaré complex. A* normal map to X with respect to the tangent bundle *is a tuple $(M^n, f, \bar{f}, \xi, k)$, consisting of the following data:*

(i) A map $f : M^n \to X$, where M is a closed n-dimensional manifold;

(ii) A vector bundle ξ over X;

(iii) A bundle map $\bar{f} : TM \oplus \epsilon^a \to \xi$ covering f, for some positive integer a, where TM is the tangent bundle of M and ϵ^a denotes the trivial bundle of rank a.

Definition 2.5.11. *The degree of a map $f : M^n \to X$ from an n-dimesional manifold to an n-dimensional Poincaré complex is the integer $\deg(f)$ such that*

$$f_*[M] = \deg(f)[X].$$

Lemma 2.5.12. *Suppose for a Poincaré complex X, there exists a degree one normal map with target X. Then $\mathcal{S}_h^{Diff}(X)$ is non-empty.*

Theorem 2.5.13. *Let $(f, \bar{f}) : M^n \to X$ be a normal map. Suppose $w \in \pi_{k+1}(f)$ is represented by an embedding $\varphi_0 : \mathbb{S}^k \hookrightarrow M$. Then*

(i) ν_{φ_0} is stably framed.

(ii) If $2(k+1) < n$, then ν_{φ_0} is framed.

2.5.2 Surgery Steps and Surgery Below the Middle Dimension

Next, we examine when an element of $\pi_{k+1}(f)$ admits a representation by an embedding.

Theorem 2.5.14. *Let $(f, \bar{f}) : M^n \to X$ be a normal map and $x \in \pi_{k+1}(f)$.*

(i) The element x determines a regular homotopy class $[\varphi_x : \mathbb{S}^k \times \mathbb{D}^{n-k} \to M^n]$ of immersions provided $k \leq n-2$.

(ii) If $2k \leq (n-1)$, then the regular homotopy class $[\varphi_x]$ contains an embedding.

Moreover, an immersion $q : \mathbb{S}^k \times \mathbb{D}^{n-k} \to M^n$ belongs to $[\varphi_x]$ if and only if q satisfies the following conditions:

1. *There exists a map* $Q : \mathbb{D}^{k+1} \times \mathbb{D}^{n-k} \to X$ *such that following diagram commutes:*

$$\begin{tikzcd}
\mathbb{S}^k \times \mathbb{D}^{n-k} \arrow[r, "q"] \arrow[d, hook] & M^n \arrow[d, "f"] \\
\mathbb{D}^{k+1} \times \mathbb{D}^{n-k} \arrow[r, "Q"'] & X.
\end{tikzcd}$$

2. *The above diagram is covered by a commutative diagram of vector bundles*

$$\begin{tikzcd}
T(\mathbb{S}^k \times \mathbb{D}^{n-k}) \oplus \epsilon^{a+b} \arrow[r, "\bar{q}"] \arrow[d, "Tj \oplus \eta_v \oplus Id_{\mathbb{R}^{a+b-1}}"'] & TM^n \oplus \epsilon^{a+b} \arrow[d, "\bar{f}"] \\
T(\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}) \oplus \epsilon^{a+b-1} \arrow[r, "\bar{Q}"'] & \xi \oplus \epsilon^b,
\end{tikzcd}$$

   where the bundle map $Tj \oplus \eta_v : T(\mathbb{S}^k \times \mathbb{D}^{n-k}) \oplus \epsilon \to T(\mathbb{D}^{k+1} \times \mathbb{D}^{n-k})$ *comes from the differential of the inclusion* $j : \mathbb{S}^k \times \mathbb{D}^{n-k} \hookrightarrow \mathbb{D}^{k+1} \times \mathbb{D}^{n-k}$ *and outward normal field of the boundary of* $\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}$.

3. *The pair* $(q|_{\mathbb{S}^k \times \{0\}}, Q|_{\mathbb{D}^{k+1} \times \{0\}})$ *represents the element* $x \in \pi_{k+1}(f)$.

The next proposition describes the effect of surgery on the relative homotopy group of f.

Proposition 2.5.15. *Let* $f : M^n \to X$ *be a map and* $k \geq 0$ *be a fixed integer such that* $2(k+1) \leq n$. *Let* $\varphi : \mathbb{S}^k \times \mathbb{D}^{n-k} \hookrightarrow M$ *be an embedding with an extension to a map* $\Phi : \mathbb{D}^{k+1} \times \mathbb{D}^{n-k} \to X$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{S}^k \times \mathbb{D}^{n-k} \arrow[r, hook, "\varphi"] \arrow[d, hook] & M^n \arrow[d, "f"] \\
\mathbb{D}^{k+1} \times \mathbb{D}^{n-k} \arrow[r, "\Phi"'] & X.
\end{tikzcd}$$

If $x = (\varphi|_{\mathbb{S}^k \times \{0\}}, \Phi|_{\mathbb{D}^{k+1} \times \{0\}})$ *represents an element of* $\pi_{k+1}(f)$ *and* $f' : M' \to X$ *is the result of surgery on* φ, *then* $\pi_{k+1}(f') \cong \pi_{k+1}(f)/\langle x \rangle$ *and* $\pi_{j+1}(f') \cong \pi_{j+1}(f)$ *for* $j < k$.

The next proposition shows that bundle data carries over to the trace of the surgery.

Proposition 2.5.16. *Let* $(f, \bar{f}) : M^n \to X$ *be a normal map and* $w \in \pi_{k+1}(f)$*. If* $\varphi : \mathbb{S}^k \times \mathbb{D}^{n-k} \hookrightarrow M$ *is an embedding lying in the regular homotopy class of immersions determined by* w*, then the normal map extends to a normal map on the trace* W *of the surgery along* φ*.*

This proposition motivates us to define bordism relation between two rank k normal maps.

Definition 2.5.17 (Bordism of Normal Maps with respect to the Tangent Bundle and Cyclindrical Target)**.** *Let* X *be a finite* n*-dimensional Poincaré complex, and let* $(M_m, f_m, \bar{f}_m, \xi, k_m)$ *for* $m = 0, 1$ *be rank* k *normal maps with respect to the tangent bundle, each with target* X *for* $m = 0, 1$*. A* normal bordism *between these maps, denoted* $(W, F, \overline{F}, \Xi, b)$*, consists of the following data:*

- *(i) A compact* $(n+1)$*-dimensional manifold* W *with boundary* $\partial W = \partial_0 W \sqcup \partial_1 W$*;*
- *(ii) A cylinder* $Y = X \times [0,1]$ *with boundary* $\partial_m Y = X \times \{m\}$, $m = 0, 1$ *;*
- *(iii) Diffeomorphisms* $u_m : M_m \to \partial_m W$ *and isomorphism of CW complexes* $v_m : X \to \partial_m Y$ *sending* x *to* (x, m) *for* $m = 0, 1$*;*
- *(iv) A vector bundle* $\Xi = Pr_X^*(\xi)$ *over* Y*, given by pulling back* ξ *along the projection map* $Pr_X : X \times [0,1] \to X$*;*
- *(v) A map* $F : W \to Y = X \times [0,1]$ *satisfying:*

$$\begin{tikzcd}
M_m \arrow[r, "u_m"] \arrow[d, "f_m"'] & \partial_m W \arrow[r, hook, "k_m"] & W \arrow[d, "F"] \\
X \arrow[r, "v_m"'] & \partial_m Y = X \times \{m\} \arrow[r, hook, "l_m"'] & Y = X \times [0,1],
\end{tikzcd}$$

  where $k_m : \partial_m W \to W$ *and* $l_m : \partial_m Y \to Y$ *are inclusions;*

- *(vi) A bundle map* $\overline{F} : TW \oplus \epsilon^b \to \Xi$ *covering* $F : W \to Y$ *for some integer* $b \geq \max\{a_0, a_1\}$*;*

(vii) Bundle maps $\overline{l_m \circ v_m} : \xi \oplus \epsilon^{b+1-a_m} \to \Xi|_{\partial_m W}$ *covering* $l_m \circ v_m$ *such that*

$$\begin{tikzcd}
TM_m \oplus \epsilon^{b+1} = TM_m \oplus \epsilon^{a_m} \oplus \epsilon^{b+1-a_m} \arrow[r, "Tu_m \oplus n_m \oplus Id_{\epsilon^b}"] \arrow[d, "\bar{f}_m \oplus Id_{\epsilon^{b+1-a_m}}"'] & TW|_{\partial_m W} \oplus \epsilon^b \arrow[d, "\overline{F}"] \\
\xi \oplus \epsilon^{b-a_m+1} \arrow[r, "\overline{l_m \circ v_m}"'] & \Xi|_{\partial_m W},
\end{tikzcd}$$

where $Tu_m : TM_m \to TW|_{\partial_m W}$ *is given by differential of* $u_m : M_m \to \partial_m W$ *and* $n_m : \epsilon \to TW|_{\partial_m W}$ *is the bundle monomorphism given by an inward (respectively, outward) normal field of* $TW|_{\partial_0 W}$ *(respectively,* $TW|_{\partial_1 W}$*).*

Remark 2.5.18. *If* $(M_j, i_j, f_j, \bar{f}_j, \xi_j)$ *is rank* k *normal map of degree one for* $j = 1, 2,$ *then* $(W, F, \bar{F}, \Xi)$ *as above is called normal bordism of rank* k *normal maps of degree one if the image of the intrinsic fundamental class of* $(W, \partial W)$ *under the map* $H_{n+1}(W, \partial W) \to H_{n+1}(Y, \partial Y)$ *is* $[Y, \partial Y]$.

Denote by $\mathcal{N}(X, k)$ the set of normal bordism classes of rank k normal maps of degree one to X. We can define $\mathcal{N}(X, k) \to \mathcal{N}(X, k+1)$ by mapping $[(M, f, \bar{f}, \xi, a)]$ to $[(M, f, \bar{f}', \xi \oplus \epsilon, a)]$, where $\bar{f}' : TM \oplus \epsilon^{a+1}\mathbb{R} \xrightarrow{\bar{f} \oplus Id} \xi \oplus \epsilon$. Hence we can define

$$\mathcal{N}(X) := \operatorname*{colim}_{k \to \infty} \mathcal{N}(X, k).$$

The notion of a normal map with respect to the tangent bundle, as introduced in Definition 2.5.10, extends naturally to the setting of compact manifolds with boundary. In this setting, the target is a finite n-dimensional Poincaré pair $(X, \partial X)$, and the map $f : M \to X$ is replaced by a map of pairs $(f, \partial f) : (M, \partial M) \to (X, \partial X)$, where $\partial f : \partial M \to \partial X$ is a homotopy equivalence. The bundle data is defined analogously to the closed case.

A normal bordism between normal maps of degree one with respect to the tangent bundle relative to the bundary is a quituple $(W, F, \overline{F}, \Xi, b)$ from $(M_0, \partial M_0, f_0, \bar{f}_0, \xi, a_0)$ to $(M_1, \partial M_1, f_1, \bar{f}_1, \xi, a_1)$ and consists of datas from the closed case with the additional following data

- W is a compact $(n+1)$-dimensional manifold with boundary

  $$\partial W = \partial_0 W \cup \partial_1 W \cup \partial_2 W,$$

  where $\partial_m W$ is a codimension zero submanifold of ∂W possibly with non-empty boundary $\partial\partial_m W$ satisfying

  $$\partial_0 W \cap \partial_1 W = \emptyset;$$
  $$\partial_2 W \cap \partial_m W = \partial\partial_m W \text{ for } m = 0, 1;$$
  $$\partial\partial_2 W = \partial\partial_0 W \cap \partial\partial_1 W$$

- $\partial Y = \partial_0 Y \cup \partial_1 Y \cup \partial_2 Y$, where $\partial_m Y = X \times \{m\}$ for $m = 0, 1$ and $\partial_2 Y = \partial X \times [0, 1]$.
- The map $F : W \to Y$ restricts to maps $\partial_m F : \partial_m W \to \partial_m Y$ for $m = 0, 1, 2$, with $\partial_2 F : \partial_2 W \to \partial_2 Y$ a homotopy equivalence.
- For $m = 0, 1$ we have the identifications of pairs:

  $$(M_m, \partial M_m) \xrightarrow{\cong} (\partial_m W, \partial\partial_m W), \text{ and } (X_m, \partial X_m) \xrightarrow{\cong} (\partial_m Y, \partial Y_m).$$

We denote by $\mathcal{N}(X, \partial X)$ the set of such normal maps to the pair $(X, \partial X)$ up to the normal bordism defined above.

If X is a compact manifold, we additionally assume that the boundary map $\partial f : \partial M \to \partial X$ and the restriction $\partial_2 F : \partial_2 W \to \partial_2 Y$ are diffeomorphisms. If ∂X is empty, then $\mathcal{N}(X, \emptyset) = \mathcal{N}(X)$.

With this terminology, we may carry out surgery below the middle dimension.

Theorem 2.5.19. *Any normal map $(f, \bar{f}) : M^n \to X$ is normal bordant to a normal map that is k-connected if $n = 2k$ or $2k + 1$.*

2.5.3 The Spivak Normal Structure and Normal Invariant

In order to make an n-dimensional Poincaré complex like manifold, we describe a homotopy theoretic analogue of the stable normal bundle of a manifold. Furthermore, we investigate the existence and uniqueness of bundle data.

Now we generalize the Pontrjagin-Thom construction to a Poincaré complex.

Definition 2.5.20 (Spivak Normal Fibration for Poincaré Complex). *A* Spivak normal fibration *on a finite n-dimensional Poincaré complex X is a $(k-1)$-spherical fibration $\nu_X : E \to X$ with a pointed map $c : \mathbb{S}^{n+k} \to \mathrm{Th}(\nu_X)$ such that there is a Thom class $U_{\nu_X} \in H^k(DE, E)$ and a fundamental class $[X] \in H_n(X)$ with*

$$[X] = (\nu_X)_*(U_{\nu_X} \cap H(c)),$$

where $H : \pi_{n+k}(\mathrm{Th}(\nu_X)) \to H_{n+k}(\mathrm{Th}(\nu_X))$ is the Hurewicz homomorphism.

Example 2.5.21. *Let M be a closed n-dimensional manifold and p be the spherical fibration corresponding to the normal bundle ν_M of an embedding $M \hookrightarrow \mathbb{R}^{n+k}$. This spherical fibration together with the Pontryagin-Thom collapse map $c : \mathbb{S}^{n+k} \to \mathrm{Th}(p)$ is a Spivak normal $(k-1)$-structure for M, considering M as a finite n-dimensional Poincaré complex.*

Theorem 2.5.22 (Existence and Uniqueness of Spivak normal fibration). *Let X be an n-dimensional finite Poincaré complex.*

(i) If $k \geq \dim(X) + 1$, then there exists a $(k-1)$-spherical normal fibration over X.

(ii) All Spivak normal structures on X are stably fiber homotopy equivalent.

We provide necessary condition for a Poincaré complex X to be homotopy equivalent to a closed manifold.

Definition 2.5.23 (Vector Bundle Reduction of Spivak normal fibration). *A Spivak normal fibration (ξ, c) over a finite n-dimensional Poincaré complex X has a* vector bundle reduction *if there is vector bundle* η *over* X *such that the corresponding sphere bundle* $S(\eta)$ *is stably fiber homotopy equivalent to* ξ.

Let $J_k = l_k \circ g_k : BO(k) \to BG(k)$, as defined in Section 2.2, denote the classifying map of the universal k-dimensional vector bundle, regarded as a spherical fibration. Now taking Whitney sum with trivial bundle and fiberwise join with trivial spherical fibration gives rise to the stabilisation map $J : BO \to BG$.

Proposition 2.5.24. *The following are equivalent*

(a) A stable spherical fibration $\nu_X : X \to BG$ *admits a vector bundle reduction.*

(b) there exists a lift $\tau : X \to BO$ *such that the following daigram commutes up to homotopy:*

$$\begin{tikzcd} & BO \arrow[d, "J"] \\ X \arrow[ur, dotted, "\tau"] \arrow[r, "\nu_X"'] & BG. \end{tikzcd}$$

(c) The composition $X \xrightarrow{\nu_X} BG \to B(G/O)$ *is null homotopic.*

Here, G/O *denotes the homotopy fiber of* $J : BO \to BG$.

Now, from the existence of vector bundle reduction of Spivak normal fibration to different choices of vector bundle reduction of Spivak normal fibration, we analyze the set of normal invariants of a finite n-dimensional Poincaré complex.

Definition 2.5.25 (Rank k Normal Invariant)**.** *A* rank k normal invariant *of an* n*-dimensional finite Poincaré complex* X *consists of a* k*-dimensional vector bundle* $\xi : X \to BO(k)$ *with orientation character* $w(X) \in H^1(X; \mathbb{Z}/2)$*, along with a map* $\rho : \mathbb{S}^{n+k} \to \mathrm{Th}(\xi)$ *such that*

$$H(\rho) \cap U_\xi = [X] \in H_n(X; \mathbb{Z}^{w(X)}),$$

where $U_\xi \in H^k(\mathrm{Th}(\xi))$ *is the Thom class and* $H : \pi_{n+k}(\mathrm{Th}(\xi)) \to H_{n+k}(\mathrm{Th}(\xi))$ *is the Hurewicz homomorphism.*

We call two rank k-normal invariants (ξ_1, ρ_1) and (ξ_2, ρ_2) equivalent if there is a bundle isomorphism $\bar{f} : \xi_1 \to \xi_2$ covering $Id : X \to X$ such that the map $(\mathrm{Th}(\bar{f}))_* : \pi_{n+k}(\mathrm{Th}(\xi_1)) \to \pi_{n+k}(\mathrm{Th}(\xi_2))$ sends $[\rho_1]$ to $[\rho_2]$.

We denote the set of equivalence classes of rank k-normal invariants of X as $\mathcal{NI}_n(X, k)$.

Note that given a rank k normal invariant (ξ, ρ), we can construct a rank $(k+1)$-normal invariant $(\xi \oplus \epsilon, \Sigma\rho)$, where $\Sigma : \pi_k(\mathrm{Th}(\xi)) \to \pi_{k+1}(\Sigma\mathrm{Th}(\xi)) \cong \pi_{k+1}(\mathrm{Th}(\xi \oplus \epsilon))$ is the suspension homomorphism. Now define the set of normal invariants $\mathcal{NI}_n(X)$ of X as $\operatorname*{colim}_{k\to\infty} \mathcal{NI}_n(X, k)$. Using the Pontrjagin-Thom construction, we establish the following:

Theorem 2.5.26 (Relation between degree one Normal Maps and Normal Invariants)**.** *Let X be a finite n-dimensional Poincaré complex. Then there is a bijection between $\mathcal{N}(X)$ and $\mathcal{NI}_n(X)$.*

Theorem 2.5.27 (Browder-Novikov Normal Invariant Theorem)**.** *Let X be a finite n-dimensional Poincaré complex. The following conditions are equivalent.*

(a) The set $\mathcal{N}(X)$ is non-empty.

(b) There exists a degree one normal map $(f, \bar{f}) : M^n \to X$.

(c) The Spivak normal fibration $\nu_X : X \to BG$ admits a vector bundle reduction $\eta : X \to BO$.

(d) The composition $X \xrightarrow{\nu_X} BG \to B(G/O)$ is null homotopic.

As a consequence we obtain

Theorem 2.5.28. *An n-dimensional finite Poincaré complex X is homotopy equivalent to an closed n-dimensional manifold if and only of there exists a lift of the Spivak normal fibration to a map such that the resulting normal bordism class of degree one normal maps contains a homotopy equivalence.*

Theorem 2.5.29. *Let X be a finite n-dimensional Poincaré complex such that $\mathcal{N}(X)$ is non-empty. Then the group $[X, G/O]$ acts freely and transitively on $\mathcal{N}(X)$.*

Using the above theorem, we obtain the following:

Theorem 2.5.30. *Let X be a finite n-dimensional Poincaré Complex. Then the set of normal invariants $\mathcal{N}(X)$ is in bijective correspondence with the set $[X, G/O]$ of fiber homotopy trivialised stable vector bundles over X, if $\mathcal{N}(X)$ is non-empty.*

An analogous statement holds in the relative setting: if $\mathcal{N}(X, \partial X)$ is non-empty for a finite Poincaré pair $(X, \partial X)$, then there is a natural bijection

$$\mathcal{N}(X, \partial X) \cong [X/\partial X, G/O].$$

2.5.4 Surgery in the middle dimension

Let $(f, \bar{f}) : M \to X$ be a k-connected degree one normal map. If $n = 2k$, then in view of Theorem 2.5.19, our main goal is to kill the elements of $\pi_{k+1}(f)$ to make f into $(k+1)$-connected. We want to decide whether it is possible to do further surgery on M.

If two elements of $\pi_{k+1}(f)$ correspond to immersions, a necessary condition for eliminating them via surgery is that their images are disjoint. However, even if all such elements have disjoint images, surgery may still not be possible, as the corresponding immersions might not be regularly homotopic to embeddings. To resolve this issue, we investigate the intersection pairing for immersions.

Define $I_k(M)$ be the set of immersions $f : \mathbb{S}^k \looparrowright M$ together with a path w in M from b to $f(s)$, up to regular homotopy of f and homotopy of w, where $b \in M$ and $s \in \mathbb{S}^k$ are basepoints. The fundamental group $\pi_1(M, b)$ acts on $I_k(M)$ by composing the path w with a loop based at b. Hence $I_k(M)$ has a $\mathbb{Z}[\pi_1(M)]$-module structure. The following lemma relates $\pi_{k+1}(f)$ with $I_k(M)$.

Lemma 2.5.31. *Let $(f, \bar{f}) : M^{2k} \to X$ be a k-connected degree one normal map. Then there is a natural $\mathbb{Z}[\pi_1(M)]$-module homomorphism $\alpha : \pi_{k+1}(f) \to I_k(M)$ such that following diagram commutes:*

$$\begin{tikzcd}
\pi_{k+1}(f) \arrow[rr, "\alpha"] \arrow[dr, "\delta"'] & & I_k(M) \arrow[dl, "r"] \\
& \pi_k(M), &
\end{tikzcd}$$

where δ is the connecting homomorphism in the long exact sequence of the homotopy groups and $r([f, w]) = [f], [(f, w)] \in I_k(M)$.

Definition 2.5.32 (Intersection Pairing)**.** *Let M be a compact $2k$-dimensional manifold. The intersection pairing*

$$\lambda \colon I_k(M) \times I_k(M) \to \mathbb{Z}[\pi_1(M)]$$

is defined by

$$\lambda([(f_1, w_1)], [(f_2, w_2)]) = \sum_{p \in f_1(\mathbb{S}^k) \pitchfork f_2(\mathbb{S}^k)} \epsilon(p) g(p),$$

where $\epsilon(p) \in \{\pm 1\}$ is the sign determined by the orientations and transversality, and $g(p) \in \pi_1(M)$ is the homotopy class of the loop

$$g(p) = w_1 * f_1(\gamma_1) * \overline{f_2(\gamma_2)} * \overline{w_2}.$$

Here, γ_i are paths in $\mathbb{S}^k$ from the basepoint s to x_i, where $f_1(x_1) = f_2(x_2) = p$.

Remark 2.5.33. *If a framed, based immersion f is regularly homotopic to an embedding φ, then $\lambda(f, f) = 0$.*

While the vanishing of the intersection pairing is a necessary condition for an immersion to be regularly homotopic to an embedding, a sufficient condition requires analysis of its self-intersections.

Let π be a group. Then define an abelian group

$$Q_\epsilon(\mathbb{Z}[\pi]) := \frac{\mathbb{Z}[\pi]}{\{x \sim \epsilon\bar{x}; x \in \mathbb{Z}[\pi]\}},$$

where $x \to \bar{x}$ denotes the w-twisted involution on $\mathbb{Z}[\pi]$.

Definition 2.5.34 (Self-intersection number)**.** *The* self-intersection number *is a homomorphism*

$$\mu : I_k(M) \to Q_{(-1)^k}(\mathbb{Z}[\pi_1(M)])$$

that assigns to a regular homotopy class $[(f, w)]$ *the sum over double points of the immersion* $f : \mathbb{S}^k \looparrowright M$*. Specifically, for each point* $p \in M$ *such that* $|f^{-1}(p)| = 2$*, we associate a term* $\epsilon(p) \cdot g(p)$*, where* $\epsilon(p)$ *is the sign determined by comparing local orientations, and* $g(p)$ *is the group element in* $\pi_1(M)$ *represented by the loop*

$$w * f(\gamma_1) * \overline{f(\gamma_2)} * \overline{w},$$

with γ_1 *and* γ_2 *being paths in* $\mathbb{S}^k$ *from the basepoint* s *to the preimages* x_1 *and* x_2 *of* p*, respectively, satisfying* $f(x_1) = f(x_2) = p$*.*

Theorem 2.5.35 (Wall Embedding Theorem)**.** *Let* M *be a compact, connected* $2k \geq 6$ *dimensional manifold and* $[(f, w)] \in I_k(M)$*. Then* (f, w) *is regular homotopic to an embedding if and only if* $\mu([f, w]) = 0$*.*

We now present the general algebraic framework that subsumes intersection pairings as a particular instance. Before that we recall the notion of oriented cover. An oriented cover $(\widetilde{X}, \pi, \omega)$ of a connected space X with an orientation character $\omega(X) \in H^1(X; \mathbb{Z}/2)$ consist of a regular covering of X with group of covering translation π, together with an orinetation character $\omega : \pi \to \mathbb{Z}/2$ such that

$$\omega(X) : \pi_1(X) \to \pi \xrightarrow{\omega} \mathbb{Z}/2,$$

so that the singular chain complex of X with twisted coefficient is given by $S(X; \mathbb{Z}^\omega) =$

$\mathbb{Z}^{\omega} \otimes_{\mathbb{Z}\pi} S(X)$.

Definition 2.5.36 (Homology Kernel)**.** *Let X be a space with an oriented cover $(\widetilde{X}, \pi, \omega)$ and $f : M \to X$ be map with π-equivariant lift $\widetilde{f} : \widetilde{M} \to \widetilde{X}$, where $\widetilde{M} = (\widetilde{f})^*(\widetilde{X})$. The homology kernel is a $\mathbb{Z}[\pi]$-module defined as*

$$K_j(M) := Ker\left(\widetilde{f}_* \colon H_j(\widetilde{M}; \mathbb{Z}) \to H_j(\widetilde{X}; \mathbb{Z})\right).$$

Lemma 2.5.37 (Properties of Homology Kernels)**.** *Let $k \geq 2$ and $f : M \to X$ be a map of CW complexes with $\pi_1(X)$-equivariant lift $\widetilde{f} : \widetilde{M} \to \widetilde{X}$.*

(i) The map f is k-connected if and only if $f_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism and $K_j(M)$ vanishes for all $j \leq (k-1)$.

(ii) If M has a finite k-skeleton and X has a finite $(k+1)$-skeleton, then $\pi_{k+1}(f) \cong K_k(M)$ is a finitely generated $\mathbb{Z}[\pi_1(X)]$-module.

(iii) If M and X are both finite CW complexes and $\operatorname{coker}(\widetilde{f}^* : H^l(\widetilde{X}; \mathbb{Z}) \to H^l(\widetilde{M}; \mathbb{Z})) = 0$ *for all $l \geq (k+1)$, the $K_k(M)$ is a stably finitely generated free $\mathbb{Z}[\pi_1(X)]$-module.*

(iv) If f is k-connected, then there is an isomorphism of $\mathbb{Z}[\pi_1(M)]$-module $h_k : \pi_{k+1}(f) \to K_k(M)$ obtained through the composition $\pi_{k+1}(f) \cong \pi_{k+1}(\widetilde{f}) \cong H_{k+1}(\widetilde{f}) \cong K_k(M)$. This isomorphism makes the following diagram commute:

$$\begin{tikzcd}
\pi_k(\widetilde{M}) \arrow[d, "H"'] & \pi_{k+1}(\widetilde{f}) \arrow[l, "\partial_k"'] \arrow[r, "\cong"] & \pi_{k+1}(f) \arrow[d, "\cong"', "h_k"] \\
H_k(\widetilde{M}; \mathbb{Z}) & & K_k(M), \arrow[ll]
\end{tikzcd}$$

where $H : \pi_k(\widetilde{M}) \to H_k(\widetilde{M}; \mathbb{Z})$ is the Hurewicz homomorphism and ∂_k is the connecting homomorphism from the long exact sequence of homotopy groups.

Definition 2.5.38 (ϵ-Symmetric Form)**.** *An* ϵ-symmetric *form over an associative ring R with unit and involution consists of an R-module P together with a function*

$\lambda : P \times P \to R$ *satisfying*

(i) $\lambda(p_1 + p_2, q) = \lambda(p_1, q) + \lambda(p_2, q)$,

(ii) $\lambda(p, q_1 + q_2) = \lambda(p, q_1) + \lambda(p, q_2)$,

(iii) $\lambda(ap, bq) = b\lambda(p, q)\bar{a}$,

(iv) $\lambda(p, q) = \epsilon\overline{\lambda(q, p)}$

for all $p, p_i, q, q_i \in P$ *and* $a, b \in R$.

Example 2.5.39. *(a) Let* X *be a* $2k$*-dimensional finite Poincaré complex with orientation cover* $(\widetilde{X}, \pi, w)$*, and let* $[X] \in H_{2k}(X; \mathbb{Z}^{w(X)})$ *denote its fundamental class. Then the pairing*

$$\begin{aligned} \lambda : H^k(\widetilde{X}) \times H^k(\widetilde{X}) &\longrightarrow \mathbb{Z}[\pi_1(X)] \\ (p, q) &\longmapsto p([X] \cap q) \end{aligned}$$

defines a $(-1)^k$*-symmetric bilinear form over the group ring* $\mathbb{Z}[\pi_1(X)]$.

(b) For a k*-connected degree one normal map* $(f, \bar{f}) :: M^{2k} \to X$*, Lemmas 2.5.31 and 2.5.37 yield a composite map* $\beta = \alpha \circ h_k^{-1} \colon K_k(M) \to I_k(M)$*. Using the pairing defined in 2.5.32, we construct a bilinear form*

$$\tilde{\lambda} \colon K_k(M) \times K_k(M) \xrightarrow{\beta \times \beta} I_k(M) \times I_k(M) \xrightarrow{\lambda} \mathbb{Z}[\pi_1(M)]$$

which makes $(K_k(M) \cong \pi_{k+1}(f), \tilde{\lambda})$ *into a* $(-1)^k$*-symmetric form over* $\mathbb{Z}[\pi_1(M)]$.

Now, the following algebraic notion captures the self-intersection of the immersions.

Definition 2.5.40 (ϵ-Quadartic Form)**.** *An* ϵ-quadratic form (P, λ, μ) *over a ring* R *with unit and involution is an* ϵ*-symmetric form* (P, λ) *together with a map* $\mu : P \to Q_\epsilon(R)$ *such that for all* $z, y \in P$ *and* $a \in R$ *the following hold.*

(i) $\mu(ax) = a\mu(x)\bar{a}$,

(ii) $\mu(x) + \epsilon\overline{\mu(x)} = \lambda(x, x)$,

(iii) $\mu(x + y) - \mu(x) - \mu(y) = pr(\lambda(x, y))$, *where* $pr : \mathbb{Z}\pi \to Q_\epsilon(\mathbb{Z}\pi)$ *is the projection map.*

We call an ϵ-quadratic form (P, λ, μ) non–singular, *if $\lambda : P \to P^*$ is a R-module isomorphism.*

Definition 2.5.41. *Two ϵ-quadratic forms (P_1, λ_1, μ_1) and (P_2, λ_2, μ_2) over a ring with involution R are isomorphic if there exists an R-module isomorphism $f : P_1 \to P_2$ such that*

$$\lambda_2(f(x), f(y)) = \lambda_1(x, y) \text{ and } \mu_2(f(x)) = \mu_1(X),\ \forall x, y \in P_1.$$

Definition 2.5.42 (Hyperbolic ϵ-Quadratic Form)**.** *Let P be a finitely generated projective R-module. The* hyperbolic ϵ-quadratic form *$H_\epsilon(P)$ over R consists of $(P \oplus P^*, \lambda, \mu)$ with*

$$\lambda = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} : P \oplus P^* \longrightarrow (P \oplus P^*)^* = P^* \oplus P$$

$$(x, f) \mapsto [(y, g) \mapsto f(y) + \epsilon\overline{g(x)}],$$

and

$$\mu : P \oplus P^* \longrightarrow Q_\epsilon(R)$$

$$(x, f) \mapsto f(x).$$

Example 2.5.43. *The composition $\widetilde{\mu} = \mu \circ \beta \colon K_k(M) \to Q_{(-1)^k}(\mathbb{Z}[\pi_1(M)])$ yields a non-singular $(-1)^k$-quadratic form $(K_k(M), \widetilde{\lambda}, \widetilde{\mu})$ over $\mathbb{Z}[\pi_1(M)]$, refining the symmetric structure from Example 2.5.39(b). Here μ is the self intersection number defined in 2.5.34.*

This non-singular $(-1)^k$-quadratic form $(K_k(M) \cong \pi_{k+1}(f), \widetilde{\lambda}, \widetilde{\mu})$ is known as kernel form of the k-connected degree one normal map $(f, \bar{f}) : M^{2k} \to X$.

Proposition 2.5.44 (Realization of forms)**.** *Let N be a $(2k-1)$-dimensional manifold with $k \geq 2$. Then every $(-1)^k$-quadratic form (K, λ, μ) over $\mathbb{Z}[\pi_1(N)]$, where K is a* *finitely generated free $\mathbb{Z}[\pi_1(N)]$-module, arises as the kernel form of a k-connected degree one normal bordism*

$$(F, \overline{F}) : (W, \partial_1 W, \partial_2 W) \to (N \times [0,1], N \times \{0\}, N \times \{1\}),$$

satisfying $K_k(W) = K$, with $(F, \overline{F})|_{\partial_1 W} = Id : N \to N$, and $(F, \overline{F})|_{\partial_2 W} : \partial_2 W \to N$ being $(k-1)$-connected.

Furthermore, the form (K, λ, μ) is non-singular if and only if $(F, \overline{F})|_{\partial_2 W}$ is a homotopy equivalence.

We now state a lemma that illustrates the impact of surgery on the surgery kernel.

Lemma 2.5.45. *Let $(f, \bar{f}) : M^{2k} \to X$ be a k-connected degree one normal map. Then performing surgery on the zero element of $K_{k-1}(M) = \pi_k(f)$ replaces the surgery kernel $(K_k(M), \lambda, \mu)$ with $(K_k(M), \lambda) \oplus H_\epsilon(\mathbb{Z}[\pi_1(M)])$.*

We next introduce the stable hyperbolic quadratic form, which will be used to express a sufficient condition under which a highly connected degree one normal map is normally bordant to a homotopy equivalence.

Definition 2.5.46. *An ϵ-quadratic form (P, λ, μ) over a ring R is* stably hyperbolic *if*

$$(K, \lambda, \mu) \oplus H_\epsilon(R)^u \cong H_\epsilon(R)^v$$

for some integers $u, v \geq 0$.

The following theorem establishes that the stably hyperbolic surgery kernel form characterizes when a highly connected degree one normal map is normal bordant to a homotopy equivalence.

Theorem 2.5.47. *Let $(f, \bar{f}) : M^{2k} \to X$ be a k-connected degree one normal map. Suppose $k \geq 3$. Then $(K_k(M), \widetilde{\lambda}, \widetilde{\mu})$ is stably hyperbolic if and only if f is normal bordant to a homotopy equivalence.*

2.5.5 Even dimensional L-group and Surgery Obstruction

In even-dimensional surgery theory, the primary obstruction to converting a degree one normal map into a homotopy equivalence lies in the *Wall L-groups* $L_{2k}(\mathbb{Z}[\pi_1(X)])$, which classify $(-1)^k$-quadratic forms over the group ring $\mathbb{Z}[\pi_1(X)]$. These groups algebraically encode whether a Poincaré complex can be realized by a manifold. The surgery kernel

$$K_k(M) := Ker\left(\widetilde{f}_*\colon H_k(\widetilde{M}) \to H_k(\widetilde{X})\right),$$

equipped with its intersection form λ and quadratic refinement μ, detects non-trivial obstructions when the stable class $[\lambda, \mu] \in L_{2k}$ fails to vanish.

Proposition 2.5.48. *If* $2 \in R$ *is invertible, then*

$$\{\epsilon\text{-quadratic form over } R\} = \{\epsilon\text{-symmetric form over } R\}.$$

Definition 2.5.49 (The Even Dimensional L-Groups)**.** *Let R be a ring with unit and involution. The* $2k$-dimensional L-group $L_{2k}(R)$ *is the abelian group of equivalence classes of non-singular $(-1)^k$-quadratic forms $[(P, \lambda, \mu)]$on finitely generated free R-modules. We call (P_1, λ_1, μ_1) and (P_2, λ_2, μ_2) equivalent, if there exist positive integers u, v and an isomorphism of non-singular ϵ-quadratic forms*

$$(P_1, \lambda_1, \mu_1) \oplus H_\epsilon(R)^u \cong (P_2, \lambda_2, \mu_2) \oplus H_\epsilon(R)^v.$$

The addition and inverse in $L_{2k}(R)$ are given by

$$(P_1, \lambda_1, \mu_1) \oplus (P_2, \lambda_2, \mu_2) = (P_1 \oplus P_2, \lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2),$$
$$-(P_1, \lambda_1, \mu_1) = (P_1, -\lambda_1, -\mu_1).$$

Example 2.5.50.

$$L_{2k}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \equiv 0 \pmod 2, \\ \mathbb{Z}/2 & \text{if } k \equiv 1 \pmod 2. \end{cases}$$

The isomorphisms are given by the signature divided by 8 when k is even, and by the Arf invariant when k is odd.

Definition 2.5.51 (Surgery Obstruction in Even Dimension)**.** *Let* $(f, \bar{f}) : M^{2k} \to X$ *be a degree one normal map, and suppose that* $(f, \bar{f})$ *is normally bordant to a* k*-connected degree one normal map* $(f', \bar{f}') : M'^{2k} \to X$*. Let* $(K_k(M'), \lambda', \mu')$ *be the kernel form over* $\mathbb{Z}[\pi_1(X)]$ *associated to* $(f', \bar{f}')$*. The surgery obstruction of* $(f, \bar{f})$ *is then defined as the isomorphism class of the kernel form:*

$$\sigma_{2k}(f, \bar{f}) := (K_k(M'), \lambda', \mu') \in L_{2k}(\mathbb{Z}[\pi_1(X)]).$$

Proposition 2.5.52. *Let* $(f_1, \bar{f}_1) : M_1^{2k} \to X$ *and* $(f_2, \bar{f}_2) : M_2^{2k} \to X$ *be two* k*-connected degree one normal maps that are normally bordant. Then* $\sigma_{2k}(f_1, \bar{f}_1) = \sigma_{2k}(f_2, \bar{f}_2)$*.*

This gives rise to a well defined map $\sigma_{2k} : \mathcal{NI}_{2k}(X) \to L_{2k}(\mathbb{Z}[\pi_1(X)])$*.*

Theorem 2.5.53. *Let* $(f, \bar{f}) : M^{2k} \to X$ *be a degree one normal map and* $2k \geq 5$*. Then* $\sigma_{2k}(f, \bar{f}) = 0 \in L_{2k}(\mathbb{Z}[\pi_1(X)])$ *if and only if* $(f, \bar{f})$ *is normal bordant to a homotopy equivalence.*

2.5.6 Odd dimensional L group and Surgery Obstruction

The study of odd-dimensional manifolds reveals fundamental differences from the even-dimensional case. While even-dimensional surgery obstructions are governed by non-trivial quadratic forms, the odd-dimensional theory presents a more subtle picture. The key distinction arises from the algebraic properties of the Wall groups: for many fundamental groups, the odd L-groups $L_{2k+1}(\pi_1(X))$ *vanish completely.*

This vanishing implies that any degree one normal map $(f, \bar{f}) : M^{2k+1} \to X$ *is normally bordant to a homotopy equivalence when* $\pi_1(X)$ *satisfies appropriate conditions. However, this apparent simplicity masks an important geometric structure. The surgery kernel* $K_k(M) = Ker\left(\widetilde{f}^* : H_k(\widetilde{M}; \mathbb{Z}) \to H_k(\widetilde{X}; \mathbb{Z})\right)$*, though not present-*

ing primary obstructions, still carries crucial information about the topology of the manifold.

The interaction between this kernel and secondary invariants determines finer distinctions in the classification of odd-dimensional manifolds.

Definition 2.5.54 (Lagrangian)**.** *Le (P, λ, μ) be an ϵ-quadratic form over a ring with involutions R. Then a lagrangian of (P, λ, μ) over R is a direct summand $L \subseteq P$ such that $\mu(L) = 0$ with $L = L^{\perp} = \{y \in P : \lambda(x, y) = 0, \quad \forall x \in L\}$.*

Definition 2.5.55 (Quadratic Formation)**.** *An ϵ-quadratic formation is a tuple $(P, \lambda, \mu; F, G)$, where (P, λ, μ) is a non-singular ϵ-quadratic form over a ring with involution, and F, G are an ordered pair of lagrangians.*

Definition 2.5.56. *A morphism $f : (P, \lambda, \mu; F, G) \to (P', \lambda', \mu'; F', G')$ is called an isomorphism of ϵ-quadratic formations over R if $f : (P, \lambda, \mu) \to (P', \lambda', \mu')$ is an isomorphism of ϵ-quadratic forms such that $f(F) = F'$ and $f(G) = G'$.*

Definition 2.5.57 (Boundary quadratic formation)**.** *A boundary of an ϵ-quadratic form (P, λ, μ), denoted by $\partial(P, \lambda, \mu)$, is the ϵ-quadratic formation $(H_{\epsilon}(P); P, \Gamma_{(P,\lambda)})$ with $\Gamma_{(P,\lambda)} = \{(x, \lambda(x)) \in P \oplus P^* : x \in P\}$.*

Definition 2.5.58 (The odd Dimensional L-Groups)**.** *Let R be a ring with unit and involution. The* $(2k+1)$-dimensional L-group *$L_{2K+1}(R)$ is the abelian group of equivalence classes of ϵ-quadratic formations $(P, \lambda, \mu; F, G)$, where P, F, G are finitely generated free R-modules. We call $(P_1, \lambda_1, \mu_1; F_1, G_1)$ and $(P_2, \lambda_2, \mu_2; F_2, G_2)$ equivalent, if there is a stable isomorphism of formations between $(P_1, \lambda_1, \mu_1; F_1, G_1) \oplus \partial(Q, \tilde{\lambda}, \tilde{\mu})$ and $(P_2, \lambda_2, \mu_2; F_2, G_2) \oplus \partial(Q', \tilde{\lambda}', \tilde{\mu}')$ for some $(-\epsilon)$-quadratic forms $(Q, \tilde{\lambda}, \tilde{\mu})$ and $(Q', \tilde{\lambda}', \tilde{\mu}')$.*

The addition and inverse in $L_{2k+1}(R)$ are given by

$$\begin{aligned}
&(P_1, \lambda_1, \mu_1; F_1, G_1) \oplus (P_2, \lambda_2, \mu_2; F_2, G_2) \\
&\qquad\qquad = (P_1 \oplus P_2, \lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2; F_1 \oplus F_2, G_1 \oplus G_2)
\end{aligned}$$

$$-(P_1, \lambda_1, \mu_1; F_1, G_1) = (P_1, -\lambda_1, -\mu_1; F_1, G_1).$$

Example 2.5.59. $L_{2k+1}(\mathbb{Z}) = 0$ *[63, §5 and §6].*

Example 2.5.60. *[113]*

$$L_n(\mathbb{Z}[\mathbb{Z}]) = \begin{cases} \mathbb{Z} & \textit{if } n \equiv 0 \pmod 4, \\ \mathbb{Z} & \textit{if } n \equiv 1 \pmod 4, \\ \mathbb{Z}/2 & \textit{if } n \equiv 2 \pmod 4, \\ \mathbb{Z}/2 & \textit{if } n \equiv 3 \pmod 4. \end{cases}$$

Example 2.5.61. *[113]*

$$L_n(\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & \textit{if } n \equiv 0 \pmod 4, \\ \mathbb{Z} \oplus \mathbb{Z} & \textit{if } n \equiv 1 \pmod 4, \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \textit{if } n \equiv 2 \pmod 4, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \textit{if } n \equiv 3 \pmod 4. \end{cases}$$

We associate to any degree one normal map $(f, \overline{f})\colon M^{2k+1} \to X$ a $(-1)^k$-quadratic formation, generalizing the Heegaard splitting of 3-manifolds.

Definition 2.5.62 (Heegaard Splitting)**.** *A* Heegaard splitting *of a k-connected degree one normal map $(f, \bar{f}) : M^{2k+1} \to X$ is a decomposition*

$$(f, \bar{f}) = (f_0, \bar{f}_0) \cup (e, \bar{e}) : M^{2k+1} = M_0 \cup U^{2k+1} \to X = X_0 \cup \mathbb{D}^{2k+1},$$

where $U^{2k+1} = \#_{i=1}^{m} g_i(\mathbb{S}^k \times \mathbb{D}^{k+1}) \subset M^{2k+1}$, $M_0 = \overline{M - U}$, and $M_0 \cap U = \partial M_0 = \partial U = \#_{i=1}^{m} g_i(\mathbb{S}^k \times \mathbb{S}^k)$. The maps $g_i : \mathbb{S}^k \times \mathbb{D}^{k+1} \hookrightarrow M$ are framed k-embedding with null homotopies in X and correspond to a generating set $\{x_1, x_2, \cdots, x_m\} \subset K_k(M)$ as a $\mathbb{Z}[\pi_1(X)]$-module.

Definition 2.5.63 (Kernel Formation)**.** *The* kernel formation of a k-connected degree one normal map $(f, \bar{f}) : M^{2k+1} \to X$ *with respect to a Heegaard splitting is the* $(-1)^k$-*quadratic formation over* $\mathbb{Z}[\pi_1(X)]$

$$(K, \lambda, \mu; F, G) = (K_k(\partial U), \lambda, \mu; K_{k+1}(U, \partial U), K_{k+1}(M_0, \partial U)),$$

where $(K_k(\partial U), \lambda, \mu) = H_{(-1)^k}(K_{k+1}(U, \partial U))$ *denotes the hyperbolic* $(-1)^k$-*quadratic kernel form over* $\mathbb{Z}[\pi_1(X)]$*, associated to the* k*-connected degree one normal map* $\partial U \to \mathbb{S}^{2k}$*. The lagrangians* F *and* G *are defined using the* k*-connected degree one normal maps* $(e, \bar{e}) : (U, \partial U) \to (\mathbb{D}^{2k+1}, \mathbb{S}^{2k})$ *and* $(f_0, \bar{f}_0) : (M_0, \partial U) \to (X_0, \mathbb{S}^{2k})$*, and are given by*

$$F = Im\left(\partial : K_{k+1}(U, \partial U) \to K_k(\partial U)\right), \quad G = Im\left(\partial : K_{k+1}(M_0, \partial U) \to K_k(\partial U)\right).$$

Proposition 2.5.64. *Let* $(f, \overline{f}) : M^{2k+1} \to X$ *be a* k*-connected degree one normal map. Then:*

- *(i) All kernel formations arising from Heegaard splittings of* $(f, \overline{f})$ *are stably isomorphic.*
- *(ii) If* $k \geq 2$*, then every formation within this stable isomorphism class can be realized by some Heegaard splitting of* $(f, \overline{f})$*.*
- *(iii) For* $k \geq 2$*, the kernel formation is trivial if and only if* $(f, \overline{f})$ *is a homotopy equivalence.*

Proposition 2.5.65 (Realization of Formations). *Let* N^{2k} *be a* $2k$*-dimensional manifold with* $k \geq 2$*. Then every* $(-1)^k$*-quadratic formation* $(K, \lambda, \mu; F, G)$ *over* $\mathbb{Z}[\pi_1(N)]$ *is realized as the kernel formation of a* k*-connected degree one normal bordism* $(\mathcal{F}, \overline{\mathcal{F}}) : (W^{2k+1}, \partial_1 W, \partial_2 W) \to (N \times [0, 1], N \times \{0\}, N \times \{1\})$ *where* $(\mathcal{F}, \overline{\mathcal{F}})|_{\partial_1 W} = Id : N \to N$ *and* $(\mathcal{F}, \overline{\mathcal{F}})|_{\partial_2 W} : \partial_2 W \to N$ *a homotopy equivalence.*

Definition 2.5.66 (Surgery Obstruction in odd dimension). *Let* $(f, \bar{f}) : M^{2k+1} \to X$ *be a degree one normal map and suppose that* $(f, \bar{f})$ *is normal bordant to a* k*-connected* *degree one normal map* $(f', \bar{f}') : M'^{2k+1} \to X$. *Let* $(K', \lambda', \mu'; F', G')$ *be the kernel formation over* $\mathbb{Z}[\pi_1(X)]$ *associated with* $(f', \bar{f}')$. *The surgery obstruction of* $(f, \bar{f})$ *is defined as the isomorphism class of kernel formation over* $\mathbb{Z}[\pi_1(X)]$:

$$\sigma_{2k+1}(f, \bar{f}) := (K', \lambda', \mu'; F', G') \in L_{2k+1}(\mathbb{Z}[\pi_1(X)]).$$

Proposition 2.5.67. *Let* $(f_1, \bar{f}_1) : M_1^{2k+1} \to X$ *and* $(f_2, \bar{f}_2) : M_2^{2k+1} \to X$ *be two degree one normal maps which are normal bordant. Then*

$$\sigma_{2k+1}(f_1, \bar{f}_1) = \sigma_{2k+1}(f_2, \bar{f}_2).$$

The above proposition gives rise to a well defined map $\sigma_{2k+1} : \mathcal{NI}_{2k+1}(X) \to L_{2k+1}(\mathbb{Z}[\pi_1(X)])$.

Theorem 2.5.68. *Let* $(f, \overline{f}) : M^{2k+1} \to X$ *be a degree one normal map with* $k \geq 2$. *Then the odd-dimensional surgery obstruction* $\sigma_{2k+1}(f, \overline{f}) \in L_{2k+1}(\pi_1(X))$ *vanishes if and only if* $(f, \overline{f})$ *is normally bordant to a homotopy equivalence.*

2.5.7 The Smooth Surgery Exact Sequence

The surgery exact sequence provides a powerful algebraic framework for the classification problem of manifolds in high dimensions. Introduced by Browder, Novikov, Sullivan, and Wall in the 1960s as part of the surgery theory programme, this exact sequence organizes the obstructions involved in modifying a homotopy equivalence into systematic, computable algebraic invariants.

Throughout the section X is a compact manifold of dimension $n \geq 5$ and Whitehead torsion $Wh(\pi_1(X)) = 0$.

Definition 2.5.69 (The manifold set)**.** *Let* X *be a closed manifold of dimension* n. *The* manifold set $\mathcal{S}^h(X)$ *of* X *is the set of diffeomorphism classes of closed manifolds homotopy equivalent to* X.

To transition from the smooth structure set $\mathcal{S}_h^{Diff}(X)$ to the manifold set $\mathcal{S}^h(X)$, it is important to understand the action of the group of homotopy classes of self-homotopy equivalences of X, denoted $\mathcal{E}(X)$ on $\mathcal{S}^{Diff}(X)$. The group $\mathcal{E}(X)$ acts on $\mathcal{S}_h^{Diff}(X)$ in the following manner:

$$\begin{aligned}\mathcal{E}(X) \times \mathcal{S}_h^{Diff}(X) &\twoheadrightarrow \mathcal{S}_h^{Diff}(X)\\ ([g],[(N,f)]) &\mapsto [(N, g\circ f)].\end{aligned}$$

Moreover, there is a natural forgetful surjective map

$$\begin{aligned}\mathcal{G} : \mathcal{S}_h^{Diff}(X) &\twoheadrightarrow \mathcal{S}^h(X)\\ [(N,f)] &\mapsto [N].\end{aligned}$$

Combining the above two maps we obtain a natural surjective map

$$\mathcal{A} : \mathcal{S}_h^{Diff}(X)/\mathcal{E}(X) \twoheadrightarrow \mathcal{S}^h(X).$$

Now we show that $\mathcal{A}$ is injective.

Let $[(M,f_1)],[(M,f_2)] \in \mathcal{S}_h^{Diff}(X)$ be in the same orbit under the action of $\mathcal{E}(X)$. Note that $f_1 \simeq ((f_1 \circ f_2^{-1}) \circ f_2$, where $f_1 \circ f_2^{-1} \in \mathcal{E}(X)$. Hence

$$\mathcal{A} : \mathcal{S}_h^{Diff}(X)/\mathcal{E}(X) \xrightarrow{\cong} \mathcal{S}^h(X)$$

is a bijection.

Now we define a map

$$\begin{aligned}\eta^{Diff} : \mathcal{S}_h^{Diff}(X) &\longrightarrow \mathcal{N}(X)\\ [(N,f)] &\mapsto [(N,f,\bar{f},\xi)],\end{aligned} \tag{2.6}$$

where ξ is the bundle $(f^{-1})^*(TN)$ over X and $\bar{f} : TN \cong f^*(\xi) \to \xi$ is the bundle map covering f.

A crucial factor in the successful application of surgery is our ability to calculate

$\mathcal{N}(X)$ using Theorem 2.5.30.

Now, we define the action of the abelian group $L_{n+1}(\mathbb{Z}][\pi_1(X)])$ on $\mathcal{S}_h^{Diff}(X)$:

$$\omega^{Diff} : L_{n+1}(\mathbb{Z}[\pi_1(X)]) \times \mathcal{S}_h^{Diff}(X) \to \mathcal{S}_h^{Diff}(X). \tag{2.7}$$

Suppose $x \in L_{n+1}(\mathbb{Z}[\pi_1(X)])$, and consider an element $[(N, f)] \in \mathcal{S}_h^{Diff}(X)$. Under the correspondence in (2.6), this element is represented by a degree one normal map $(f, \overline{f})$. Then, by Theorems 2.5.44 and 2.5.65, there exists a degree one normal bordism

$$(F, \overline{F}) : (W, \partial_1 W = X, \partial_2 W) \to (X \times [0,1], X \times \{0\}, X \times \{1\})$$

such that $(F, \overline{F})|_{\partial_1 W} = Id_X$, the restriction $(F, \overline{F})|_{\partial_2 W} : \partial_2 W \to X$ is a homotopy equivalence, and the surgery obstruction satisfies $\sigma_{n+1}(F, \overline{F}) = x$.

We construct a new bordism

$$W' := W \cup_{f : N \times \{1\} \to X \times \{0\}} (N \times [0,1])$$

which produces a normal bordism

$$(F', \overline{F'}) : (W', \partial_1 W', \partial_2 W') \to (X \times [0,1], X \times \{0\}, X \times \{1\})$$

between degree one normal maps $(F', \overline{F'})|_{\partial_1 W'} = (f, \bar{f}) : N \to X$ and $(F', \overline{F'})|_{\partial_2 W'} = (F, \overline{F})|_{\partial_2 W}$ with $\sigma_{n+1}(F', \overline{F'}) = x$.

Now we define $\omega^{Diff}(x, [(N, f)]) = [(\partial_2 W, F|_{\partial_2 W})]$.

We have to show that the class $[(N, f)] \in \mathcal{S}_h^{Diff}(X)$ is independent of choice of normal bordism $(F', \overline{F'}) : (W', \partial_1 W', \partial_2 W') \to (X \times [0,1], X \times \{0\}, X \times \{1\})$.

Let $(F_i, \overline{F_i}) : (W_i, \partial_1 W_i, \partial_2 W_i) \to (X \times [0,1], X \times \{0\}, X \times \{1\})$ for $i = 1, 2$ be two degree one normal bodisms between $(F_i, \overline{F_i})|_{\partial_1 W_i} = f : N \to X$, $(F_i, \overline{F_i})|_{\partial_2 W_i} : \partial_2 W_i \to X$ with $\sigma_{n+1}(F_1, \overline{F_1}) = \sigma_{n+1}(F_2, \overline{F_2}) = x$ and $(F_i, \overline{F_i})|_{\partial_2 W_i} : \partial_2 W_i \to X$ is a homotopy equivalence. Gluing these two normal bordisms, we obtain a degree one normal bordism between $(\partial_2 W_1, F_1|_{\partial_2 W_1}, \overline{F_1}|_{\partial_2 W_1})$ and $(\partial_2 W_2, F_2|_{\partial_2 W_2}, \overline{F_2}|_{\partial_2 W_2})$ whose surgery obstruction vanishes. Then, by Theorems 2.5.53 and 2.5.68, together with the s-cobordism theorem, the pairs $(\partial_2 W_1, F_1|_{\partial_2 W_1})$ and $(\partial_2 W_2, F_2|_{\partial_2 W_2})$ are equivalent in $\mathcal{S}_h^{Diff}(X)$.

Theorem 2.5.70. *Let X be a closed smooth manifold of dimension $n \geq 5$, and let $\pi = \pi_1(X)$. Then there exists the smooth surgery exact sequence*

$$\begin{aligned} \cdots \to \mathcal{N}(X \times [0,1], \partial(X \times [0,1])) \xrightarrow{\sigma_{n+1}^{Diff}} L_{n+1}(\mathbb{Z}[\pi]) \xrightarrow{\omega^{Diff}} \mathcal{S}_h^{Diff}(X) \xrightarrow{\eta^{Diff}} \\ \to \mathcal{N}(X) \cong [X, G/O] \xrightarrow{\sigma_n^{Diff}} L_n(\mathbb{Z}[\pi]) \end{aligned} \tag{2.8}$$

which is exact in the sense described below.

*(i) (**Exactness at** $\mathcal{N}(X)$) Any $\alpha \in \mathcal{N}(X)$ lies in the image of $\eta^{Diff} : \mathcal{S}_h^{Diff}(X) \to \mathcal{N}(X)$ if and only if $\sigma_n^{Diff}(\alpha) = 0$.*

*(ii) (**Exactness at** $\mathcal{S}_h^{Diff}(X)$) Let $[(N_1, f_1)]$ and $[(N_2, f_2)]$ be two elements of $\mathcal{S}^{Diff}(X)$. Now $\eta^{Diff}([(N_1, f_1)]) = \eta^{Diff}([(N_2, f_2)])$ if and only if $\omega^{Diff}(x, [(N_1, f_1)]) = [(N_2, f_2)]$.*

*(iii) (**Exactness at** $L_{n+1}(\mathbb{Z}[\pi])$) An element $x \in L_{n+1}(\mathbb{Z}[\pi])$ satisfies $\omega^{Diff}(x, [(X, Id_X)]) = [(X, Id_X)] \in \mathcal{S}_h^{Diff}(X)$ if and only if there exists a normal bordism class $(F, \overline{F}) \in \mathcal{N}(X \times [0,1], \partial(X \times [0,1]))$ such that $\sigma_{n+1}^{Diff}(F, \overline{F}) = x$.*

Proof. Statement (i) follows from the definition of surgery obstrction given in 2.5.51 and 2.5.66 along with Theorems 2.5.53 and 2.5.68.

(ii) Let $\eta^{Diff}([(N_1, f_1)]) = \eta^{Diff}([(N_2, f_2)])$. Then there is a normal bordism $(F, \overline{F}) : (W, N_1, N_2) \to (X \times [0,1], X \times \{0\}, X \times \{1\})$ between $(N_1, f_1, \bar{f}_1, \xi_1)$ and $(N_2, f_2, \bar{f}_2, \xi_2)$ where $\xi_i = (f_i^{-1})^*(TN_i)$. Let $\sigma_*^{Diff}(F, \overline{F}) = x$. Hence, by definition (2.7), $\omega^{Diff}(x, [(N_1, f_1)]) = [(N_2, f_2)]$.

Conversely, let $\omega^{Diff}(x, [(N_1, f_1)]) = [(N_2, f_2)]$. Then there exists a degree one normal bordism $(F, \overline{F}) : (W, N_1, N_2) \to (X \times [0,1], X \times \{0\}, X \times \{1\})$ such that

$(F,\overline{F})|_{\partial_1 W} = (f_1, \bar{f}_1) : N_1 \to X$ and $(F,\overline{F})|_{\partial_2 W} = (f_2, \bar{f}_2) : N_2 \to X$ with $\sigma_*^{Diff}(F,\overline{F}) = x$, where $\bar{f}_i$ is the bundle ismorphism $f_i^* \circ (f_i^{-1})^*(TN_i) \overset{\cong}{\to} (f_i^{-1})^*(TN_i)$ covering f_i. Hence $(N_1, f_1, \bar{f}_1, (f_1^{-1})^*(TN_1))$ is equivalent to $(N_2, f_2, \bar{f}_2, (f_2^{-1})^*(TN_2))$ in $\mathcal{N}(X)$, which implies that $\eta^{Diff}([(N_1,f_1)]) = \eta^{Diff}([(N_2,f_2)])$.

(iii) Let $x \in L_{n+1}(\mathbb{Z}[\pi])$ be such that

$$\omega^{Diff}(x, [(X, Id_X]) = [(X, Id_X)].$$

Then, by definition of the surgery obstruction map, there exists a degree one normal bordism

$$(F, \overline{F}, \partial F) : (W, \partial W) \to (X \times [0,1], \partial(X \times [0,1]))$$

from $(X, \partial X, Id_X)$ to itself, with surgery obstruction equal to x. Since Id_X is a diffeomorphism, this bordism defines an element of $\mathcal{N}(X \times [0,1], \partial(X \times [0,1]))$, and we have

$$\sigma_{n+1}^{Diff}([(F, \overline{F}, \partial F)]) = x.$$

Conversely, let $(F, \overline{F}, \partial F) \in \mathcal{N}(X \times [0,1], \partial(X \times [0,1]))$. Since ∂F is a diffeomorphism, this again defines a degree one normal bordism from $(X, \partial X, Id_X)$ to itself. Then, by the definition of ω^{Diff}, we obtain

$$\omega^{Diff}(\sigma_{n+1}^{Diff}([(F, \overline{F}, \partial F)]), [(X, \partial X, Id_X)]) = [(X, \partial X, Id_X)].$$

This completes the proof. □

Note 2.5.71. *Although the surgery obstruction map* $\sigma_n^{Diff} : \mathcal{N}(X) \to L_n(\mathbb{Z}[\pi_1(C)])$ *is defined between groups, it is not a group homomorphism.*

Note 2.5.72. *[114, Proposition 2.2] Let* $f : N \to X$ *and* $g : M \to N$ *be homotopy equivalence between closed manifolds. Then*

$$\eta^{Diff}(g \circ f) = \eta^{Diff}(g) + (g^{-1})^* \eta^{Diff}(f).$$

Let CAT denote either Top or PL. Then, for any closed CAT manifold X of dimension $n \geq 5$, there is a sequence of pointed sets

$$\cdots \to [\Sigma X, G/CAT] \xrightarrow{\sigma_{n+1}^{CAT}} L_{n+1}(\mathbb{Z}[\pi]) \xrightarrow{\omega^{CAT}} \mathcal{S}_h^{CAT}(X) \xrightarrow{\eta^{CAT}} \\ \to [X, G/CAT] \xrightarrow{\sigma_n^{CAT}} L_n(\mathbb{Z}[\pi])$$

which is exact in the sense described in the preceding theorem. In the case $CAT = Top$, it is shown in [64, pp. 277–283] that the map $\eta^{Top} : \mathcal{S}_h^{Top}(X) \to [X, G/Top]$ is a group homomorphism, and that $\mathcal{S}_h^{Top}(X)$ admits a group structure. As a consequence, the topological surgery exact sequence becomes a long exact sequence of abelian groups.

Suppose X is a compact smooth manifold of dimension $n \geq 5$. Then the *relative structure set* $\mathcal{S}_h^{Diff}(X \ \text{rel}\, \partial X)$ fits into the following exact sequence

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi]) \xrightarrow{\omega_{\text{rel}}^{Diff}} \mathcal{S}_h^{Diff}(X \ \text{ rel } \ \partial X) \xrightarrow{\eta_{\text{rel}}^{Diff}} [X/\partial X, G/O] \xrightarrow{\sigma_{\text{rel}}^{Diff}} L_n(\mathbb{Z}[\pi]).$$

We consider the product manifold $M \times \mathbb{D}^k$, where M is a closed, smooth n-dimensional manifold. The relative smooth structure set $\mathcal{S}_h^{Diff}(M \times \mathbb{D}^k \text{ rel } M \times \mathbb{S}^{k-1})$ is denoted by $\mathcal{S}_k^{Diff}(M)$. As shown in [101, 105, 23, 18], $\mathcal{S}_k^{Diff}(M)$ carries a group structure. Furthermore, when $n + k \geq 5$, there exists a relative surgery exact sequence of groups [18, Page 24], which takes the form:

(2.9)

$$\cdots \to L_{n+k+1}(\mathbb{Z}[\pi_1(M)]) \xrightarrow{\omega_{\text{rel}}^{Diff}} \mathcal{S}_k^{Diff}(M) \xrightarrow{\eta_{\text{rel}}^{Diff}} [\Sigma^k M_+, G/O] \xrightarrow{\sigma_{\text{rel}}^{Diff}} L_{n+k}(\mathbb{Z}[\pi_1(M)]),$$

where $\eta_{\text{rel}}^{Diff} : \mathcal{S}_k^{Diff}(M) \to [\Sigma^k M_+, G/O]$ and $\sigma_{\text{rel}}^{Diff} : [\Sigma^k M_+, G/O] \to L_{n+k}(\pi_1(M))$ are group homomorphism.

We now recall the notion of a tangential structure set, which will be useful later in the thesis.

Definition 2.5.73. *[33, Page 103] Let X be a closed, smooth or PL manifold of dimension n, with stable normal bundle ν_X of sufficiently high rank. The* smooth (or PL) tangential structure set *$\mathcal{S}^t_{Diff}(X)$ (or $\mathcal{S}^t_{PL}(X)$) of X consists of equivalence classes* *of triples* $(N, f, \bar{f})$*, where* N *is a smooth (or PL) manifold,* $f : N \to X$ *is a homotopy equivalence, and* $\bar{f} : \nu_N \to \nu_X$ *is a bundle map between the stable normal bundles. Two such triples* $(N_1, f_1, \bar{f}_1)$ *and* $(N_2, f_2, \bar{f}_2)$ *are considered equivalent if there exists an s-cobordism* $(W, N_1, N_2, F, \bar{F})$*, where* $F : W \to X$ *is a simple homotopy equivalence and* $\bar{F} : \nu_W \to \nu_X$ *is a bundle map that restricts to* $\bar{f}_1$ *and* $\bar{f}_2$ *on the respective boundaries.*

The tangential structure set $\mathcal{S}^t_{CAT}(X)$ also fits into exact sequence of four terms

$$L_{n+1}(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\omega^t} \mathcal{S}^t_{CAT}(X) \xrightarrow{\eta^t} [X, G] \xrightarrow{\sigma^t} L_n(\mathbb{Z}[\pi_1(X)]).$$

A relative version of the tangential structure set also exists. Let $X = M \times \mathbb{D}^k$ where M is a closed, smooth (or PL) n-manifold. Then the relative smooth (or PL) tangential structure set $\mathcal{S}^t_k(M) = \mathcal{S}^t_{CAT}(M \times \mathbb{D}^k,\ \mathrm{rel}\, M \times \mathbb{S}^{k-1})$ fits into the following exact sequence

$$\cdots \to L_{n+k+1}(\mathbb{Z}[\pi_1(M)]) \xrightarrow{\omega^t} \mathcal{S}^t_k(M) \xrightarrow{\eta^t} [\Sigma^k M_+, G] \xrightarrow{\sigma^t} L_{n+k}(\mathbb{Z}[\pi_1(M)]). \tag{2.10}$$

Moreover, there are natural maps from (2.10) to (2.9) such that following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & L_{n+k+1}(\mathbb{Z}[\pi_1(M)]) \arrow[r, "\omega^t"] \arrow[d, "="'] & \mathcal{S}^t_k(M) \arrow[r, "\eta^t"] \arrow[d, "\mathcal{F}"] & {[\Sigma^k M_+, G]} \arrow[r, "\sigma^t"] \arrow[d, "j_*"] & L_{n+k}(\mathbb{Z}[\pi_1(M)]) \arrow[d, "="] \\
\cdots \arrow[r] & L_{n+k+1}(\mathbb{Z}[\pi_1(M)]) \arrow[r, "\omega^{Diff}_{\mathrm{rel}}"'] & \mathcal{S}^{Diff}_k(M) \arrow[r, "\eta^{Diff}_{\mathrm{rel}}"'] & {[\Sigma^k M_+, G/O]} \arrow[r, "\sigma^{Diff}_{\mathrm{rel}}"'] & L_{n+k}(\mathbb{Z}[\pi_1(M)]).
\end{tikzcd}$$

As per [107], there is a forgetful map $\mathcal{H} : [\Sigma^k M_+, PL] \to \mathcal{S}^t_k(M)$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & {[\Sigma^{k+1} M_+, G/PL]} \arrow[r] \arrow[d, "\sigma^{PL}_{n+k+1}"'] & {[\Sigma^k M_+, PL]} \arrow[r] \arrow[d, "\mathcal{H}"] & {[\Sigma^k M_+, G]} \arrow[r] \arrow[d, "\cong"] & {[\Sigma^k M_+, G/PL]} \arrow[d, "\sigma^{PL}_{n+k}"] \\
\cdots \arrow[r] & L_{n+k+1}(\mathbb{Z}[\pi_1(M)]) \arrow[r, "\omega^t"'] & \mathcal{S}^t_k(M) \arrow[r, "\eta^t"'] & {[\Sigma^k M_+, G]} \arrow[r, "\sigma^t"'] & L_{n+k}(\mathbb{Z}[\pi_1(M)]),
\end{tikzcd}$$

where the top row is induced from the fiber sequence $\cdots \to PL \to G \to G/PL$.

We now combine the two diagrams above to obtain the following diagram, which will be useful later.

(2.11)

$$\begin{tikzcd}
\cdots \arrow[r] & {[\Sigma^{k+1}M_+, G/PL]} \arrow[r] \arrow[d, "\sigma^{PL}_{n+k+1}"'] & {[\Sigma^k M_+, PL]} \arrow[r] \arrow[d, "\mathcal{F}\circ\mathcal{H}"] & {[\Sigma^k M_+, G]} \arrow[r] \arrow[d, "j_*"] & {[\Sigma^k M_+, G/PL]} \arrow[d, "\sigma^{PL}_{n+k}"] \\
\cdots \arrow[r] & L_{n+k+1}(\mathbb{Z}[\pi_1(M)]) \arrow[r, "\omega^{Diff}_{\mathrm{rel}}"'] & \mathcal{S}^{Diff}_k(M) \arrow[r, "\eta^{Diff}_{\mathrm{rel}}"'] & {[\Sigma^k M_+, G/O]} \arrow[r, "\sigma^{Diff}_{\mathrm{rel}}"'] & L_{n+k}(\mathbb{Z}[\pi_1(M)]).
\end{tikzcd}$$

2.6 Some Background from Homotopy Theory

In this section, we recall definitions and results from homotopy theory that will be used later.

2.6.1 The Spanier–Whitehead category

Let $\mathbf{HoCW}_*$ denote the pointed homotopy category of pointed CW complexes. Its objects are pointed CW complexes, and its morphisms are homotopy classes of basepoint-preserving maps. In general, for pointed spaces X and Y, the set $[X, Y]_*$ carries only the structure of a pointed set.

To work in a more tractable framework, Spanier and Whitehead introduced the notion of *stable homotopy classes of maps*. For pointed CW complexes X and Y, these are defined by

$$\{X, Y\} := \operatorname*{colim}_n [\Sigma^n X, \Sigma^n Y]_*.$$

For $n \geq 2$, this colimit is an abelian group.

When X and Y are finite CW complexes, the Freudenthal Suspension Theorem implies that the suspension homomorphism

$$\Sigma : [\Sigma^k X, \Sigma^k Y]_* \xrightarrow{\cong} [\Sigma^{k+1} X, \Sigma^{k+1} Y]_*$$

is an isomorphism for sufficiently large k. Thus, suspension becomes invertible in the stable range.

Motivated by this observation, one may view the passage to stable homotopy classes as a form of localization of $\mathbf{HoCW}_*$ in which suspension is forced to be an equivalence. This is achieved formally by allowing desuspensions of spaces.

Definition 2.6.1. *The* Spanier–Whitehead category $\mathscr{SW}$ *is defined as follows. Its objects are pairs* (X, m)*, where* X *is a pointed CW complex and* $m \in \mathbb{Z}$*. The morphisms are given by*

$$\{(X,m),(Y,n)\} := \operatorname*{colim}_k [\Sigma^{m+k}X, \Sigma^{n+k}Y]_*.$$

2.6.2 Spectra and Basic Properties

Definition 2.6.2. *A* spectrum E *is a sequence of based topological spaces* $\{E_n\}$ *with structure map*

$$\varepsilon_n : \Sigma E_n \to E_{n+1}.$$

Since there is a natural equivalence

$$Map_*(\Sigma X, Y) \cong Map_*(X, \Omega Y),$$

the structure map ε_n is equivalent to $\varepsilon'_n : E_n \to \Omega E_{n+1}$.

Definition 2.6.3. *A spectrum* E *is called* Ω-spectrum *if, for each* n*, the structure map* $\varepsilon'_n : E_n \to \Omega E_{n+1}$ *is a weak equivalence.*

Example 2.6.4.

(i) Given a CW complex X*, take*

$$E_n = \begin{cases} \Sigma^n X & \text{if } n \geq 0, \\ pt & \text{if } n < 0, \end{cases}$$

with structure map $Id : \Sigma(\Sigma^n X) \to \Sigma^{n+1}X$*. This is called the suspension spectrum.*

In particular, for any abelian group G, the Moore spectrum MG is the suspension spectrum of the Moore space $M(G,0)$.

(ii) The Thom spectrum, denoted by MO has its n-th term given by $MO(n) = Th(\gamma_n)$, where γ_n is the universal n-plane bundle over $BO(n)$. The structure map is given by the homotopy equivalence $\Sigma Th(\gamma_n) \simeq Th(\gamma_n \oplus \epsilon^1)$.

(iii) Let G be an abelian group. Then the Eilenberg spectrum HG is given by

$$HG_n = \begin{cases} K(G,n) & \text{if } n \geq 0, \\ pt & \text{if } n < 0, \end{cases}$$

with structure maps $\varepsilon'_n : HG_n \to \Omega HG_{n+1}$ given by the homotopy equivalences $K(G,n) \simeq \Omega K(G,n+1)$.

Homotopy Group of a Spectrum

Let $E = (E_n, \varepsilon_n)$ be a spectrum. Then its k-th homotopy group is defined as follows:

$$\pi_k(E) := \lim_{n\to\infty} \pi_{n+k}(E_n),$$

where the limit is taken over the composition of the homomorphisms

$$\begin{tikzcd}
\pi_{n+k}(E_n) \arrow[rr] \arrow[dr, "\Sigma"'] & & \pi_{n+k+1}(E_{n+1}) \\
& \pi_{n+k+1}(\Sigma E_n). \arrow[ur, "(\varepsilon_n)_*"'] &
\end{tikzcd}$$

Note that if E is an Ω-spectrum, then

$$\pi_{n+k}(E_n) \xrightarrow{(\varepsilon'_n)_*} \pi_{n+k}(\Omega E_{n+1}) \cong \pi_{n+k+1}(E_{n+1})$$

is an isomorphism for $n+k \geq 1$, hence the corresponding direct limit stabilizes. Thus, in this case $\pi_k(E) = \pi_{n+k}(E_n)$ for all $n+k \geq 1$.

Theorem 2.6.5. *[9] A reduced cohomology theory $\widetilde{K}^*$ determines and is determined* *by an Ω-spectrum (E_n, ε'_n). The spaces E_n and the maps are unique up to homotopy equivalence.*

Example 2.6.6. *Let G be an abelian group. The Eilenberg–MacLane spectrum HG represents reduced singular cohomology with coefficients in G; that is, for every based CW complex X and integer $n \geq 0$, there is a natural isomorphism*

$$\widetilde{H}^n(X; G) \cong \{X, HG_n\}.$$

Let $f : E = (E_n, \varepsilon_n) \to F = (F_n, \bar{\varepsilon}_n)$ be a map of spectra; that is, there is a sequence of maps $f_n : E_n \to F_n$ such that the following diagram commutes for each n:

$$\begin{tikzcd}
\Sigma E_n \arrow[r, "\varepsilon_n"] \arrow[d, "\Sigma f_n"'] & E_{n+1} \arrow[d, "f_{n+1}"] \\
\Sigma F_n \arrow[r, "\bar{\varepsilon}_n"'] & F_{n+1}.
\end{tikzcd}$$

Before introducing the notion of homotopy between spectra, we recall the definition of the cylinder spectrum. Let $X = \{X_n, \varepsilon_n\}$ be a spectrum. The *cylinder spectrum* $\mathrm{Cyl}(X)$ is defined levelwise by

$$\mathrm{Cyl}(X)_n := [0,1]_+ \wedge X_n,$$

with structure maps given by the composition

$$S^1 \wedge [0,1]_+ \wedge X_n \xrightarrow{\mathrm{flip} \wedge \mathrm{id}_{X_n}} [0,1]_+ \wedge S^1 \wedge X_n \xrightarrow{\mathrm{id}_{[0,1]_+} \wedge \varepsilon_n} [0,1]_+ \wedge X_{n+1}.$$

Definition 2.6.7. *Let X and Y be spectra. A homotopy between two maps $f, g : X \to Y$ of spectra is a map $H : \mathrm{Cyl}(X) \to Y$ of spectra such that $H \circ i_0 = f$ and $H \circ i_1 = g$, where i_0 and i_1 are the two end poin inclusions of $\mathbb{S}^0$ into $[0,1]_+$. This defines $\{X, Y\}$ for spectra X and Y.*

Definition 2.6.8. *We call a map of spectra $f : X \to Y$ a π_*-isomorphism if the* *induced map* $f_* : \pi_n(X) \to \pi_n(Y)$ *is an isomorphism for all* $n \in \mathbb{Z}$.

Definition 2.6.9. *The* homotopy category of spectra*, denoted* **HoSp***, is the category whose objects are spectra and whose morphisms are homotopy classes of maps of spectra.*

Definition 2.6.10. *The* stable homotopy category*, denoted* $\mathscr{SHC}$*, is defined to be the localization of the homotopy category of spectra* **HoSp** *with respect to the class of* π_**-isomorphisms. That is,*

$$\mathscr{SHC} := \mathbf{HoSp}[\pi_*^{-1}],$$

where all π_**-isomorphisms are formally inverted. The objects of* $\mathscr{SHC}$ *are spectra, and A morphism from* X *to* Y *in* **SH** *is represented by a finite zigzag*

$$X \xleftarrow{w_1} X_1 \to X_2 \xleftarrow{w_2} \cdots \to Y,$$

where each map w_i *is a* π_**-isomorphism. Two such zigzags define the same morphism if they agree in the localization.*

The cofiber of f, denoted by $C_f = F \cup_f Cone(E)$, is the spectrum with n-th space $(C_f)_n = F_n \cup_{f_n} Cone(E_n)$ and structure maps induced from those of E and F.

Theorem 2.6.11. *Let* $E \xrightarrow{f} F \xhookrightarrow{i} C_f$ *be a cofiber sequence in the stable homotopy category. Then for any spectrum* Z*, the following hold:*

(i) The sequence

$$[C_f, Z] \xrightarrow{i^*} [F, Z] \xrightarrow{f^*} [E, Z]$$

is exact.

(ii) The sequence

$$[Z, E] \xrightarrow{f_*} [Z, F] \xrightarrow{i_*} [Z, C_f]$$

is exact.

Lemma 2.6.12. *Let $f : X \to Y$ be a map of spectra. Then there exists a natural map*

$$Ff \longrightarrow \Omega C_f$$

which is a π_*-isomorphism, where Ff and C_f denote the homotopy fibre and homotopy cofiber of f, respectively.

2.6.3 Stable Homotopy groups of Moore Spectrum and its Mapping Cone

The following lemma is a consequence of the Blakers–Massey theorem, which yields a truncated long exact sequence in homotopy associated with a cofiber sequence.

Lemma 2.6.13. *[49, 7] Let $A \xrightarrow{i} X \xrightarrow{q} Q$ be a cofiber sequence in which A is k-connected and Q is t-connected, for integers $k, t \geq 1$. Then, for each integer i with $2 < i \leq k+t$, there exists a connecting homomorphism $\partial_i : \pi_i(Q) \to \pi_{i-1}(A)$ such that the sequence*

$$\pi_{k+t}(A) \xrightarrow{i_*} \pi_{k+t}(X) \xrightarrow{q_*} \pi_{k+t}(Q) \xrightarrow{\partial_{k+t}} \pi_{k+t-1}(A) \to \cdots$$
$$\cdots \to \pi_2(A) \xrightarrow{i_*} \pi_2(X) \xrightarrow{q_*} \pi_2(Q) \to 0$$

is exact.

A related formulation of the long exact sequence applies when a space is formed by attaching a cell via a map. In this setting, consider the cofiber sequence

$$\mathbb{S}^n \xrightarrow{h} Y \xhookrightarrow{i} Y \cup_h \mathbb{D}^{n+1} \xrightarrow{q} \mathbb{S}^{n+1} \xrightarrow{\Sigma h} \Sigma Y \to \cdots .$$

This gives rise to the following:

Lemma 2.6.14. *[53, Lemma A] Let Y be a k-connected topological space, and suppose $n \geq k \geq 2$. Then the associated long exact sequences in homotopy groups are:*

(a) For $i \leq n+k-1$, the sequence

$$\cdots \to \pi_i(\mathbb{S}^n) \xrightarrow{h_*} \pi_i(Y) \xrightarrow{i_*} \pi_i(Y \cup_h \mathbb{D}^{n+1}) \xrightarrow{\partial_i} \pi_{i-1}(\mathbb{S}^n) \xrightarrow{h_*} \pi_{i-1}(Y) \to \cdots$$

is exact.

(b) For $i \leq 2k-1$, the cofiber sequence induces the exact sequence

$$\cdots \to \pi_i(\mathbb{S}^n) \xrightarrow{h_*} \pi_i(Y) \xrightarrow{i_*} \pi_i(Y \cup_h \mathbb{D}^{n+1}) \xrightarrow{q_*} \pi_i(\mathbb{S}^{n+1}) \xrightarrow{(\Sigma h)_*} \pi_i(\Sigma Y) \to \cdots .$$

The mapping cone of $b : \mathbb{S}^0 \to \mathbb{S}^0$ in spectra is denoted by $M(\mathbb{Z}/b)$. Hence, there is a cofiber sequence of spectra

$$\cdots \to \mathbb{S}^{3+k} \xrightarrow{b} \mathbb{S}^{3+k} \xhookrightarrow{\Sigma^{3+k} i_b} \Sigma^{3+k} M(\mathbb{Z}/b) \xrightarrow{q_{4+k}} \mathbb{S}^{4+k} \to \cdots . \tag{2.12}$$

Before stating the following calculations, we adopt the following conventions. For each prime power p^r, let $i_{p^r} : \mathbb{S}^0 \hookrightarrow M(\mathbb{Z}/p^r)$ denote the inclusion of the bottom cell. For a stable homotopy element $x \in \{\eta, \eta^2, \alpha_1, \nu\}$, the symbol $\widetilde{x}$ denotes a lift of x to the Moore space $M(\mathbb{Z}/p^r)$ whose composition with the canonical map $M(\mathbb{Z}/p^r) \to \mathbb{S}^1$ represents x.

Using the cofiber sequence (2.12) in Lemma 2.6.14, we obtain the following computation of the homotopy groups of the Moore spectrum $M(\mathbb{Z}/b)$.

$$\pi_1(M(\mathbb{Z}/p^r)) = \begin{cases} \mathbb{Z}/2\{i_{2^r} \circ \eta\} & \text{if } p = 2, \\ 0 & \text{if } p \text{ is odd prime.} \end{cases}$$

$$\pi_2(M(\mathbb{Z}/p^r)) = \begin{cases} \mathbb{Z}/4\{\widetilde{\eta}_2\} & \text{if } p = 2 \text{ and } r = 1, \\ \mathbb{Z}/2\{\widetilde{\eta}_{2^r}\} \oplus \mathbb{Z}/2\{i_{2^r} \circ \eta^2\} & \text{if } p = 2 \text{ and } r \geq 2, \\ 0 & p \geq 3 \text{ is prime.} \end{cases}$$

$$\pi_3(M(\mathbb{Z}/p^r)) = \begin{cases} 0 & \text{if } p > 3 \text{ is odd prime,} \\ \mathbb{Z}/3\{i_{3^r} \circ \alpha_1\} & \text{if } p = 3, \\ \mathbb{Z}/2\{i_2 \circ 4\nu\} \oplus \mathbb{Z}/2\{\widetilde{\eta_2^2}\} & \text{if } p = 2 \text{ and } r = 1, \\ \mathbb{Z}/4\{i_{2^2} \circ 2\nu\} \oplus \mathbb{Z}/2\{\widetilde{\eta_{2^2}^2}\} & \text{if } p = 2 \text{ and } r = 2, \\ \mathbb{Z}/2\{i_{2^r} \circ \nu\} \oplus \mathbb{Z}/2\{\widetilde{\eta_{2^r}^2}\} & \text{if } p = 2 \text{ and } r \geq 3. \end{cases}$$

$$\pi_4(M(\mathbb{Z}/p^r)) = \begin{cases} 0 & \text{if } p > 3 \text{ is odd prime,} \\ \mathbb{Z}/3\{\widetilde{\alpha_1}\} & \text{if } p = 3, \\ \mathbb{Z}/2^r\{2^{3-r}\widetilde{\nu}\} & \text{if } p = 1 \text{ and } 1 \leq r \leq 3, \\ \mathbb{Z}/8\{\widetilde{\nu}\} & \text{if } p = 2 \text{ and } r \geq 4. \end{cases}$$

$$\pi_5(M(\mathbb{Z}/b)) = 0.$$

Let $C_{\Sigma^2 i_{2^r} \circ \eta}$ be the cofiber of the composition $\mathbb{S}^3 \xrightarrow{\eta} \mathbb{S}^2 \xhookrightarrow{\Sigma^2 i_{2^r}} \Sigma^2 M(\mathbb{Z}/2^r)$. Then

$$\pi_5^s(C_{\Sigma^2 i_{2^r} \circ \eta}) = \begin{cases} \mathbb{Z}/2\{i^r \circ \widetilde{\eta_2^2}\} & \text{if } r = 1, \\ \mathbb{Z}/2\{i^r \circ \Sigma^2 i_{2^r} \circ 4\nu\} \oplus \mathbb{Z}/2\{i^r \circ \widetilde{\eta_{2^r}^2}\} & \text{if } r = 2, \\ \mathbb{Z}/4\{i^r \circ \Sigma^2 i_{2^r} \circ 2\nu\} \oplus \mathbb{Z}/2\{i^r \circ \widetilde{\eta_{2^r}^2}\} & \text{if } r \geq 3, \end{cases}$$

where $i^r : \Sigma^2 M(\mathbb{Z}/2^r) \hookrightarrow C_{\Sigma^2 i_{2^r} \circ \eta}$ is the inclusion.

Additionally, we need to understand the homotopy class of the map between $\Sigma^n M(\mathbb{Z}/p^r)$ and $\Sigma^n M(\mathbb{Z}/p^t)$, for some $r, t \in \mathbb{N}$.

Let's consider $\chi_t^r : \mathbb{Z}/p^r \to \mathbb{Z}/p^t$ as the generator of $Hom(\mathbb{Z}/p^r, \mathbb{Z}/p^t)$. Here, $\chi_t^r(1) = 1$, if $r \geq t$ and $\chi_t^r(1) = p^{t-r}$, if $t \geq r$. In this case, there exists a unique element

$$B(\chi_t^r) : M(\mathbb{Z}/p^r) \to M(\mathbb{Z}/p^t) \tag{2.13}$$

such that the map induced by $\Sigma^n B(\chi^r_t)$ at the n-th homology level is χ^r_t for each $n \geq 3$ [15]. Moreover, $(\Sigma^n B(\chi^r_t))_* : \pi_{n+2}(\Sigma^n M(\mathbb{Z}/p^r)) \to \pi_{n+2}(\Sigma^n M(\mathbb{Z}/p^t))$ is given by

$$(\Sigma^n B(\chi^r_t))_*(\widetilde{\eta}_{2^r}) = \widetilde{\eta}_{2^t} \text{ and } (\Sigma^n B(\chi^r_r))_*(\widetilde{\eta}_{2^r}) = \widetilde{\eta}_{2^r}. \tag{2.14}$$

Also, $B(\chi^r_t)$ satisfies the following relations (cf. [15]):

(2.15)

$$\Sigma^n B(\chi^r_t) \circ \Sigma^n i_{p^r} = \begin{cases} \Sigma^n i_{p^t}, & r \geq t, \\ 2^{t-r}\Sigma^n i_{p^t}, & r \leq t, \end{cases} \qquad q_{n+1} \circ \Sigma^n B(\chi^r_t) = \begin{cases} 2^{r-t} q_{n+1}, & r \geq t, \\ q_{n+1}, & r \leq t, \end{cases}$$

where $\Sigma^n i_{p^s} : \mathbb{S}^n \hookrightarrow \Sigma^n M(\mathbb{Z}/p^s)$ and $q_{n+1} : \Sigma^n M(\mathbb{Z}/p^s) \to \mathbb{S}^{n+1}$ are the inclusion and pinching map for $\Sigma^n M(\mathbb{Z}/p^s)$ and any integer $s \geq 1$ respectively.

2.6.4 Detecting Non-Trivial Maps Using Cohomology Operations

We study how cohomology operations, including Steenrod squares, secondary operations, and higher Bocksteins, detect non-triviality of maps between spheres and their mapping cones.

Let $\alpha \in \pi_{n+i-1}(\mathbb{S}^n)$. Consider the mapping cone

$$Cone(\alpha) = \mathbb{S}^n \cup_\alpha \mathbb{D}^{n+i},$$

which is uniquely determined by α up to homotopy. We define a group homomorphism

$$H_2 \colon \pi_{n+i-1}(\mathbb{S}^n) \longrightarrow \mathbb{Z}/2$$

by declaring that $H_2(\alpha) \not\equiv 0 \pmod 2$ if and only if the Steenrod square

$$Sq^i \colon H^n(Cone(\alpha); \mathbb{Z}/2) \longrightarrow H^{n+i}(Cone(\alpha); \mathbb{Z}/2)$$

is an isomorphism. It follows from [1, Theorem 1.1.1] that H_2 is trivial unless $i = 1, 2, 4$, or 8.

The following proposition describes the image of the homomorphism H_2 in terms of generators of the homotopy groups of spheres.

Proposition 2.6.15 ([125]). *Let $i \in \{2, 4, 8\}$.*

1. *The homomorphism*
$$H_2 \colon \pi_{n+i-1}(\mathbb{S}^n) \longrightarrow \mathbb{Z}/2$$
is surjective if and only if $n > i - 1$.

2. *Suppose $n > i$. Then $H_2(\alpha) \not\equiv 0 \pmod 2$ if and only if*
$$\alpha \equiv \eta_n,\ \nu_n,\ \textit{or}\ \sigma_n \pmod{2\,\pi_{n+i-1}(\mathbb{S}^n)},$$
corresponding to $i = 2, 4, 8$, respectively.

3. *Suppose $n = i$. Then $H_2(\alpha) \not\equiv 0 \pmod 2$ if and only if*
$$\alpha \equiv \eta_2,\ \nu_4,\ \textit{or}\ \sigma_8 \pmod{2\,\pi_{2i-1}(\mathbb{S}^i) + \pi_{2i-1}(\mathbb{S}^i)},$$
corresponding to $i = 2, 4, 8$, respectively.

As a consequence of the above proposition, we obtain the following result.

Corollary 2.6.16. *The stable map $\eta_n : \mathbb{S}^{n+1} \to \mathbb{S}^n$, representing the generator $\eta \in \pi_1^s$, is detected by the Steenrod square Sq^2.*

Corollary 2.6.17. *Let $Cone(\widetilde{\eta}_{2^r}) = \Sigma^n M(\mathbb{Z}/2^r) \cup_{\widetilde{\eta}_{2^r}} \mathbb{D}^{n+3}$. Then the Steenrod Square $Sq^2 : H^{n+1}(Cone(\widetilde{\eta}_{2^r}); \mathbb{Z}/2) \to H^{n+3}(Cone(\widetilde{\eta}_{2^r}); \mathbb{Z}/2)$ is an isomorphism.*

Next, we recall the definition of the secondary cohomology operation associated to steenrod square.

Definition 2.6.18. *Let X be a topological space, and let $x \in H^n(X;\mathbb{Z}/2)$ satisfy* $\mathrm{Sq}^i(x) = 0$. *A* secondary cohomology operation *associated to* Sq^i *is a class*

$$\Phi(x) \in H^{n+i-1}(X;\mathbb{Z}/2)/\mathrm{Indet},$$

where Indet *denotes the indeterminacy generated by other Steenrod operations. It measures the obstruction to lifting x through* Sq^i.

Lemma 2.6.19. *The map $\eta_n^2 = \eta_n \circ \eta_{n+1} : \mathbb{S}^{n+2} \to \mathbb{S}^n$ is null homotopic if and only if the secondary cohomology operation*

$$\Theta : Ker(Sq^2) \cap Ker(Sq^3)(\subset H^n(Cone(\eta^2);\mathbb{Z}/2)) \to H^{n+3}(Cone(\eta^2);\mathbb{Z}/2)$$

associated to the Adem relation $Sq^2Sq^2 = Sq^3Sq^1$ in the mod 2 Steenrod algebra is trivial.

Let X be a CW complex. For each integer $r \geq 1$, there is a higher Bockstein operation

$$\beta_r : H^n(X;\mathbb{Z}/2) \to H^{n+1}(X;\mathbb{Z}/2),$$

defined inductively. The first, β_1, is the classical Bockstein homomorphism arising from the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

For $r \geq 2$, the operation β_r is defined on the intersection of the kernels of β_i for $i < r$ and takes values in the quotient by the sum of their images.

Lemma 2.6.20. *[89] Let X be a CW complex.*

(i) For each $r \geq 1$, there is exactly one non-trivial higher Bockstein

$$\beta_r : H^n(\Sigma^n M(\mathbb{Z}/2^r);\mathbb{Z}/2) \to H^{n+1}(\Sigma^n M(\mathbb{Z}/2^r);\mathbb{Z}/2).$$

(ii) *If* $x \in H^*(X; \mathbb{Z}/2)$ *lies in the image of the reduction map from the free part of* $H^*(X; \mathbb{Z})$*, then* $\beta_r(x) = 0$ *for all* $r \geq 1$*.*

(iii) *Suppose* $x \in H^{n+1}(X; \mathbb{Z}/2)$ *generates a direct summand isomorphic to* $\mathbb{Z}/2^r$*. Then there exist elements* $\tilde{x} \in H^n(X; \mathbb{Z}/2)$ *and* $\tilde{\tilde{x}} \in H^{n+1}(X; \mathbb{Z}/2)$ *such that*

$$\beta_r(\tilde{x}) = \tilde{\tilde{x}}, \quad \beta_i(\tilde{x}) = \beta_i(\tilde{\tilde{x}}) = 0 \text{ for all } i < r.$$

2.6.5 Some Facts about Self-Homotopy Equivalences of Product Spaces

Let X be a CW complex and denote by $\mathcal{E}(X)$ the group of homotopy classes of self-homotopy equivalences of X. Given another CW complex Y, the group $\mathcal{E}(X \times Y)$ consists of homotopy classes of self-homotopy equivalences of the product space $X \times Y$. Within this group, we consider the following subgroups: $\mathcal{E}_X(X \times Y)$ consisting of those $f \in \mathcal{E}(X \times Y)$ that fix X. In other words, these maps take the form $f = (p_X, p_Y \circ f)$, where p_X and p_Y are the projection maps from $X \times Y$ to X and Y respectively. The subgroup $\mathcal{E}_Y(X \times Y)$ is defined analogously.

We recall that a self-homotopy equivalence $f : X \times Y \to X \times Y$ is *diagonalizable* if the compositions $p_X \circ f \circ i_X : X \to X$ and $p_Y \circ f \circ i_Y : Y \to Y$ are self-homotopy equivalences of X and Y, respectively. where, $i_X : X \hookrightarrow X \times Y$ and $i_Y : Y \hookrightarrow X \times Y$ are the canonical inclusions.

The following result provides a criterion under which all self-homotopy equivalences of a product space $X \times Y$ are diagonalizable.

Proposition 2.6.21. *[99, Proposition 2.1] Let* X *and* Y *be connected CW complexes. Suppose that for every integer* $n \geq 1$ *and for every pair of maps* $f : X \to Y$ *and* $g : Y \to X$*, at least one of the induced homomorphisms*

$$f_* : \pi_n(X) \to \pi_n(Y) \quad \text{or} \quad g_* : \pi_n(Y) \to \pi_n(X)$$

is trivial. Then every element of $\mathcal{E}(X \times Y)$ is diagonalizable.

Under the diagonalizability assumption, the following result from [99] describes the structure of $\mathcal{E}(X \times Y)$ in terms of the subgroups $\mathcal{E}_X(X \times Y)$ and $\mathcal{E}_Y(X \times Y)$.

Proposition 2.6.22. *[99, Theorem 2.5, Proposition 2.3(d)] Let X and Y be CW complexes.*

(a) If every element of $\mathcal{E}(X \times Y)$ is diagonalizable, then any $f \in \mathcal{E}(X \times Y)$ can be written as a composite $f_1 \circ f_2$, where $f_1 \in \mathcal{E}_X(X \times Y)$ and $f_2 \in \mathcal{E}_Y(X \times Y)$.

(b) If X is connected, then $\mathcal{E}_X(X \times Y)$ fits into a short exact sequence

$$0 \to [X, E_1(Y)] \to \mathcal{E}_X(X \times Y) \to \mathcal{E}(Y) \to 0,$$

where $E_1(Y)$ denotes the identity component of the space of self-maps of Y.

Remark 2.6.23.

(a) $\mathcal{E}(\mathbb{S}^n) = \mathbb{Z}/2$, generated by the reflection map.

(b) Any self-homotopy equivalence of $\mathbb{C}P^n (n \geq 3)$ is homotopic to the identity or the conjugation [121, Theorem 8(i)].

The above remark shows that every self-homotopy equivalence of $\mathbb{S}^n$ or $\mathbb{C}P^n$ is homotopic to a diffeomorphism.

Chapter 3

On Stable Homotopy Types and Attaching Maps

Throughout this thesis, all manifolds under consideration are assumed to be closed, connected, oriented, and smooth. In this chapter, we determine the stable homotopy types of the following: closed 4-manifolds, simply connected 5-manifolds, and 3-connected 8-manifolds. We also analyze the top cell attaching map of simply connected 6-manifolds, highlighting how certain cohomology operations constrain its stable homotopy class. These results are instrumental in analyzing the concordance inertia group and the concordance structure set of the product of these manifolds with a sphere.

3.1 Stable Homotopy Type of a 4-dimensional Manifold

Let M be a closed, smooth, and oriented 4-manifold. By applying Poincaré duality and the Universal Coefficient Theorem, we obtain

$$H_l(M;\mathbb{Z}) = \begin{cases} \oplus\mathbb{Z}^m \oplus \bigoplus_{j=1}^{n} \mathbb{Z}/b_j & \text{for } l=1, \\ \oplus\mathbb{Z}^d \oplus \bigoplus_{j=1}^{n} \mathbb{Z}/b_j & \text{for } l=2, \\ \oplus\mathbb{Z}^m & \text{for } l=3, \\ \mathbb{Z} & \text{for } l=0 \text{ or } 4, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where $m, d \geq 0$ are non-negetive integers and each b_j is a prime power. We make certain choices in the above decomposition. Suppose that $u_1, \cdots, u_m$ be a basis of the free part of $H_1(M;\mathbb{Z})$, and $v_1, \cdots, v_d$ be a basis of the free part of $H_2(M;\mathbb{Z})$. Choose a basis $\mu_1, \cdots, \mu_m$ of $H^1(M;\mathbb{Z}) \cong \mathbb{Z}^m$ such that $\langle \mu_i, u_j \rangle = \delta_{ij}$. The basis of $H_3(M;\mathbb{Z})$ is then fixed as $[M] \cap \mu_1, \cdots, [M] \cap \mu_m$. Let a_i denote a generator of the $\mathbb{Z}/b_i$ factor of $H_1(M;\mathbb{Z})$. These come equipped with maps $\hat{q}_i : M \to K(\mathbb{Z}/b_i, 1)$ such that a_i maps to 1 in $H_1(K(\mathbb{Z}/b_i, 1);\mathbb{Z})$, and the u_j map to 0. Define $\alpha_i \in H^2(M)$ as the image

$$1 \in \mathbb{Z}/b_i \cong H^2(K(\mathbb{Z}/b_i, 1);\mathbb{Z}) \xrightarrow{\hat{q_i}^*} H^2(M;\mathbb{Z}).$$

Define $k_i = [M] \cap \alpha_i$. With $\mathbb{Z}/2$-coefficients, define ν_i to be a dual basis of v_i which evaluates to 0 on the torsion classes. If $2 \mid b_i$, let $c_i \in H_2(M;\mathbb{Z}/2)$ denote a choice of a class such that $\hat{q}_{i*}(c_i)$ is a generator of $H_2(K(\mathbb{Z}/b_i, 1);\mathbb{Z}/2)$. In this case, also define $\chi_i \in H^2(M;\mathbb{Z}/2)$ by the equation

$$\langle \chi_i, k_j \rangle = \delta_{ij}, \ \langle \chi_i, v_j \rangle = 0, \ \langle \chi_i, c_j \rangle = 0.$$

We thus have for some $r \leq n$,

$$H^2(M;\mathbb{Z}/2) \cong \mathbb{Z}/2\{\nu_1, \cdots, \nu_d, \alpha_1, \cdots, \alpha_r, \chi_1, \cdots, \chi_r\}.$$

In terms of the cup product $H^2(M;\mathbb{Z}/2)\otimes H^2(M;\mathbb{Z}/2)\to H^4(M;\mathbb{Z}/2)\cong\mathbb{Z}/2$, we have

$$\alpha_i\nu_j=0,\ \chi_i\alpha_j=\delta_{ij}.$$

We use a minimal cell structure on the stable homotopy type of M, as described in [47, §4.C]. Under this cell structure, the 2-skeleton is clearly stably homotopy equivalent to

$$M^{(2)}\simeq\bigvee_{i=1}^{m}\mathbb{S}^1\vee\bigvee_{j=1}^{n}\left(\Sigma M(\mathbb{Z}/b_j)\vee\mathbb{S}^2\right)\vee\bigvee_{l=1}^{d}\mathbb{S}^2.$$

We now examine the attachment of the 3-cells, which stably corresponds to a sum of maps onto the wedge summands of the 2-skeleton. Note that $Sq^2:H^1(M;\mathbb{Z}/2)\to H^3(M;\mathbb{Z}/2)$ is trivial. This implies that the attaching maps do not involve any η-components. Observe that

$$\pi_2(\Sigma M(\mathbb{Z}/b))=\begin{cases}\mathbb{Z}/2\{\Sigma i_b\circ\eta\} & \text{if } b \text{ is even,}\\ 0 & \text{otherwise,}\end{cases}$$

and that $H^2(M;\mathbb{Z})\cong H^2(M^{(3)};\mathbb{Z})$. Hence, the 3-skeleton has the following stable homotopy type:

$$M^{(3)}\simeq\bigvee_{i=1}^{m}\left(\mathbb{S}^1\vee\mathbb{S}^3\right)\vee\bigvee_{j=1}^{n}\left(\Sigma M(\mathbb{Z}/b_j)\vee\Sigma^2M(\mathbb{Z}/b_j)\right)\vee\bigvee_{l=1}^{d}\mathbb{S}^2.$$

Finally, the stable homotopy type of M is determined by the attaching map $\phi_3:\mathbb{S}^3\to M^{(3)}$ for the 4-cell of M. The formula for the homology implies that the attaching map becomes trivial after quotienting out the 2-skeleton. Thus, we consider the factorization of ϕ_3 as

$$\mathbb{S}^3\xrightarrow{\phi}M^{(2)}\to M^{(3)}.$$

The possibilities for the map ϕ are computed via two observations. As both Sq^2 and Sq^3 are identically the 0-operation on a 1-dimensional class, the secondary cohomology operation $\Theta : H^1(M;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2)$ based on the relation $Sq^3Sq^1 + Sq^2Sq^2 = 0$ is 0. This implies that ϕ maps trivially onto the factors given by $\mathbb{S}^1$ and the 0-cell of $\Sigma M(\mathbb{Z}/b_j)$. Therefore, the nontrivial factors of the map ϕ may only be

(1) As a nonzero multiple of η onto some of the d-copies of $\mathbb{S}^2$.

(2) As a nonzero multiple of $\Sigma^2 i_{b_j} \circ \eta$ onto one of the copies of $\Sigma^2 M(\mathbb{Z}/b_j)$.

(3) As a nonzero multiple of $\widetilde{\eta}_{b_j}$ onto one of the copies of $\Sigma M(\mathbb{Z}/b_j)$.

If any of the cases above occur, the operation $Sq^2 : H^2(M;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2)$ is nontrivial, which means by Wu's formula that the manifold is not spin. We also observe that if $2 \mid b_j$ but $4 \nmid b_j$, the last case cannot arise for a 4-manifold, as then the composition

$$H^1(M;\mathbb{Z}/2) \overset{Sq^1}{\to} H^2(M;\mathbb{Z}/2) \overset{Sq^2}{\to} H^4(M;\mathbb{Z}/2)$$

would be nontrivial. However, this composition maps a class $\alpha \mapsto \alpha^4$, and we then have $Sq^1(\alpha^3) = \alpha^4$, which contradicts $w_1(M) = 0$ by Wu's formula. In fact, we will now observe that 3) cannot occur at all and if 2) occurs, 1) will also occur. The latter case is clear as the cohomology classes χ_j which evaluate nontrivially on the factors corresponding to the $\Sigma^2 M(\mathbb{Z}/b_j)$ factors for $2 \mid b_j$ are not integral, and the fact that $w_2(M)$ is the (mod 2) reduction of an integral class [111, Page 170] and [123].

Suppose that the composite

$$\mathbb{S}^3 \to M^{(3)} \overset{q_j}{\to} \Sigma M(\mathbb{Z}/b_j)$$

equals $\widetilde{\eta}_{b_j}$ for some j such that $2 \mid b_j$. Viewing unstably, the map q_j yields $\hat{q}_j : M^{(3)} \to K(\mathbb{Z}/b_j, 1)$ which classifies $1 \in \mathrm{Hom}(\mathbb{Z}/b_j, \mathbb{Z}/b_j) \subset \mathrm{Hom}(H_1(M), \mathbb{Z}/b_j)$. As $Sq^2(\alpha) = \alpha^2$ for a degree 2 class α, the assumption implies that the class $a_j \in H^2(M;\mathbb{Z}/2)$ corresponding to the generator of $\mathrm{Ext}(\mathbb{Z}/b_j, \mathbb{Z}/2)\Big(\subset \mathrm{Ext}(H_1(M), \mathbb{Z}/2) \subset H^2(M;\mathbb{Z}/2)\Big)$ satisfies a_j^2 is the generator of $H^4(M;\mathbb{Z}/2) \cong \mathbb{Z}/2$. The entire information fits into the following diagram:

$$\begin{tikzcd}
H^2(K(\mathbb{Z}/b_j,1);\mathbb{Z}/2) \arrow[r, "\hat{q}^*"] \arrow[d, "Sq^2"] & H^2(M;\mathbb{Z}/2) \arrow[d, "Sq^2"] & H^2(M;\mathbb{Z}) \arrow[l] \arrow[d, "(\ )^2"] \\
H^4(K(\mathbb{Z}/b_j,1);\mathbb{Z}/2) \arrow[r, "\hat{q}^*"] & H^4(M;\mathbb{Z}/2) & H^4(M;\mathbb{Z}). \arrow[l]
\end{tikzcd}$$

As $\mathrm{Ext}(\mathbb{Z}/b_j,\mathbb{Z}) \to \mathrm{Ext}(\mathbb{Z}/b_j,\mathbb{Z}/2)$ is surjective, the class a_j lifts to a b_j-torsion class in $H^2(M;\mathbb{Z})$. Therefore, it's square is 0, and thus $Sq^2(a_j) = 0$. This rules out the option 3) for an orientable non-spin manifold M.

We conclude that for a non-spin 4-manifold M, ϕ may map by η to more than one copy of $\mathbb{S}^2$, as well as the maps of the type (2). However, we may change the wedge sum decomposition of $M^{(3)}$, so that ϕ maps to exactly one copy of $\mathbb{S}^2$. This is done by considering the homotopy class $\mathbb{S}^2 \to M^{(3)}$ as the sum of the nontrivial $\mathbb{S}^2$-factors and the inclusions coming from the $M(\mathbb{Z}/b_j)$-factors of type (2). This means that after such a choice ϕ maps by η to exactly one copy of $\mathbb{S}^2$.

Therefore, we obtain exactly two cases (noting that $Cone(\eta : \mathbb{S}^3 \to \mathbb{S}^2) \simeq \mathbb{C}P^2$).

(a) If the manifold M is spin, then the stable homotopy type of M is

$$M \simeq \mathbb{S}^4 \vee \bigvee_{i=1}^{m} \left(\mathbb{S}^1 \vee \mathbb{S}^3\right) \vee \bigvee_{j=1}^{n} (\Sigma M(\mathbb{Z}/b_j) \vee \Sigma^2 M(\mathbb{Z}/b_j)) \vee \bigvee_{l=1}^{d} \mathbb{S}^2. \tag{3.2}$$

(b) If the manifold M is non-spin, then its stable homotopy type is

$$M \simeq \mathbb{C}P^2 \vee \bigvee_{i=1}^{m} \left(\mathbb{S}^1 \vee \mathbb{S}^3\right) \vee \bigvee_{j=1}^{n} (\Sigma M(\mathbb{Z}/b_j) \vee \Sigma^2 M(\mathbb{Z}/b_j)) \vee \bigvee_{l=1}^{d-1} \mathbb{S}^2. \tag{3.3}$$

3.2 Stable Homotopy Type of a Simply Connected 5-Manifold

Let M be a simply connected, closed 5-dimensional smooth manifold. Then the homology groups of M follow from Poincaré duality and the Universal Coefficient Theorem, and are given by:

$$H_l(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } \ l=0 \text{ or } 5, \\ \mathbb{Z}^d \oplus \bigoplus_{j=1}^{t} \mathbb{Z}/p_j^{r_j} & \text{for } \ l=2, \\ \mathbb{Z}^d & \text{for } \ l=3, \\ 0 & \text{otherwise}, \end{cases} \tag{3.4}$$

where d and t are non-negative integers, r_j is a non-negative integer for each $1 \leq j \leq t$, and each p_j is a prime number.

We now consider a minimal cell structure on M, as described in [47, Proposition 4C.1], under which the stable homotopy type of the 3-skeleton $M^{(3)}$ is given by

$$M^{(3)} \simeq \bigvee_{i=1}^{d} (\mathbb{S}^2 \vee \mathbb{S}^3) \vee \bigvee_{j=1}^{t} \Sigma^2 M(\mathbb{Z}/p_j^{r_j}). \tag{3.5}$$

The manifold M is then constructed from $M^{(3)}$ by attaching a 5-cell along a map $g : \mathbb{S}^4 \to M^{(3)}$, yielding a cofiber sequence of spectra:

$$\mathbb{S}^4 \xrightarrow{g} M^{(3)} \overset{\widehat{i}}{\hookrightarrow} M \xrightarrow{f_M} \mathbb{S}^5 \to \cdots . \tag{3.6}$$

We analyze the possibilities for the attaching map g by distinguishing between the spin and non-spin cases. As in the 4-dimensional case, we prove that the stable attaching map is trivial if M is spin.

M is a spin manifold

From the classification results of Smale [117] and Barden [8], we note that M is completely classified as a connected sum of some atomic pieces. These building blocks include $\mathbb{S}^2 \times \mathbb{S}^3$, $\mathbb{S}^5$, and M_k with $H_2(M_k;\mathbb{Z}) = \mathbb{Z}/k \oplus \mathbb{Z}/k$ for $k \neq 1$ and ∞ [117, Theorem A], [8, Page 373]. We have to show that η^2 does not appear in the attaching map in each case, and this will imply the proposition. This is clear for $\mathbb{S}^2 \times \mathbb{S}^3$ and $\mathbb{S}^5$.

We now prove this for M_k from an explicit cell complex construction of it following [8, Page-373].

Let N be the manifold $(\mathbb{S}^2 \times \mathbb{S}^2)\#(\mathbb{S}^2 \times \mathbb{S}^2)$. Write $N = \partial D_i$ for $i = 1, 2$ with each

$$D_i \cong \mathbb{D}^3 \times \mathbb{S}^2 \setminus (\mathbb{D}^2 \times \mathbb{D}^2) \cup_{\partial\mathbb{D}^4 \times \mathbb{D}^1} \mathbb{D}^3 \times \mathbb{S}^2 \setminus (\mathbb{D}^2 \times \mathbb{D}^2).$$

Now consider $\phi : N \to N$ which is described by the matrix A_k on the 2-skeleton.

$$A_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ -k & 0 & 0 & 1 \end{bmatrix}$$

The manifold M_k is defined as $D_1 \cup_{\phi : N \to N} D_2$. Observe that $\mathbb{D}^3 \times \mathbb{S}^2$ can be constructed from $\mathbb{S}^2 \times \mathbb{S}^2$ by first attaching a 3-cell along the inclusion of the first factor. This yields the quotient space $\mathbb{S}^2 \times \mathbb{S}^2/(\mathbb{S}^2 \times *) \simeq \mathbb{S}^2 \vee \mathbb{S}^4$, to which a 5-cell is subsequently attached along the copy of $\mathbb{S}^4$. Let u_i, $1 \leq i \leq 4$, be the four 2-spheres in N. Analogously, each is built from N by attaching 3-cells along u_1 and u_3, giving $Cone(\mathbb{S}^2 \vee \mathbb{S}^2 \overset{u_1 \vee u_3}{\to} N) \simeq \mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^4$, followed by a 5-cell attachment along the copy of $\mathbb{S}^4$. Finally putting all together, we have that M_k is constructed out of N by attaching 3-cells along u_1, u_3, $u_1 - ku_4$, $ku_2 + u_3$, followed by two 5-cells along the copies of $\mathbb{S}^4$. It follows that the stable homotopy type of M_k is

$$\Sigma^2 M(\mathbb{Z}/k) \vee \Sigma^2 M(\mathbb{Z}/k) \vee \mathbb{S}^5.$$

Hence, if M is a spin manifold, then M is stably homotopy equivalent to

$$M \simeq \mathbb{S}^5 \vee \bigvee_{i=1}^{d} \left(\mathbb{S}^2 \vee \mathbb{S}^3\right) \vee \bigvee_{j=1}^{t} \Sigma^2 M(\mathbb{Z}/p_j^{r_j}).$$

M is a non-spin manifold

In this case, the second Stiefel–Whitney class $\omega_2(M)$ is nontrivial. By Wu's formula, this implies that η appears in the stable attaching map of the top cell.

If the stable attaching map

$$\mathbb{S}^4 \to M^{(4)} \simeq \bigvee_{i=1}^{d} \left(\mathbb{S}^2 \vee \mathbb{S}^3\right) \vee \bigvee_{j=1}^{t} \Sigma^2 M(\mathbb{Z}/p_j^{r_j})$$

includes a component mapping to a $\mathbb{S}^2$ summand via η^2, then the homotopy class of the stable attaching map from $\mathbb{S}^3$ can be modified so that this component becomes trivial. $M^{(4)}$ is adjusted by a stable self-homotopy equivalence by adding the following composites

$$\mathbb{S}^3 \xrightarrow{\eta} \mathbb{S}^2, \quad \Sigma^2 M(\mathbb{Z}/2^s) \xrightarrow{\hat{\eta}} \mathbb{S}^2,$$

where the map $\hat{\eta}$ is given as $\Sigma^2 M(\mathbb{Z}/2^s) \to \mathbb{S}^3 \xrightarrow{\eta} \mathbb{S}^2$. On composing with $\eta : \mathbb{S}^4 \to \mathbb{S}^3$ (and respectively, $\widetilde{\eta}_{2^s} : \mathbb{S}^4 \to \Sigma^2 M(\mathbb{Z}/2^s)$), we may adjust the η^2 in the expression to 0.

If the map η hits the $\mathbb{S}^3$-factor as in the 4-dimensional case, it cones off to give a copy of the suspension of $\mathbb{C}P^2$. However η may also map via $\widetilde{\eta}_{2^s}$ for some $s \in \mathbb{N}$ onto the Moore spectrum part.

Thus, we may have two cases both of which may occur as demonstrated in the examples of [8, Page-373]. Thus, if M is not spin, the stable homotopy type is either

$$M \simeq \Sigma\mathbb{C}P^2 \vee \bigvee_{i=1}^{d-1} \left(\mathbb{S}^2 \vee \mathbb{S}^3\right) \vee \mathbb{S}^2 \vee \bigvee_{j=1}^{t} \Sigma^2 M(\mathbb{Z}/p_j^{r_j}),$$

or

$$M \simeq Cone(\widetilde{\eta}_{2^r}) \vee \bigvee_{i=1}^{d} \left(\mathbb{S}^2 \vee \mathbb{S}^3\right) \vee \bigvee_{j=2}^{t} \Sigma^2 M(\mathbb{Z}/p_j^{r_j}),$$

where $p_1^r = 2^r$ is the factor which supported the $\widetilde{\eta}_{2^r}$ map. The value of r denotes the smallest power of 2 present in the Moore spectrum segment.

Theorem 3.2.1. *Let M be a simply connected, closed, oriented, smooth, 5-dimensional manifold with homology as in (3.4).*

(a) *If M is a spin manifold, then we have a stable equivalence*

$$M \simeq \mathbb{S}^5 \vee \bigvee_{i=1}^{d} \left(\mathbb{S}^2 \vee \mathbb{S}^3\right) \vee \bigvee_{j=1}^{t} \Sigma^2 M(\mathbb{Z}/p_j^{r_j}). \tag{3.7}$$

(b) *If M is a non-spin manifold, then its stable homotopy type is given by one of the following two forms:*

$$M \simeq \Sigma\mathbb{C}P^2 \vee \bigvee_{i=1}^{d-1} \left(\mathbb{S}^2 \vee \mathbb{S}^3\right) \vee \mathbb{S}^2 \vee \bigvee_{j=1}^{t} \Sigma^2 M(\mathbb{Z}/p_j^{r_j}), \tag{3.8}$$

or

$$M \simeq Cone(\widetilde{\eta}_{2^r}) \vee \bigvee_{i=1}^{d} \left(\mathbb{S}^2 \vee \mathbb{S}^3\right) \vee \bigvee_{j=2}^{t} \Sigma^2 M(\mathbb{Z}/p_j^{r_j}), \tag{3.9}$$

where r is the least positive integer such that $\mathbb{Z}/2^r \subseteq H_2(M;\mathbb{Z})$.

3.3 The Top Cell Attaching Map of a Simply Connected 6-Manifold

Let M be a simply connected, closed, smooth 6-manifold. By Poincaré duality and the Universal coefficient theorem, the homology of M is given by

$$H_q(M;\mathbb{Z}) = \begin{cases} \mathbb{Z}^m \oplus \bigoplus_{j=1}^{\tilde{s}_1} \mathbb{Z}/3^{r_j} \oplus \bigoplus_{j=\tilde{s}_1+1}^{s_1} \mathbb{Z}/p_j^{r_j} \oplus \bigoplus_{j=1}^{s_2} \mathbb{Z}/2^{r_j}, & \text{if } q = 2; \\ \mathbb{Z}^{2d} \oplus \bigoplus_{j=1}^{\tilde{s}_1} \mathbb{Z}/3^{r_j} \oplus \bigoplus_{j=\tilde{s}_1+1}^{s_1} \mathbb{Z}/p_j^{r_j} \oplus \bigoplus_{j=1}^{s_2} \mathbb{Z}/2^{r_j}, & \text{if } q = 3; \\ \mathbb{Z}^m, & \text{if } q = 4; \\ \mathbb{Z}, & \text{if } q = 0, 6; \\ 0, & \text{otherwise}; \end{cases} \tag{3.10}$$

where m, d, s_2, and $\tilde{s}_1 \leq s_1$ are nonnegative integers, and each $p_j > 3$ is an odd prime with $r_j \in \mathbb{N}\cup\{0\}$. It follows from [133] that $M \cong N\#\left(\#_{l=1}^{d}\mathbb{S}^3\times\mathbb{S}^3\right)$ where N is a simply connected, closed, smooth manifold of dimension 6 satisfying $H_q(N;\mathbb{Z}) = H_q(M;\mathbb{Z})$ for $q \neq 3$, $H_3(N;\mathbb{Z}) = \bigoplus\limits_{t=1}^{\tilde{s}_1} \mathbb{Z}/3^{r_t} \oplus \bigoplus\limits_{t=\tilde{s}_1+1}^{s_1} \mathbb{Z}/p_t^{r_t} \oplus \bigoplus\limits_{t=1}^{s_2} \mathbb{Z}/2^{r_t}$. Moreover, N is unique up to oriented diffeomorphism. By fixing minimal CW-structure on N, it can easily be seen that $N\#\big(\#_{l=1}^{d}\mathbb{S}^3 \times \mathbb{S}^3\big)$ can be obtained by a cofiber sequence

$$\mathbb{S}^5 \xrightarrow{f} N^{(5)} \vee (\bigvee_{l=1}^{2d} \mathbb{S}^3) \hookrightarrow N\#\big(\#_{l=1}^{d}\mathbb{S}^3 \times \mathbb{S}^3\big),$$

where the attaching map $f : \mathbb{S}^5 \to N^{(5)} \vee (\bigvee\limits_{l=1}^{2d} \mathbb{S}^3)$ is homotopic to the composition $(f' \vee \omega) \circ c : \mathbb{S}^5 \to N^{(5)} \vee (\bigvee\limits_{l=1}^{2d} \mathbb{S}^3)$. Here, the map $c : \mathbb{S}^5 \to \mathbb{S}^5 \vee \mathbb{S}^5$ is the suspension comultiplication, $\omega : \mathbb{S}^5 \to \bigvee\limits_{l=1}^{2d} \mathbb{S}^3$ and $f' : \mathbb{S}^5 \to N^{(5)}$ are the attaching maps of the top cell in $\#_{l=1}^{d}(\mathbb{S}^3 \vee \mathbb{S}^3)$ and N, respectively. Now by using the fact that $\Sigma\omega$ is null homotopic, we obtain that M is stably homotopic to

$$M \simeq N \vee \bigvee_{l=1}^{2d} \mathbb{S}^3. \tag{3.11}$$

Henceforth, we assume M is a simply connected, closed, smooth, 6-dimensional manifold with its third Betti number $b_3(M) = 0$. Based on the homology given in (3.10), the 3-skeleton of M can be described as follows:

$$M^{(3)} \simeq \bigvee_{i=1}^{m} \mathbb{S}^2 \vee \bigvee_{j=1}^{\tilde{s}_1} \Sigma^2 M(\mathbb{Z}/3^{r_j}) \vee \bigvee_{j=\tilde{s}_1+1}^{s_1} \Sigma^2 M(\mathbb{Z}/p_j^{r_j}) \vee \bigvee_{j=1}^{s_2} \Sigma^2 M(\mathbb{Z}/2^{r_j}) \vee \bigvee_{j=1}^{s_1+s_2} \mathbb{S}^3. \tag{3.12}$$

The stable homotopy type of the 4-skeleton $M^{(4)}$ of M is determined by the attaching map $\phi_4 : \bigvee\limits_{i=1}^{m+s_1+s_2} \mathbb{S}^3 \to M^{(3)}$. From (3.12), we note that the stable nontrivial components of ϕ_4 may include:

- A nonzero multiple of η onto some of the copies of $\mathbb{S}^2$.
- A nontrivial multiple of $\Sigma^2 i_{2^r} \circ \eta$ onto one of the copies of $\Sigma^2 M(\mathbb{Z}/2^{r_j})$.
- Maps of degree 3^{r_j} onto the $\mathbb{S}^3$-summands, for $1 \leq j \leq \tilde{s}_1$.

- Maps of degree $p_j^{r_j}$ onto $\mathbb{S}^3$-summands, for $\tilde{s}_1 + 1 \leq j \leq s_1$.
- Maps of degree 2^{r_j} onto $\mathbb{S}^3$-summands, for $1 \leq j \leq s_2$.

Consequently, based on (3.10), the stable homotopy type of the 4-skeleton $M^{(4)}$ can be expressed as

$$\begin{aligned}M^{(4)} = M^{(5)} \simeq & \bigvee_{i=1}^{c} \mathbb{C}P^2 \vee \bigvee_{i=1}^{m-c} \mathbb{S}^2 \vee \bigvee_{j=\tilde{s}_1+1}^{s_1} \Sigma^2 M(\mathbb{Z}/p_j^{r_j}) \vee \bigvee_{j=1}^{\tilde{s}_1} \Sigma^2 M(\mathbb{Z}/3^{r_j}) \\ & \vee \bigvee_{j=1}^{s_2-l} \Sigma^2 M(\mathbb{Z}/2^{r_j}) \vee \bigvee_{j=1}^{\tilde{s}_1} \Sigma^3 M(\mathbb{Z}/3^{r_j}) \vee \bigvee_{j=\tilde{s}_1+1}^{s_1} \Sigma^3 M(\mathbb{Z}/p_j^{r_j}) \\ & \vee \bigvee_{j=1}^{s_2} \Sigma^3 M(\mathbb{Z}/2^{r_j}) \vee \bigvee_{w=1}^{m} \mathbb{S}^4 \vee \bigvee_{j=1}^{l} C_{\Sigma^2 i_{2^{r_j}} \circ \eta},\end{aligned} \tag{3.13}$$

where $0 \leq c \leq m,\ 0 \leq l \leq s_2$, are integers, and $C_{\Sigma^2 i_{2^r} \circ \eta} = Cone(\Sigma^2 i_{2^r} \circ \eta)$. We now analyze the individual components of the stable attaching map $\phi_6 : \mathbb{S}^5 \to M^{(5)}$ for the top cell.

Since the Steenrod operation $Sq^4 : H^2(M; \mathbb{Z}/2) \to H^6(M; \mathbb{Z}/2)$ is trivial, stably the map $\phi_6 : \mathbb{S}^5 \to M^{(5)}$ can be written as follows:

$$\begin{aligned}\phi_6 = & \sum_{i=1}^{c} (a_i(i' \circ 2\nu) + a_i'(i' \circ \alpha_1)) + \sum_{i=1}^{m-c} (2b_i\nu + b_i'\alpha_1) + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^2 i_{3^{r_j}} \circ \alpha_1) \\ & + \sum_{j=1}^{s_2-l} \Big(2x_j(\Sigma^2 i_{2^{r_j}} \circ 2^{\zeta(r_j)}\nu) + \tilde{x}_j \widetilde{\eta^2_{2^{r_j}}}\Big) + \sum_{j=1}^{s_2} \big(y_j \widetilde{\eta}_{2^{r_j}} + y_j'(\Sigma^3 i_{2^{r_j}} \circ \eta^2)\big) \\ & + \sum_{w=1}^{m} z_w \eta + \sum_{j=1}^{l} \Big(d_j(i^{r_j} \circ \widetilde{\eta^2_{2^{r_j}}}) + 2d^{r_j}(i^{r_j} \circ \Sigma^2 i_{2^{r_j}} \circ 2^{1+\delta_{r_j}} \nu)\Big),\end{aligned} \tag{3.14}$$

Where the coefficients in the above expression are as follows:

- $a_i, b_i \in \mathbb{Z}/4$;
- $a_i', b_i', c_j \in \mathbb{Z}/3$;
- $\tilde{x}_j, z_w, d_j \in \mathbb{Z}/2$;
- $d^{r_j} = 0$ when $r_j = 1$, lies in $\mathbb{Z}/2$ if $r_j = 2$, and in $\mathbb{Z}/4$ for $r_j \geq 3$;

- δ_{r_j} denotes the indicator function equal to 1 when $r_j = 2$, and 0 otherwise;

- $\zeta(r_j)$ is defined as

$$\zeta(r_j) = \begin{cases} 2 & \text{if } r_j = 1, \\ 1 & \text{if } r_j = 2, \\ 0 & \text{if } r_j \geq 3; \end{cases}$$

- $x_j \in \mathbb{Z}/2$ when $r_j = 1$, $x_j \in \mathbb{Z}/4$ when $r_j = 2$, and $x_j \in \mathbb{Z}/8$ for $r_j \geq 3$;

- $y_j \in \mathbb{Z}/4$ when $r_j = 1$, and $y_j \in \mathbb{Z}/2$ when $r_j \geq 2$;

- $y'_j \in (1 - \widetilde{\delta_{r_j}}) \cdot \mathbb{Z}/2$, where $\widetilde{\delta_{r_j}} = \begin{cases} 1 & \text{if } r_j = 1, \\ 0 & \text{otherwise.} \end{cases}$.

Based on Corollary 2.6.16, 2.6.17, and Lemma 2.6.19, 2.6.20, we now introduce the following conditions on cohomology operations introduced in 2.6.4:

Condition A: For any $u, v \in H^3(M; \mathbb{Z}/2)$ satisfying $\Theta(u) \neq 0,\ \Theta(v) = 0,\ u + v \notin \mathrm{Im}(\beta_s)$ for any $s \geq 1$, while there exist $u',\ v' \in H^3(M; \mathbb{Z}/2)$ with $\Theta(u') \neq 0,\ \Theta(v') = 0$ such that $\beta_r(u' + v') \neq 0$ for some r.

Condition B: There exist $u, v \in H^3(M; \mathbb{Z}/2)$ satisfying $\Theta(u) \neq 0,\ \Theta(v) = 0$ and $u + v \in \mathrm{Im}(\beta_r)$ for some r.

Condition C: For every $u \in H^4(M; \mathbb{Z}/2)$ with $Sq^2(u) \neq 0$, any $v \in Ker(Sq^2)$ satisfies $u + v \notin \mathrm{Im}(\beta_r)$ for any $r \geq 1$.

Condition D: There exist $u, v \in H^4(M; \mathbb{Z}/2)$ such that $Sq^2(u) \neq 0,\ Sq^2(v) = 0,\ u + v \in \mathrm{Im}(\beta_r)$ for some r.

M is a spin manifold

In this case, $\omega_2(M)$ is zero, implying that $Sq^2 : H^4(M; \mathbb{Z}/2) \to H^6(M; \mathbb{Z}/2)$ is trivial. Hence, by Corollary 2.6.16 and 2.6.17, the elements y_j, z_w in (3.14) must be zero. Consequently, (3.14) simplifies to

$$\begin{aligned}\phi_6 = {} & \sum_{i=1}^{c}(a_i(i' \circ 2\nu) + a'_i(i' \circ \alpha_1)) + \sum_{i=1}^{m-c}(2b_i\nu + b'_i\alpha_1) + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^2 i_{3^{r_j}} \circ \alpha_1) \\ & + \sum_{j=1}^{s_2-l}\left(2x_j(\Sigma^2 i_{2^{r_j}} \circ 2^{\zeta(r_j)}\nu) + \tilde{x}_j\widetilde{\eta^2_{2^{r_j}}}\right) + \sum_{j=1}^{s_2} y'_j(\Sigma^3 i_{2^{r_j}} \circ \eta^2) \\ & + \sum_{j=1}^{l}\left(d_j(i^{r_j} \circ \widetilde{\eta^2_{2^{r_j}}}) + 2d^{r_j}(i^{r_j} \circ \Sigma^2 i_{2^{r_j}} \circ 2^{1+\delta_{r_j}}\nu)\right).\end{aligned} \tag{3.15}$$

We further distinguish two cases depending on whether the secondary cohomology operation Θ acts trivially or nontrivially on $H^3(M;\mathbb{Z}/2)$.

Suppose the secondary cohomology operation Θ is trivial on $H^3(M;\mathbb{Z}/2)$. Then, by [46] and [72, Lemma 3.3], all coefficients $d_j, \tilde{x}_j, y'_j$ in (3.15) vanish, reducing (3.15) to:

$$\begin{aligned}\phi_6 = {} & \sum_{i=1}^{c}(a_i(i' \circ 2\nu) + a'_i(i' \circ \alpha_1)) + \sum_{i=1}^{m-c}(2b_i\nu + b'_i\alpha_1) + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^2 i_{3^{r_j}} \circ \alpha_1) \\ & + \sum_{j=1}^{s_2-l} 2x_j(\Sigma^2 i_{2^{r_j}} \circ 2^{\zeta(r_j)}\nu) + \sum_{j=1}^{l} 2d^{r_j}(i^{r_j} \circ \Sigma^2 i_{2^{r_j}} \circ 2^{1+\delta_{r_j}}\nu).\end{aligned} \tag{3.16}$$

If the secondary cohomology operation Θ acts nontrivially on $H^3(M;\mathbb{Z}/2)$, then at least one of $d_j, \tilde{x}_j, y'_j$ in (3.15) is nonzero. We now consider the following two possibilities, based on which of these terms are nonzero:

(a) If M satisfies Condition A, then $y'_j = 1$ for some $1 \leq j \leq s_2$, while $d_j = 0$ for all $1 \leq j \leq l$ and $\tilde{x}_j = 0$ for all $1 \leq j \leq s_2 - l$. Then by [72, Lemma 4.1], (3.15) further reduces to

(3.17)

$$\begin{aligned}\phi_6 = {} & \sum_{i=1}^{c}(a_i(i' \circ 2\nu) + a'_i(i' \circ \alpha_1)) + \sum_{i=1}^{m-c}(2b_i\nu + b'_i\alpha_1) + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^2 i_{3^{r_j}} \circ \alpha_1) \\ & + \sum_{j=1}^{s_2-l} 2x_j(\Sigma^2 i_{2^{r_j}} \circ 2^{\zeta(r_j)}\nu) + \Sigma^3 i_{2^{r_{j_0}}} \circ \eta^2 + \sum_{j=1}^{l} 2d^{r_j}(i^{r_j} \circ \Sigma^2 i_{2^{r_j}} \circ 2^{1+\delta_{r_j}}\nu),\end{aligned}$$

where j_0 is the index such that $r_{j_0} = \max\{r_j : y'_j = 1,\ 1 \leq j \leq s_2\} = \max\{r_j : \beta_{r_j}(u'+v') \neq 0,\ 1 \leq j \leq s_2\}$.

(b) If M satisfies Condition B, then either $d_j = 1$ or $\tilde{x}_j = 1$ in (3.15) for some j. By [72, Lemma 4.1], $y'_j = 0$ for all $1 \leq j \leq s_2$.

  (i) If $d_j = 1$ for some j and $\tilde{x}_j = 0$ for all $1 \leq j \leq s_2 - l$, then using [72, Lemma 4.2] we may assume that $d_{j_1} = 1$ and $d_j = 0$ for $j \neq j_1$, where j_1 is the index for which $r_{j_1} = \min\{r_j : d_j = 1 \text{ for } 1 \leq j \leq l\} = \min\{r_j : u+v \in \operatorname{Im}(\beta_{r_j}) \text{ for } 1 \leq j \leq l\}$.

  (ii) If $\tilde{x}_j = 1$ for some j and $d_j = 0$ for all $1 \leq j \leq l$, then by [72, Lemma 4.1] we may assume $\tilde{x}_{j_2} = 1$, and $\tilde{x}_j = 0$ for all $j \neq j_2$, where j_2 is the index satisfying $r_{j_2} = \min\{r_j : \tilde{x}_j = 1 \text{ for } 1 \leq j \leq s_2 - l\} = \min\{r_j : u+v \in \operatorname{Im}(\beta_{r_j}) \text{ for } 1 \leq j \leq s_2 - l\}$.

  (iii) Suppose $d_{j_3} = 1$ and $\tilde{x}_{j_4} = 1$ for some j_3, j_4. If $\min(r_{j_3}, r_{j_4}) = r_{j_3}$, then by [72, Lemma 4.1], we may assume $d_{j_3} = 1$ and $d_j = 0$ for $j \neq j_3$, and $\tilde{x}_j = 0$ for all $1 \leq j \leq s_2 - l$. If $\min(r_{j_3}, r_{j_4}) = r_{j_4}$, then again [72, Lemma 4.1] implies $\tilde{x}_{j_4} = 1$, and $\tilde{x}_j = 0$ for all $j \neq j_4$ and $d_j = 0$ for all $1 \leq j \leq l$.

Now we define r as follows:

$$r = \begin{cases} r_{j_1} & \text{if } \tilde{x}_j = 0 \text{ for all } 1 \leq j \leq s_2 - l, \\ r_{j_2} & \text{if } d_j = 0 \text{ for all } 1 \leq j \leq l, \\ \min(r_{j_3}, r_{j_4}) & \text{if } d_{j_3} = 1 \text{ and } \tilde{x}_{j_4} = 1. \end{cases} \tag{3.18}$$

If $r = r_{j_1}$ or r_{j_3}, then (3.15) simplifies to

$$\begin{aligned}\phi_6 = \sum_{i=1}^{c}(a_i(i'\circ 2\nu) + a_i'(i'\circ\alpha_1)) + \sum_{i=1}^{m-c}(2b_i\nu + b_i'\alpha_1) + (i^r\circ\widetilde{\eta_{2^r}^2})\\ + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^2 i_{3^{r_j}}\circ\alpha_1) + \sum_{j=1}^{s_2-l}(2x_j(\Sigma^2 i_{2^{r_j}}\circ 2^{\zeta(r_j)}\nu)\\ + \sum_{j=1}^{l} 2d^{r_j}(i^{r_j}\circ\Sigma^2 i_{2^{r_j}}\circ 2^{1+\delta_{r_j}}\nu).\end{aligned}\tag{3.19}$$

If $r = r_{j_2}$ or r_{j_4}, then (3.15) can be written as

$$\begin{aligned}\phi_6 = \sum_{i=1}^{c}(a_i(i'\circ 2\nu) + a_i'(i'\circ\alpha_1)) + \sum_{i=1}^{m-c}(2b_i\nu + b_i'\alpha_1)\\ + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^2 i_{3^{r_j}}\circ\alpha_1) + \sum_{j=1}^{s_2-l} 2x_j(\Sigma^2 i_{2^{r_j}}\circ 2^{\zeta(r_j)}\nu) + \widetilde{\eta_{2^r}^2}\\ + \sum_{j=1}^{l} 2d^{r_j}(i^{r_j}\circ\Sigma^2 i_{2^{r_j}}\circ 2^{1+\delta_{r_j}}\nu).\end{aligned}\tag{3.20}$$

M is a non-spin manifold

In this case, the second Stiefel–Whitney class $\omega_2(M)$ is nonzero. Consequently, the Steenrod square $Sq^2 : H^4(M;\mathbb{Z}/2) \to H^6(M;\mathbb{Z}/2)$ is nontrivial, which forces at least one of the coefficients y_j or z_w in (3.14) to be equal to 1. Based on the previously stated conditions, we now determine when y_j or z_w is nonzero, as follows:

If M satisfies Condition C, then $y_j = 0$ for all $1 \leq j \leq s_2$ and $z_w = 1$ for at least one $1 \leq w \leq m$. Using [72, Lemma 4.1], we may further assume that $d_j, \tilde{x}_j, y_j'$ are zero for all j. In addition, following [52, §3], we can take $z_{w_0} = 1$ for some w_0, and $z_w = 0$ for all $w \neq w_0$. Under these assumptions, equation (3.14) simplifies to:

$$\begin{aligned}\phi_6 = \sum_{i=1}^{c}(a_i(i'\circ 2\nu) + a_i'(i'\circ\alpha_1)) + \sum_{i=1}^{m-c}(2b_i\nu + b_i'\alpha_1) + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^2 i_{3^{r_j}}\circ\alpha_1)\\ + \sum_{j=1}^{s_2-l} 2x_j(\Sigma^2 i_{2^{r_j}}\circ 2^{\zeta(r_j)}\nu) + \eta + \sum_{j=1}^{l} 2d^{r_j}(i^{r_j}\circ\Sigma^2 i_{2^{r_j}}\circ 2^{1+\delta_{r_j}}\nu).\end{aligned}\tag{3.21}$$

If M satisfies Condition D, then $y_j = 1$ for some $1 \leq j \leq s_2$. Let j_5 be the index for which $r_{j_5} = \min\{r_j : y_j = 1 \text{ for } 1 \leq j \leq s_2\} = \min\{r_j : u{+}v \in \mathrm{Im}(\beta_{r_j}) \text{ for } 1 \leq j \leq s_2\}$. Then by [72, Lemma 4.1], we may assume that $y_{j_5} = 1, y_j = 0$ for all $j \neq j_5$, and that all other coefficients $d_j, \tilde{x}_j, y'_j, z_w$ in (3.14) are zero. Under these assumptions, (3.14) takes the form:

$$\begin{aligned}\phi_6 = {} & \sum_{i=1}^{c} (a_i(i' \circ 2\nu) + a'_i(i' \circ \alpha_1)) + \sum_{i=1}^{m-c} (2b_i\nu + b'_i\alpha_1) + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^2 i_{3^{r_j}} \circ \alpha_1) \\ & + \sum_{j=1}^{s_2-l} 2x_j(\Sigma^2 i_{2^{r_j}} \circ 2^{\zeta(r_j)}\nu) + \widetilde{\eta}_{2^{r_{j_5}}} + \sum_{j=1}^{l} 2d^{r_j}(i^{r_j} \circ \Sigma^2 i_{2^{r_j}} \circ 2^{1+\delta_{r_j}}\nu).\end{aligned} \tag{3.22}$$

3.4 Stable Homotopy Type of 3-Connected 8-Manifold

Let M be a closed, smooth, 3-connected 8-dimensional manifold with fourth Betti number $b_4(M) = m$, referred to as the rank of M. Wall [126] classified such manifolds M, up to connected sum with a homotopy sphere, by analyzing the triple $(H^4(M;\mathbb{Z}), \lambda_M, \beta_M)$. Here,

$$\lambda_M : H^4(M;\mathbb{Z}) \times H^4(M;\mathbb{Z}) \to \mathbb{Z}$$

is the intersection form defined by $\lambda_M(x, y) = \langle x \cup y, [M]\rangle$, where $[M]$ is the fundamental class of M, and

$$\beta_M : H^4(M;\mathbb{Z}) \to \pi_3(SO(4)) \cong \mathbb{Z}$$

is the stable tangential invariant. The map β_M is defined by first realizing the cohomology classes of $H^4(M;\mathbb{Z})$ by embeddings of 4-spheres in M and then taking the clutching functions of the normal bundles of the 4-spheres in M.

We denote by $\tau_{4+l} \in \pi_{7+l}(\mathbb{S}^{4+l})$, for $l \geq 1$, the element representing the stable generator of π_3^s, such that $(\tau_{4+l})_{(2)} = \nu_{4+l}$ and $(\tau_{4+l})_{(3)} = \alpha_1$.

According to [126], M is homotopy equivalent to a CW complex $\left(\bigvee_{i=1}^{m}\mathbb{S}^4\right)\cup_g \mathbb{D}^8$, where $g:\mathbb{S}^7\to\bigvee_{i=1}^{m}\mathbb{S}^4$ is the attaching map of the top cell $\mathbb{D}^8$.
In the case $m=1$, it follows from [40, §6] that for $l\geq 1$,

$$\Sigma^l g = t\tau_{4+l}, \quad \text{where } t\in\mathbb{Z}/24.$$

Therefore, when the rank of M is one, its stable homotopy type is given by

$$M\simeq Cone\,(t\tau_4)\,. \tag{3.23}$$

Suppose $m\geq 2$. Then, following [39], for any $l\geq 1$, the map $\Sigma^l g$ admits a decomposition of the form

$$\Sigma^l g=\sum_{i=1}^{m}a_i\tau_{4+l}, \tag{3.24}$$

where each $a_i\in\mathbb{Z}/12$. Applying the methods in [52, §3], (3.24) can be further refined as

$$\Sigma^l g=\mathrm{ind}(M)\tau_{4+l},$$

where $\mathrm{ind}(M)$ is the index the subgroup $\mathrm{Im}(\chi)$ in $\mathrm{Im}(J)$ if the homomorphism χ is nontrivial and is zero otherwise. Here the homomorphism χ is given by the composition

$$\chi: H_4(M;\mathbb{Z})\xrightarrow{\beta_M}\pi_3(SO(4))\xrightarrow{J}\pi_7(S^4)\xrightarrow{\Sigma}\pi_3^s,$$

where J is the classical J-homomorphism and Σ is the suspension map. This leads to the following stable homotopy type for M [52, Theorem 1.1]:

$$M\simeq(\bigvee_{i=1}^{m-1}\mathbb{S}^4)\vee Cone\,(\mathrm{ind}(M)\tau_4)\,,\ \textit{if}\ m\geq 2. \tag{3.25}$$

Chapter 4

Concordance Inertia Group of $M \times \mathbb{S}^k$

In this chapter, we study the concordance inertia group of manifolds of the form $M \times \mathbb{S}^k$, where M is a closed, smooth, oriented manifold of dimension $4, 5, 6$, or 8, as considered in Chapter 2. Understanding this group plays an important role in the classification of smooth structures up to concordance. We use the homotopy-theoretic information obtained in the previous chapter to compute this group.

Recall from Proposition 2.4.15 that for any closed, oriented, smooth manifold M of dimension $n \geq 5$,

$$I_c(M) = Ker\left(\Theta_n \xrightarrow{f_M^*} \mathcal{C}(M)\right).$$

Now, for the product manifold $M \times \mathbb{S}^k$, the concordance inertia group is described as follows.

Lemma 4.0.1. *Let M be a closed, oriented, smooth n-manifold and let $f_M : M \to \mathbb{S}^n$ be a degree one collapse map. Then, for $n + k \geq 5$, the concordance inertia group of the product $M \times \mathbb{S}^k$ is given by*

$$I_c(M \times \mathbb{S}^k) = Ker\left(\pi_{n+k}(Top/O) \xrightarrow{(\Sigma^k f_M)^*} [\Sigma^k M, Top/O]\right),$$

where $\Sigma^k f_M : \Sigma^k M \to \mathbb{S}^{n+k}$ is the k-fold suspension of f_M.

Proof. We observe that there is a split short exact sequence along an H-group Y

$$0 \to [\Sigma^k M, Y] \xrightarrow{p^*} [M \times \mathbb{S}^k, Y] \xrightarrow{i^*} [M \vee \mathbb{S}^k, Y] \to 0, \tag{4.1}$$

which is induced from the cofiber sequence

$$M \vee \mathbb{S}^k \overset{i}{\hookrightarrow} M \times \mathbb{S}^k \xrightarrow{p} M \times \mathbb{S}^k / M \vee \mathbb{S}^k \simeq \Sigma^k M.$$

Note that the map p fits into the following homotopy commutative diagram:

$$\begin{tikzcd}
M \times \mathbb{S}^k \arrow[r, "p"] \arrow[d, "f_{M\times\mathbb{S}^k}"'] & \Sigma^k M \arrow[dl, "\Sigma^k f_M"] \\
\mathbb{S}^{n+k}. &
\end{tikzcd}$$

This, in turn, induces the following commutative diagram on homotopy classes of maps:

$$\begin{tikzcd}
{[M \times \mathbb{S}^k, Top/O]} & {[\Sigma^k M, Top/O]} \arrow[l, "p^*"'] \\
{[\mathbb{S}^{n+k}, Top/O].} \arrow[u, "f^*_{M\times\mathbb{S}^k}"] \arrow[ur, "(\Sigma^k f_M)^*"'] &
\end{tikzcd}$$

Since p^* is injective by (4.1), and $\Theta_{n+k} \cong \pi_{n+k}(Top/O)$ for $n+k \geq 5$ by Theorem 2.2.32, the conclusion follows. $\square$

The next lemma enables us to compute the concordance inertia group of $M \times \mathbb{S}^k$, provided the stable homotopy type of M is known.

Lemma 4.0.2. *Let M be an n-dimensional, closed, oriented, and smooth manifold. Suppose M is stably homotopy equivalent to $X \vee Z$, where X is the cofiber of a map $f : \mathbb{S}^{n-1} \to \bigvee_{i=1}^{w} A_i$ with $w \geq 1$, and both $\bigvee_{i=1}^{w} A_i$ and Z are of dimension less than n. Then, for any integer $k \geq 1$ such that $n+k \geq 5$, the concordance inertia group*

$I_c(M \times \mathbb{S}^k)$ is isomorphic to

$$Im\left(\bigoplus_{i=1}^{w}[\Sigma^{k+1}A_i, Top/O] \xrightarrow{\bigoplus\limits_{i=1}^{w}(\Sigma^{k+1}f_i)^*} \pi_{n+k}(Top/O)\right),$$

where each $f_i = Pr_{A_i} \circ f$, and Pr_{A_i} denotes the projection from $\bigvee\limits_{i=1}^{w} A_i$ onto its i-th factor.

Proof. By assumption, there exists a positive integer j such that for any $l \geq j$, we have a homotopy equivalence $\phi_l : \Sigma^l M \to \Sigma^l X \vee \Sigma^l Z$. Since Top/O is an infinite loop space, there exists a topological space Y such that Top/O is weakly equivalent to $\Omega^j Y$. Now by Lemma 4.0.1, we have

$$I_c(M \times \mathbb{S}^k) \cong Ker\left(\pi_{n+k+j}(Y) \xrightarrow{(\Sigma^{k+j}f_M)^*} [\Sigma^{k+j}M, Y]\right).$$

Observe that the $(k+j)$-fold suspension of the degree one map $\Sigma^{k+j}f_M : \Sigma^{k+j}M \to \mathbb{S}^{n+k+j}$ is homotopic to the composition $\lambda \circ \Sigma^{k+j}f_X \circ Pr_{\Sigma^{k+j}X} \circ \phi_{k+j}$, where $Pr : \Sigma^{k+j}X \vee \Sigma^{k+j}Z \to \Sigma^{k+j}X$ denotes the projection onto the first wedge summand, $\Sigma^{k+j}f_X : \Sigma^{k+j}X \to \Sigma^{k+j}\mathbb{S}^n$ is the $(k+j)$-fold suspension of the pinch map $f_X : X \to \mathbb{S}^n$ onto its top cell, and $\lambda : \mathbb{S}^{n+k+j} \to \mathbb{S}^{n+k+j}$ is either the identity or the reflection map. This implies that

$$I_c(M \times \mathbb{S}^k) \cong Ker\left(\pi_{n+k+j}(Y) \xrightarrow{(\Sigma^{k+j}f_X)^*} [\Sigma^{k+j}X, Y]\right).$$

This kernel can be equivalently expressed as the image of the homomorphism $\bigoplus\limits_{i=1}^{w}(\Sigma^{k+j+1}f_i)^* : \bigoplus\limits_{i=1}^{w}[\Sigma^{k+j+1}A_i, Y] \to \pi_{n+k+j}(Y)$, obtained from the long exact sequence induced from the cofiber sequence $\Sigma^{k+j}\mathbb{S}^{n-1} \xrightarrow{\Sigma^{k+j}f} \Sigma^{k+j}(\bigvee\limits_{i=1}^{w} A_i) \hookrightarrow \Sigma^{k+j}X$ along Y. Consequently, the group $I_c(M \times \mathbb{S}^k)$ is isomorphic to the image of the map $\bigoplus\limits_{i=1}^{w}(\Sigma^{k+1}f_i)^* : \bigoplus\limits_{i=1}^{w}[\Sigma^{k+1}A_i, Top/O] \to \pi_{n+k}(Top/O)$. This completes the proof. □

This lemma states that, to compute $I_c(M \times \mathbb{S}^k)$, one needs to examine the image of the components of the top cell attaching map in the stable homotopy type of M under the functor $[\_, Top/O]$.

We observe from Theorems 2.1.11 and 2.2.32 that certain elements of $\pi_k(Top/O)$ correspond to generators of the stable homotopy groups of spheres π_k^s. In this thesis, the generators of π_k^s are chosen according to the conventions in [125, 103] (see Theorem 2.2.28 for dimensions up to 19), and this choice is adopted consistently throughout the thesis.

4.1 Concordance Inertia Group of the Product of a 4-Manifold with a Sphere

From Section 3.1, we have $M \simeq M^{(3)} \cup_{\phi_3} \mathbb{D}^4$ as an object in the stable homotopy category, where ϕ_3 is null homotopic if M is a spin manifold and $\phi_3 = \iota \circ \eta$ if M is a non-spin manifold. Here ι refers to the map which is the inclusion of the d-th $\mathbb{S}^2$-factor. This, together with Lemma 4.0.2, allows us to compute $I_c(M \times \mathbb{S}^k)$ by analyzing only the image of $\eta_k^* : \pi_k(Top/O) \to \pi_{k+1}(Top/O)$. In particular, we prove the following lemma.

Lemma 4.1.1. *The image of* $\eta_k^* : \pi_k(Top/O) \to \pi_{k+1}(Top/O)$ *is given in the following table:*

$k =$	1 *to* 6	7	8	9	10	11	12
$Im(\eta_k^*)$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0

$k =$	13	14	15	16	17	18	19
$Im(\eta_k^*)$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	0

Proof. Since $\pi_k(Top/O)$ vanishes for all $1 \leq k \leq 6$ with $k \neq 3$, and also for $k = 12$, the result follows for $k = 1, 2, 3, 4, 5, 6, 11$, and 12.

As $\pi_{13}(Top/O)_{(2)}$ is zero, the map $\eta_{13}^* : \pi_{13}(Top/O) \to \pi_{14}(Top/O)$ is trivial.

Consider the following commutative diagram:

(4.2)
$$\begin{tikzcd}
\pi_k(Top/O) \arrow[r, "\eta_k^*"] \arrow[d, "\psi_*"'] & \pi_{k+1}(Top/O) \arrow[d, "\psi_*"] \\
\pi_k(G/O) \arrow[r, "\eta_k^*"'] & \pi_{k+1}(G/O) \\
\pi_k^s \arrow[u, "j_*"] \arrow[r, "\eta_k^*"'] & \pi_{k+1}^s . \arrow[u, "j_*"']
\end{tikzcd}$$

Since $\pi_7(G/O) = 0$ and $\psi_* : \pi_8(Top/O) \to \pi_8(G/O)$ is injective, it follows from the commutative diagram above that the map $\eta_7^* : \pi_7(Top/O) \to \pi_8(Top/O)$ is trivial.

For $k = 8$, we note from [125, Page 189] that the image of $\eta_8^* : \pi_8^s \to \pi_9^s$ is $\mathbb{Z}/2\{\nu^3\} \oplus \mathbb{Z}/2\{\eta\epsilon\}$. Since $j_*(\bar{\nu}) = j_*(\epsilon)$ in $\pi_8(G/O)$ [103], we obtain from the lower rectangle of the diagram (4.2) that the image of the map $\eta_8^* : \pi_8(G/O) \to \pi_9(G/O)$ restricted to the torsion part of $\pi_8(G/O)$ is $\mathbb{Z}/2$. Now, the result follows from the top rectangle (4.2) as $j_*(\pi_8^s) = \psi_*(\pi_8(Top/O))$.

For $k = 9$, the element $\mu \in \pi_9^s$ maps to $\eta\mu \in \pi_{10}^s$ under the map η_9^*. The image of μ under $j_* : \pi_9^s \to \pi_9(G/O)$ is nonzero [103, Theorem 1.1.14], and the map $\psi_* : \pi_9(Top/O) \to \pi_9(G/O)$ is onto. Furthermore, both $\psi_* : \pi_{10}(Top/O) \to \pi_{10}(G/O)$ and $j_* : \pi_{10}^s \to \pi_{10}(G/O)$ are isomorphisms. It then follows from the commutativity of diagram (4.2) that the image of $\eta_9^* : \pi_9(Top/O) \to \pi_{10}(Top/O)$ is isomorphic to $\mathbb{Z}/2$, generated by the element corresponding to $\eta\mu \in \pi_{10}^s$.

We now turn to the case $k = 10$. Since $Top/PL \simeq K(\mathbb{Z}/2, 3)$ [64], the natural map $PL/O \to Top/O$ induces isomorphism on the homotopy groups in degree $i \geq 4$. Hence to compute the image of the map $\eta_{10}^* : \pi_{10}(Top/O) \to \pi_{11}(Top/O)$, it suffices to determine the image of $\eta_{10}^* : \pi_{10}(PL/O) \to \pi_{11}(PL/O)$. Since $\pi_{10}(PL/O) = \mathbb{Z}/6$ and $\pi_{11}(PL/O) = \mathbb{Z}/992$, the image of $\eta_{10}^* : \pi_{10}(PL/O) \to \pi_{11}(PL/O)$ is atmost $\mathbb{Z}/2$. We claim that the image of the map $\eta_{10}^* : \pi_{10}(PL/O) \to \pi_{11}(PL/O)$ is exactly $\mathbb{Z}/2$. To prove this, we consider the following commutative diagram with exact rows induced from the fiber sequence $PL/O \to BO \to BPL$

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \pi_{10}(PL) \arrow[r, "\cong"] \arrow[d, "\eta_{10}^*"'] & \pi_{10}(PL/O) \arrow[r, "0"] \arrow[d, "\eta_{10}^*"] & \pi_{10}(BO) \arrow[d] \\
0 \arrow[r] & \pi_{12}(BO) \arrow[r] & \pi_{11}(PL) \arrow[r, "\beta"'] & \pi_{11}(PL/O) \arrow[r] & 0,
\end{tikzcd}$$

where $\pi_{10}(PL) \cong \pi_{10}(PL/O)$ [50, 51] and $\pi_{11}(PL) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ [25, Theorem 1.4]. Since the restriction of $\beta : \pi_{11}(PL) \to \pi_{11}(PL/O)$ to the torsion subgroup of $\pi_{11}(PL)$ is an inclusion [25, Page 307], it follows from the diagram that the image of $\eta_{10}^* : \pi_{10}(PL/O) \to \pi_{11}(PL/O)$ is $\mathbb{Z}/2$ if and only if the image of $\eta_{10}^* : \pi_{10}(PL) \to \pi_{11}(PL)$ is $\mathbb{Z}/2$. The map $\eta_{10}^* : \pi_{10}(PL) \to \pi_{11}(PL)$ fits into the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & \pi_{10}(PL) \arrow[r, "\cong"] \arrow[d, "\eta_{10}^*"'] & \pi_{10}^s \arrow[r] \arrow[d, "\eta_{10}^*"] & \pi_{10}(G/PL) \\
0 \arrow[r] & \pi_{12}(G/PL) \arrow[r] & \pi_{11}(PL) \arrow[r] & \pi_{11}^s \arrow[r] & 0,
\end{tikzcd}$$

where the rows are induced from the fibration $G/PL \to BPL \to BG$; the isomorphism $\pi_{10}(PL) \cong \pi_{10}^s$ holds since $\pi_{10}(PL) \cong \pi_{10}(PL/O)$. Since the image of $\eta_{10}^* : \pi_{10}^s \to \pi_{11}^s$ is $\mathbb{Z}/2$ [125], the above diagram shows that the map $\eta_{10}^* : \pi_{10}(PL) \to \pi_{11}(PL)$ has image $\mathbb{Z}/2$. This completes the proof for $k = 10$.

For $k = 14$, it follows from [125, Page 189-190] that the generators κ and σ^2 of π_{14}^s are mapped under η_{14}^* to $\eta\kappa$ and 0 in π_{15}^s, respectively. Moreover, the map $j_* : \pi_{14}^s \to \pi_{14}(G/O)$ is an isomorphism, since $\pi_{15}(BO) = 0$ and $\pi_{14}(G/O) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$; additionally, $j_* : \pi_{15}^s \to \pi_{15}(G/O) \cong \mathbb{Z}/2$ sends $\eta\kappa$ to the generator of $\pi_{15}(G/O)$ [103, Theorem 1.1.14]. Combining these facts, the lower part of diagram (4.2) shows that the image of $\eta_{14}^* : \pi_{14}(G/O) \to \pi_{15}(G/O)$ is $\mathbb{Z}/2$. Now, since $\psi_* : \pi_{14}(Top/O) \to \pi_{14}(G/O)$ is injective, and the exotic sphere in $\pi_{14}(Top/O)$ corresponds to either κ or $\kappa + \sigma^2$ in π_{14}^s [61, Page 45], it follows from the upper part of diagram (4.2) that $\eta_{14}^* : \pi_{14}(Top/O) \to \pi_{15}(Top/O)$ is a monomorphism. Hence, its image is isomorphic to $\mathbb{Z}/2$.

For $k = 15$, the generator $\eta\kappa \in \pi_{15}^s$ maps to zero in π_{16}^s under η_{15}^* [125, Theorem

14.1], but maps to the generator of $\pi_{15}(G/O) = \mathbb{Z}/2$ under the map j_* [103, Theorem 1.1.14]. Therefore, from the lower part of the commutative diagram (4.2), we obtain that $\eta^*_{15} : \pi_{15}(G/O) \to \pi_{16}(G/O)$ is the zero map. Since $\psi_* : \pi_{15}(Top/O) \to \pi_{15}(G/O)$ is surjective and $\psi_* : \pi_{16}(Top/O) \to \pi_{16}(G/O)$ is injective, the commutativity of the upper row of the diagram then implies that the map $\eta^*_{15} : \pi_{15}(Top/O) \to \pi_{16}(Top/O)$ is also trivial.

Now we consider the case $k = 16$. Note that the map η^*_{16} sends the generator η^* of π^s_{16} to the generator $\eta\eta^*$ of π^s_{17} [125, Page 189-190]. Since $j_*(\eta^*)$ generates the torsion part of $\pi_{16}(G/O)$, and $j_*(\eta\eta^*) \neq 0$ in $\pi_{17}(G/O)$ [103, Table A3.3], it follows from the lower part of the diagram (4.2) that the restriction of $\eta^*_{16} : \pi_{16}(G/O) \to \pi_{17}(G/O)$ to the torsion subgroup of $\pi_{16}(G/O)$ has image $\mathbb{Z}/2$. Since the map ψ_* maps $\pi_{16}(Top/O)$ injectively to the torsion part of $\pi_{16}(G/O)$, the upper part of the diagram (4.2) implies that the image of $\eta^*_{16} : \pi_{16}(Top/O) \to \pi_{17}(Top/O)$ is also $\mathbb{Z}/2$.

For $k = 17$, we note that the map $j_* : \pi^s_{17} \to \pi_{17}(G/O)$ is onto and its image is generated by $j_*(\eta\eta^*), j_*(\nu\kappa), j_*(\bar{\mu})$ [103, Table A3.3 and Theorem 1.1.13], where $\pi^s_{17} = \mathbb{Z}/2\{\eta\eta^*\} \oplus \mathbb{Z}/2\{\nu\kappa\} \oplus \mathbb{Z}/2\{\bar{\mu}\} \oplus \mathbb{Z}/2\{\eta^2\rho\}$ are the generators. Since $\eta^2\eta^* = 4\nu^*, \eta\nu\kappa = 0$ [125, Theorem 14.1], the image of the map $\eta^*_{17} : \pi^s_{17} \to \pi^s_{18}$, restricted to the subgroup $\mathbb{Z}/2\{\eta\eta^*\} \oplus \mathbb{Z}/2\{\nu\kappa\} \oplus \mathbb{Z}/2\{\bar{\mu}\}$, is $\mathbb{Z}/2\{4\nu^*\} \oplus \mathbb{Z}/2\{\eta\bar{\mu}\}$. As $j_* : \pi^s_{18} \to \pi_{18}(G/O)$ is an isomorphism, the lower part of diagram (4.2) implies that the image of $\eta^*_{17} : \pi_{17}(G/O) \to \pi_{18}(G/O)$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since $\psi_* : \pi_{17}(Top/O) \to \pi_{17}(G/O)$ is surjective and $\psi_* : \pi_{18}(Top/O) \to \pi_{18}(G/O)$ is an isomorphism, the upper part of the diagram shows that $\eta^*_{17} : \pi_{17}(Top/O) \to \pi_{18}(Top/O)$ also has image $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Next, consider the case $k = 18$. Since $\pi_i(Top/O) \cong \pi_i(PL/O)$ for all $i \geq 6$, computing the image of $\eta^*_{18} : \pi_{18}(Top/O) \to \pi_{19}(Top/O)$ reduces to computing the image of $\eta^*_{18} : \pi_{18}(PL/O) \to \pi_{19}(PL/O)$. This map appears in the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & \pi_{18}(PL) \arrow[r, "\cong"] \arrow[d, "\eta_{18}^*"'] & \pi_{18}(PL/O) \arrow[r] \arrow[d, "\eta_{18}^*"] & 0 \\
0 \arrow[r] & \pi_{19}(O) \cong \mathbb{Z} \arrow[r] & \pi_{19}(PL) \arrow[r, "\beta"'] & \pi_{19}(PL/O) \arrow[r] & 0.
\end{tikzcd}$$

Here, the rows are induced by the fiber sequence $PL/O \to BO \to BPL$; the map $\pi_{18}(PL) \to \pi_{18}(PL/O)$ is surjective [25, Page 291]; and $\pi_{18}(PL) \cong \pi_{18}(PL/O)$ since $\pi_{18}(O) = 0$. Moreover, $\pi_{19}(PL) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8$ [25, Theorem 1.4], and as noted in [25, Page 307], the restriction of $\beta : \pi_{19}(PL) \to \pi_{19}(PL/O)$ to the torsion summand is injective. Hence, by commutativity of the diagram, determining the image of $\eta_{18}^* : \pi_{18}(PL/O) \to \pi_{19}(PL/O)$ reduces to determining the image of $\eta_{18}^* : \pi_{18}(PL) \to \pi_{19}(PL)$. To compute this, we consider the following commutative diagram induced from the fiber sequence $PL \to G \to G/PL$.

$$\begin{tikzcd}
 & 0 \arrow[r] & \pi_{18}(PL) \arrow[r, "\cong"] \arrow[d, "\eta_{18}^*"'] & \pi_{18}^s \arrow[r, "0"] \arrow[d, "\eta_{18}^*"] & \pi_{18}(G/PL) \\
0 \arrow[r] & \pi_{20}(G/PL) \arrow[r] & \pi_{19}(PL) \arrow[r, "j_{PL}"'] & \pi_{19}^s \arrow[r] & 0.
\end{tikzcd}$$

Here, $\pi_{18}(PL) \cong \pi_{18}^s$ since $\pi_{18}(PL) \cong \pi_{18}(PL/O)$. From [125, Theorem 14.1], we get $\eta \circ \nu^* = 0$, $\eta^2 \circ \bar{\mu} = 4\bar{\zeta}$ [125, Theorem 14.1], so the image of the map $\eta_{18}^* : \pi_{18}^s \to \pi_{19}^s$ is $\mathbb{Z}/2\{4\bar{\zeta}\}$. Since the torsion components of $\pi_{19}(PL)$ maps injectively into π_{19}^s under j_{PL} [25, Page 307], the commutative diagram above implies that the map $\eta_{18}^* : \pi_{18}(PL) \to \pi_{19}(PL)$ has image $\mathbb{Z}/2$. This completes the proof for the case $k = 18$.

For $k = 19$, we note from [103, Table A3.3] that $\pi_{19}(G/O)$ is generated by $j_*(\bar{\sigma})$. But $\bar{\sigma} \in \langle \nu, \eta \circ \sigma, \sigma \rangle$ with zero indeterminacy [125]. Then by properties of Toda brackets [125, Page 33], we have

$$\eta\bar{\sigma} \in \eta\langle \nu, \eta\sigma, \sigma \rangle \subseteq \langle \eta\nu, \eta\sigma, \sigma \rangle.$$

Since $\eta\nu = 0$ [125, Theorem 14.1] and the Toda bracket $\langle \eta\nu, \eta\sigma, \sigma \rangle$ has indeterminancy zero, $\eta\bar{\sigma} = 0$. Hence, by the lower part of the commutative diagram (4.2), we have $\eta_{19}^* : \pi_{19}(G/O) \to \pi_{20}(G/O)$ is a trivial map. Since $\psi_* : \pi_{20}(Top/O) \to \pi_{20}(G/O)$ is injective, it then follows from the upper part of the diagram (4.2) that $\eta_{19}^* : \pi_{19}(Top/O) \to \pi_{20}(Top/O)$ is also the zero map. $\square$

From the proof of the above Lemma and [125], we observe the following:

Remark 4.1.2. *The image of* $\eta_k^* : \pi_k(Top/O) \to \pi_{k+1}(Top/O)$ *is generated by the exotic spheres corresponding to the generators* $\eta\epsilon$, $\eta\mu$, $\eta\kappa$, $\eta\eta^*$, $4\nu^*$, *and* $\eta\bar{\mu}$ *in* π_{k+1}^s *for* $k = 8, 9, 14, 16,$ *and* 17, *respectively.*

Corollary 4.1.3. *Let* $\eta_k^2 = \eta_k \circ \eta_{k+1}$. *Then:*

(i) The map $(\eta^2)^* : \pi_k(Top/O) \to \pi_{k+2}(Top/O)$ *is zero for* $7 \leq k \leq 19$, *except for* $k = 9, 16$ *and* 17.

(ii) The image of the map $(\eta^2)^* : \pi_k(Top/O) \to \pi_{k+2}(Top/O)$ *is* $\mathbb{Z}/2$ *for* $k = 9, 16$ *and* 17.

Using the long exact sequence induced from the cofiber sequence $\mathbb{S}^0 \xrightarrow{\times 2^r} \mathbb{S}^0 \xhookrightarrow{i_{2^r}} M(\mathbb{Z}/2^r)$ along Top/O and Lemma 4.1.1, we have the following result.

Corollary 4.1.4. *The image of* $(\Sigma^{3+k} i_{2^r} \circ \eta_{3+k})^* : [\Sigma^{3+k} M(\mathbb{Z}/2^r), Top/O] \to \pi_{4+k}(Top/O)$ *is given by:*

(i) 0, *for* $k = 1, 2, 3, 4, 8, 9, 10, 12$ *or* 16;

(ii) $\mathbb{Z}/2$, *for* $k = 5, 6, 7, 11, 13$ *or* 15;

(iii) $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, *for* $k = 14$,

where the map $\Sigma^{3+k} i_{2^r} : \mathbb{S}^{3+k} \hookrightarrow \Sigma^{3+k} M(\mathbb{Z}/2^r)$ *is the* $(3+k)$*-fold suspension of the inclusion* $i_{2^r} : \mathbb{S}^0 \hookrightarrow M(\mathbb{Z}/2^r)$.

We now determine the concordance inertia group $I_c(M \times \mathbb{S}^k)$ for all $1 \leq k \leq 16$.

Theorem 4.1.5. *Let* M *be a closed, oriented, smooth* 4*-manifold.*

(i) If M is a spin manifold, then $I_c(M \times \mathbb{S}^k) = 0$ for $k \geq 1$.

(ii) If M is a non-spin manifold, then

(a) $I_c(M \times \mathbb{S}^k) = 0$, for $k = 1, 2, 3, 4, 8, 9, 10, 12$ or 16;

(b) $I_c(M \times \mathbb{S}^k) = \mathbb{Z}/2$, for $k = 5, 6, 7, 11, 13$ or 15;

(c) $I_c(M \times \mathbb{S}^{14}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proof. The proof follows from the stable splittings (3.2) and (3.3) for M, together with Lemma 4.0.2 and Lemma 4.1.1. □

Proposition 4.1.6. *Let M be a non-spin, closed, oriented, smooth 4-dimensional manifold. Then $\mathbb{Z}/2 \subseteq I_c(M \times \mathbb{S}^k)$, for $k = 17, 18,$ or $8n - 2$ with $n \geq 3$.*

Proof. From [103, Table A3.3] and [2, Theorems 1.3 and 1.4], it follows that for $k = 17, 18$, or $k = 8n - 2$ with $n \geq 3$, there exists an element of order two, denoted by x_{3+k}, in $\pi^s_{3+k}/Im(J) \subseteq \pi_{3+k}(G/O)$ such that $x_{3+k} \circ \eta_{3+k}$ is nonzero in $\pi^s_{4+k}/Im(J) \subseteq \pi_{4+k}(G/O)$. The existence of such elements, together with the surjectivity of the map $\psi_* : \Theta_{3+k} \to \pi_{3+k}(G/O)$, implies that the image of $\eta^*_{3+k} : \pi_{3+k}(Top/O) \to \pi_{4+k}(Top/O)$ contains an exotic $(4 + k)$-sphere corresponding to $\eta_{3+k} \circ x_{3+k}$. The proposition then follows from the stable splitting (3.3) and Lemma 4.0.2. □

4.2 Concordance Inertia Group of the Product of a Simply Connected 5-Manifold with a Sphere

From Section 3.2, we have $M \simeq M^{(3)} \cup_g \mathbb{D}^5$, where, by Theorem 3.2.1, the attaching map g is either null-homotopic, homotopic to η, or homotopic to $\widetilde{\eta}_{2^r}$. Therefore, by Lemma 4.0.2, the concordance inertia group $I_c(M \times \mathbb{S}^k)$ is given by

$$(4.3)\quad I_c(M \times \mathbb{S}^k) = \begin{cases} 0, & \text{if } g \simeq *; \\ Im\left(\pi_{4+k}(Top/O) \xrightarrow{\eta^*_{4+k}} \pi_{5+k}(Top/O)\right), & \text{if } g \simeq \eta; \\ Im\left([\Sigma^{3+k} M(\mathbb{Z}/2^r), Top/O] \xrightarrow{(\widetilde{\eta}_{2^r})^*} \pi_{5+k}(Top/O)\right), & \text{if } g \simeq \widetilde{\eta}_{2^r}. \end{cases}$$

Note that the image of η^*_{4+k} is described in Lemma 4.1.1. Thus, it remains to compute the image of $(\widetilde{\eta}_{2^r})^* : [\Sigma^{3+k}M(\mathbb{Z}/2^r), Top/O] \to \pi_{5+k}(Top/O)$. Throughout the proof of the following lemma, we frequently use the long exact sequence

(4.4)

$$\cdots \xrightarrow{\times 2^r} \pi_{4+k}(Top/O) \xrightarrow{(q_{4+k})^*} [\Sigma^{3+k}M(\mathbb{Z}/2^r), Top/O] \xrightarrow{(\Sigma^{3+k}i_{2^r})^*} \pi_{3+k}(Top/O) \xrightarrow{\times 2^r} \cdots,$$

induced from the cofiber sequence (2.12).

Lemma 4.2.1. *The image of the map* $(\widetilde{\eta}_{2^r})^* : [\Sigma^{3+k}M(\mathbb{Z}/2^r), Top/O] \to \pi_{5+k}(Top/O)$ *is described as follows:*

- *(a) For each* $r \geq 1$, *the image of the map* $(\widetilde{\eta}_{2^r})^* : [\Sigma^{3+k}M(\mathbb{Z}/2^r), Top/O] \to \pi_{5+k}(Top/O)$ *is*
  - *(i) trivial, for* $k = 1, 2, 3, 7, 8$ *or* 9*;*
  - *(ii)* $\mathbb{Z}/2$*, for* $k = 5$ *or* 10.
- *(b) The image of* $(\widetilde{\eta}_{2^r})^* : [\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \to \pi_9(Top/O)$ *is*
  - *(i)* $\mathbb{Z}/2 \oplus \mathbb{Z}/2$*, if* $r = 1$ *or* 2*;*
  - *(ii)* $\mathbb{Z}/2$*, if* $r \geq 3$.
- *(c) The image of* $(\widetilde{\eta}_{2^r})^* : [\Sigma^9 M(\mathbb{Z}/2^r), Top/O] \to \pi_{11}(Top/O)$ *is*
  - *(i)* $\mathbb{Z}/4$*, if* $r = 1$*;*
  - *(ii)* $\mathbb{Z}/2$*, if* $r \geq 2$.
- *(d) The image of* $(\widetilde{\eta}_{2^r})^* : [\Sigma^{14} M(\mathbb{Z}/2^r), Top/O] \to \pi_{16}(Top/O)$ *is*
  - *(a)* $\mathbb{Z}/2$*, if* $r = 1$*;*
  - *(b) zero, if* $r \geq 2$.

Proof. We proceed by examining each value of k separately to establish the result. The definition of $\widetilde{\eta}_{2^r}$ gives rise to the following commutative diagram:

$$\begin{tikzcd}
\pi_{4+k}(Top/O) \arrow[rr, "(q_{4+k})^*"] \arrow[dr, "\eta^*_{4+k}"'] & & {[\Sigma^{3+k}M(\mathbb{Z}/2^r), Top/O]} \arrow[dl, "(\widetilde{\eta}_{2^r})^*"] \\
& \pi_{5+k}(Top/O). &
\end{tikzcd} \tag{4.5}$$

For $k = 1$ and 7, the result follows immediately from the triviality of $\pi_{5+k}(Top/O)$. By applying the long exact sequence (4.4), we see that the groups $[\Sigma^5 M(\mathbb{Z}/2^r), Top/O]$ and $[\Sigma^{11}M(\mathbb{Z}/2^r), Top/O]_{(3)}$ are trivial. Consequently, the map

$$(\widetilde{\eta}_{2^r})^* : [\Sigma^{3+k}M(\mathbb{Z}/2^r), Top/O] \to \pi_{5+k}(Top/O)$$

is trivial for $k = 2$ and 8. This completes the cases $k = 2$ and 8.

For $k = 3$ and 9, the map $\eta^*_{4+k} : \pi_{4+k}(Top/O) \to \pi_{5+k}(Top/O)$ is trivial by Lemma 4.1.1. Furthermore, by the long exact sequence (4.4), the map $(q_{4+k})^* : \pi_{4+k}(Top/O) \to [\Sigma^{3+k}M(\mathbb{Z}/2^r), Top/O]$ is surjective. Combining these facts with the commutativity of the diagram (4.5), we conclude that the result holds for these cases.

Cases $k = 5$ and 10:

In these cases, Lemma 4.1.1 asserts that the image of the map $\eta^*_{4+k} : \pi_{4+k}(Top/O) \to \pi_{5+k}(Top/O)$ is $\mathbb{Z}/2$. Moreover, the exact sequence (4.4) shows that the map $q_9^* : \pi_9(Top/O) \to [\Sigma^8 M(\mathbb{Z}/2^r), Top/O]$ is injective, with $[\Sigma^8 M(\mathbb{Z}/2^r), Top/O]_{(3)}$ trivial, and that the map $q^*_{14} : \pi_{14}(Top/O) \to [\Sigma^{13}M(\mathbb{Z}/2^r), Top/O]$ is an isomorphism. Together with the commutativity of the diagram (4.5), these facts establish the result for $k = 5$ and $k = 10$.

Case $k = 4$:

Since $[\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \cong [\Sigma^7 M(\mathbb{Z}/2^r), PL/O]$ and $\pi_9(Top/O) \cong \pi_9(PL/O)$, we may identify the image of the map $(\widetilde{\eta}_{2^r})^* : [\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \to \pi_9(Top/O)$ with that of $(\widetilde{\eta}_{2^r})^* : [\Sigma^7 M(\mathbb{Z}/2^r), PL/O] \to \pi_9(PL/O)$, and hence it suffices to compute the image of the latter. First, consider the case $r = 1$. From [14, Page 29, Corollary

1.6.12], we have the following split short exact sequence:

(4.6)

$$0 \to \operatorname{Ext}(\mathbb{Z}/2, \pi_8(PL/O)) \to [\Sigma^7 M(\mathbb{Z}/2), PL/O] \to \operatorname{Hom}(\mathbb{Z}/2, \pi_7(PL/O)) \to 0.$$

Furthermore, the map $(\widetilde{\eta}_2)^* : [\Sigma^7 M(\mathbb{Z}/2), PL/O] \to \pi_9(PL/O)$ fits into the following diagram:

$$\begin{tikzcd}
{[\Sigma^7 M(\mathbb{Z}/2), PL/O]} \arrow[rr, "(\widetilde{\eta}_2)^*"] \arrow[dr, "(\xi)^*"', "\cong"] & & \pi_9(PL/O) \\
& \Gamma_9(PL/O), \arrow[ur, hook] &
\end{tikzcd}$$

where $\Gamma_9(PL/O)$ denotes the image of the map $\pi_9(PL/O^{(8)}) \to \pi_9(PL/O^{(9)})$ induced by the inclusion $PL/O^{(8)} \hookrightarrow PL/O^{(9)}$, and the map $\xi^* : [\Sigma^7 M(\mathbb{Z}/2), PL/O] \to \Gamma_9(PL/O)$ is an isomorphism by [14, Page 264, Theorem 8.3.7]. Using the identification $[\Sigma^7 M(\mathbb{Z}/2), PL/O] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ from (4.6), we conclude that the image of $(\widetilde{\eta}_2)^* : [\Sigma^7 M(\mathbb{Z}/2), PL/O] \to \pi_9(PL/O)$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. This completes the proof for the case $k = 4$ and $r = 1$.

Assume now that $r \geq 2$ and consider the following diagram of cofiber sequences:

(4.7)

$$\begin{tikzcd}
\mathbb{S}^7 \arrow[r, "\times 2^r"] & \mathbb{S}^7 \arrow[r, hook, "\Sigma^7 i_{2^r}"] & \Sigma^7 M(\mathbb{Z}/2^r) \arrow[r, "q_8"] & \mathbb{S}^8 \arrow[r, "\times 2^r"] & \mathbb{S}^8 \\
\mathbb{S}^7 \arrow[u, equal] \arrow[r, "\times 2"'] & \mathbb{S}^7 \arrow[u, "\times 2^{r-1}"'] \arrow[r, hook, "\Sigma^7 i_{2^r}"'] & \Sigma^7 M(\mathbb{Z}/2) \arrow[u, "\Sigma^7 B(\chi_r^1)"'] \arrow[r, "q_8"'] & \mathbb{S}^8 \arrow[u, equal] \arrow[r, "\times 2"'] & \mathbb{S}^8, \arrow[u, "\times 2^{r-1}"']
\end{tikzcd}$$

where $\Sigma^7 B(\chi_r^1) : \Sigma^7 M(\mathbb{Z}/2) \to \Sigma^7 M(\mathbb{Z}/2^r)$ is the map defined in (2.13), satisfying the relations (2.15). This diagram induces a diagram of long exact sequences along PL/O.

$$\begin{tikzcd}
{} \arrow[r, "0"] & \pi_8(PL/O) \arrow[r, "(q_8)^*"] \arrow[d, equal] & {[\Sigma^7 M(\mathbb{Z}/2^r), PL/O]} \arrow[r, "(\Sigma^7 i_{2^r})^*"] \arrow[d, "(\Sigma^7 B(\chi_r^1))^*"'] & \pi_7(PL/O) \arrow[r, "\times 2^r"] \arrow[d, "\times 2^{r-1}"'] & \pi_7(PL/O) \arrow[d, equal] \\
{} \arrow[r, "0"'] & \pi_8(PL/O) \arrow[r, "(q_8)^*"'] & {[\Sigma^7 M(\mathbb{Z}/2), PL/O] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2} \arrow[r, "(\Sigma^7 i_{2^r})^*"'] & \pi_7(PL/O) \arrow[r, "\times 2"'] & \pi_7(PL/O),
\end{tikzcd}$$

This commutative diagram implies the following:

- If $r \geq 3$, then the image of

  $$(\Sigma^7 B(\chi_r^1))^* : [\Sigma^7 M(\mathbb{Z}/2^r), PL/O] \to [\Sigma^7 M(\mathbb{Z}/2), PL/O]$$

  is $\mathbb{Z}/2$.

- If $r = 2$, then the map

  $$(\Sigma^7 B(\chi_r^1))^* : [\Sigma^7 M(\mathbb{Z}/4), PL/O] \to [\Sigma^7 M(\mathbb{Z}/2), PL/O]$$

  has image $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

By combining the previous observations with the commutative diagram induced by the relation (2.14),

$$\begin{tikzcd}
{[\Sigma^7 M(\mathbb{Z}/2^r), PL/O]} \arrow[rr, "(\Sigma^7 B(\chi_r^1))^*"] \arrow[dr, "(\widetilde{\eta}_{2^r})^*"'] & & {[\Sigma^7 M(\mathbb{Z}/2), PL/O] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2} \arrow[dl, "(\widetilde{\eta}_2)^*"] \\
& \pi_9(PL/O) &
\end{tikzcd}$$

completes the proof for the case $k = 4$ and $r \geq 2$.

Case $k = 6$:

In this case, it follows from the long exact sequence (4.4) with $r = 1$ and [37, Lemma 2.3] that there is a non-split exact sequence

$$\begin{aligned} 0 \to \mathbb{Z}/2\,(\subset \pi_{10}(Top/O)) &\xrightarrow{(q_{10})^*} [\Sigma^9 M(\mathbb{Z}/2), Top/O] \\ &\cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{(\Sigma^9 i_2)^*} \pi_9(Top/O) \to 0, \end{aligned}$$

where the map $(q_{10})^* : \pi_{10}(Top/O) \to [\Sigma^9 M(\mathbb{Z}/2), Top/O]$ sends the exotic 10-sphere corresponding to the generator $\mu \circ \eta$ to an element of order 2 in the $\mathbb{Z}/4$ summand. Further, from Lemma 4.1.1, the image of $\eta_{10}^* : \pi_{10}(Top/O) \to \pi_{11}(Top/O)$ is $\mathbb{Z}/2$. Combining these facts in the diagram (4.5) implies that the image of $(\widetilde{\eta}_2)^* : [\Sigma^9 M(\mathbb{Z}/2), Top/O] \to \pi_{11}(Top/O)$ is $\mathbb{Z}/4$. This completes the proof of the statement (c)(i).

Assume $r \geq 2$ and consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & {[\Sigma^9 M(\mathbb{Z}/2^r), Top/O]} \arrow[r, "(\Sigma^9 i_{2^r})^*"] \arrow[d, "(\Sigma^9 B({}^1\chi_r))^*"'] & \pi_9(Top/O) \arrow[r, "0"] \arrow[d, "0"] & {} \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & {[\Sigma^9 M(\mathbb{Z}/2), Top/O] \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2} \arrow[r, "(\Sigma^9 i_2)^*"'] & \pi_9(Top/O) \arrow[r, "0"'] & {}
\end{tikzcd},$$

where the first and second rows are induced by the cofiber sequences of the mapping cones $\Sigma^9 M(\mathbb{Z}/2^r)$ and $\Sigma^9 M(\mathbb{Z}/2)$, respectively, as in (4.7).

Suppose now that $[\Sigma^9 M(\mathbb{Z}/2^r), Top/O] \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Then, from the commutativity of above diagram, it follows that the composition $(\Sigma^9 i_2)^* \circ (\Sigma^9 B({}^1\chi_r))^*$, when restricted to the $\mathbb{Z}/4$ summand of $[\Sigma^9 M(\mathbb{Z}/2^r), Top/O]$, is non-trivial. However, this contradicts the right most vertical map being zero, and hence,

$$[\Sigma^9 M(\mathbb{Z}/2^r), Top/O] \cong \bigoplus_{i=1}^{4} \mathbb{Z}/2.$$

Applying the same argument as in the case $r = 1$ using the diagram (4.5), we conclude that the image of the map

$$(\widetilde{\eta}_{2^r})^* : [\Sigma^9 M(\mathbb{Z}/2^r), Top/O] \to \pi_{11}(Top/O)$$

is $\mathbb{Z}/2$. This completes the proof of part (c)(ii).

Case $k = 10$:

We first consider $r = 1$. By combining [125, Proposition 1.7] and the result on [61, Page 45], computing the image of the map $(\widetilde{\eta}_2)^* : [\Sigma^{14} M(\mathbb{Z}/2), Top/O] \to \pi_{16}(Top/O)$ reduces to evaluating the Toda bracket $\langle \eta, 2, \kappa \rangle$ or $\langle \eta, 2, \kappa + \sigma^2 \rangle$, where $\pi_{14}^s = \mathbb{Z}/2\{\kappa\} \oplus \mathbb{Z}/2\{\sigma^2\}$. According to [103, Table A3.3], both Toda brackets are nontrivial. Hence, the image of $(\widetilde{\eta}_2)^*$ is $\mathbb{Z}/2$.

Since $\eta_{15}^* : \pi_{15}(Top/O) \to \pi_{16}(Top/O)$ is trivial and $\eta_{15} = q_{15} \circ \widetilde{\eta}_{2^r}$, the restriction of $(\widetilde{\eta}_{2^r})^* : [\Sigma^{14}M(\mathbb{Z}/2^r), Top/O] \to \pi_{16}(Top/O)$ to $\pi_{15}(Top/O)$ is trivial.

We note that for $r \geq 2$, there exists a unique element $B(\chi_r^1) : M(\mathbb{Z}/2) \to M(\mathbb{Z}/2^r)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{S}^{14} \arrow[r, "\times 2^r"] & \mathbb{S}^{14} \arrow[r, hook, "\Sigma^{14} i_{2^r}"] & \Sigma^{14}M(\mathbb{Z}/2^r) \arrow[r, "q_{15}"] & \mathbb{S}^{15} \arrow[r, "\times 2^r"] & \mathbb{S}^{15} \\
\mathbb{S}^{14} \arrow[u, equal] \arrow[r, "\times 2"'] & \mathbb{S}^{14} \arrow[u, "\times 2^{r-1}"'] \arrow[r, hook, "\Sigma^{14} i_2"'] & \Sigma^{14}M(\mathbb{Z}/2) \arrow[u, dashed, "\Sigma^{14}B(\chi_r^1)"'] \arrow[r, "q_{15}"'] & \mathbb{S}^{15} \arrow[u, equal] \arrow[r, "\times 2"'] & \mathbb{S}^{15}, \arrow[u, "\times 2^{r-1}"']
\end{tikzcd}$$

which induces the following commutative diagram of long exact sequences along Top/O.

$$\begin{tikzcd}
{} \arrow[r, "\times 2^r"] & \pi_{15}(Top/O) \arrow[r, "(q_{15})^*"] \arrow[d, equal] & {[\Sigma^{14}M(\mathbb{Z}/2^r), Top/O]} \arrow[r, "(\Sigma^{14} i_{2^r})^*"] \arrow[d, "(\Sigma^{14}B(\chi_r^1))^*"'] & \pi_{14}(Top/O) \arrow[r, "0"] \arrow[d, "0"'] & \pi_{14}(Top/O) \arrow[d, equal] \\
{} \arrow[r, "\times 2"'] & \pi_{15}(Top/O) \arrow[r, "(q_{15})^*"'] & {[\Sigma^{14}M(\mathbb{Z}/2), Top/O]} \arrow[r, "(\Sigma^{14} i_2)^*"'] & \pi_{14}(Top/O) \arrow[r, "0"'] & \pi_{14}(Top/O).
\end{tikzcd}$$

This diagram shows that if for any $a \in [\Sigma^{14}M(\mathbb{Z}/2^r), Top/O]$ satisfies $(\Sigma^{14} i_{2^r})^*(a) \neq 0$, then $(\Sigma^{14}B(\chi_r^1))^*(a) = 0$. Combining this with the fact that $(q_{15} \circ \widetilde{\eta}_{2^r})^* : \pi_{15}(Top/O) \to \pi_{16}(Top/O)$ is a trivial map, it follows from $\widetilde{\eta}_{2^r} = \Sigma^{14}B(\chi_r^1) \circ \widetilde{\eta}_2$ that the map $(\widetilde{\eta}_{2^r})^* : [\Sigma^{14}M(\mathbb{Z}/2^r), Top/O] \to \pi_{16}(Top/O)$ is trivial for all $r \geq 2$.

This concludes the proof of the lemma. ☐

The following theorem gives a complete description of the concordance inertia group of $M \times \mathbb{S}^k$.

Theorem 4.2.2. *Let M be a simply connected, closed, smooth 5-manifold whose homology is of the form (3.4). Then:*

(1) If M is a spin manifold, then $I_c(M \times \mathbb{S}^k) = 0$, for all $k \geq 1$.

(2) Suppose M is a non-spin manifold.

(a) $I_c(M \times \mathbb{S}^k) = 0$, for $k = 2, 3, 7, 8,$ or 9.

(b) For $k = 4, 5, 6,$ or 10,

(i) If M satisfies (3.8), then $I_c(M \times \mathbb{S}^k) = \mathbb{Z}/2$.

(ii) If M satisfies (3.9), then

A.

$$I_c(M \times \mathbb{S}^4) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2, & \text{if } r = 1 \text{ or } 2; \\ \mathbb{Z}/2, & \text{if } r \geq 3; \end{cases}$$

B. $I_c(M \times \mathbb{S}^5) = \mathbb{Z}/2$;

C.

$$I_c(M \times \mathbb{S}^6) = \begin{cases} \mathbb{Z}/4, & \text{if } r = 1; \\ \mathbb{Z}/2, & \text{if } r \geq 2; \end{cases}$$

D. $I_c(M \times \mathbb{S}^{10}) = \mathbb{Z}/2$.

Here, r denotes the smallest positive integer such that $\mathbb{Z}/2^r$ appears in the torsion subgroup of $H_2(M;\mathbb{Z})$.

Proof. The conclusions in (1) and (2) follow from (4.3) by applying Proposition 3.7, the stable splittings (3.8) and (3.9), together with the computations from Lemmas 4.1.1 and 4.2.1. $\square$

4.3 Concordance Inertia Group of the Product of a Simply Connected 6-Manifold with a Sphere

We observe from (3.11) and (3.14) that the cofiber sequence $\mathbb{S}^5 \xrightarrow{\phi_6} M^{(5)} \overset{\tilde{i}}{\hookrightarrow} M \xrightarrow{f_M} \mathbb{S}^6 \cdots$ induces the following long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & {[\Sigma^{k+1}M^{(5)}, Top/O]} \arrow[r, "(\Sigma^{k+1}\phi_6)^*"] & \pi_{6+k}(Top/O) \arrow[r, "(\Sigma^k f_M)^*"] & {[\Sigma^k M, Top/O]} \arrow[dlll, "(\Sigma^k\tilde{i})^*"] & \\
{[\Sigma^k M^{(5)}, Top/O]} \arrow[r, "(\Sigma^k\phi_6)^*"] & \pi_{5+k}(Top/O) \arrow[r] & {[\Sigma^{k-1}M, Top/O]} \arrow[r] & \cdots .
\end{tikzcd} \tag{4.8}$$

Now combining Lemma 4.0.2 and (4.8), we get

$$I_c(M \times \mathbb{S}^k) = Im\left([\Sigma^{k+1}M^{(5)}, Top/O] \xrightarrow{(\Sigma^{k+1}\phi_6)^*} \pi_{6+k}(Top/O)\right). \tag{4.9}$$

We now compute the image of each component of $\Sigma^{k+1}\phi_6$ given in (3.14) along Top/O.

Lemma 4.3.1.

(a) Let $\nu_l : \mathbb{S}^{l+3} \to \mathbb{S}^l$ *be the generator of* $\pi_{l+3}(\mathbb{S}^l)_{(2)} = (\pi_3^s)_{(2)}$*. Then the image of the induced map*

$$(\nu_l)^* : \pi_l(Top/O) \to \pi_{l+3}(Top/O)$$

is given in the following table:

$l =$	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*
$Im(\nu_l^*)$	*0*	*0*	*0*	0	*0*	*0*	*0*	$\mathbb{Z}/2$	*0*	*0*	*0*	$\mathbb{Z}/2$	$\mathbb{Z}/2$

(b) Let $\alpha_1 : \mathbb{S}^{3+l} \to \mathbb{S}^l$ *denote the generator of* $\pi_{3+l}(\mathbb{S}^l)_{(3)} = (\pi_3^s)_{(3)}$*. Then the image of the map* $\alpha_1^* : \pi_l(Top/O) \to \pi_{l+3}(Top/O)$ *is trivial for all* $7 \leq l \leq 19$ *with* $l \neq 10$*, and equals* $\mathbb{Z}/3$ *when* $l = 10$.

Proof. Since $\pi_{12}(Top/O)$ and $\pi_{16}(Top/O)_{(3)}$ are both trivial, the result follows immediately for $l = 9, 12$, and 13.

For $l = 7$ and 11, consider the commutative diagram

$$\begin{tikzcd}
 & \pi_l(Top/O) \arrow[r, "\psi_*"] \arrow[d, "\nu_l^*"'] & 0 \arrow[d] \\
0 \arrow[r] & \pi_{l+3}(Top/O) \arrow[r, "\psi_*"'] & \pi_{l+3}(G/O),
\end{tikzcd}$$

where the map $\psi_* : \pi_{l+3}(Top/O) \to \pi_{l+3}(G/O)$ is injective, since $\pi_{l+4}(G/Top) = 0$ for $l = 7$ and 11. It then follows from the commutativity of the diagram that the map

$$\nu_l^* : \pi_l(Top/O) \to \pi_{l+3}(Top/O)$$

is trivial for $l = 7$ and 11.

Since $\pi_l(Top/O) \cong \pi_l(PL/O)$ for $l \geq 5$, it suffices to show that the map $\nu_8^* : \pi_8(PL/O) \to \pi_{11}(PL/O)$ is trivial. We use the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi_8(O) \arrow[r] \arrow[d, "\nu_8^*"'] & \pi_8(PL) \arrow[r] \arrow[d, "\nu_8^*"] & \pi_8(PL/O) \arrow[r] \arrow[d, "\nu_8^*"] & 0 \\
0 \arrow[r] & \pi_{11}(O) \arrow[r] & \pi_{11}(PL) \arrow[r] & \pi_{11}(PL/O) \arrow[r] & 0,
\end{tikzcd}$$

where the rows arise from the fibration $PL/O \to BO \to BPL$, and each forms a short exact sequence, as established in [51]. Now it follows from the commutativity of the above diagram that if the map $\nu_8^* : \pi_8(PL) \to \pi_{11}(PL)$ is trivial, then so is the map $\nu_8^* : \pi_8(PL/O) \to \pi_{11}(PL/O)$. To show that $\nu_8 : \pi_8(PL) \to \pi_{11}(PL)$ is trivial, consider the following commutative diagram induced from the fiber sequence $G/PL \to BPL \to BG$.

$$\begin{tikzcd}
 & \pi_8(PL) \arrow[r, "\cong"] \arrow[d, "\nu_8^*"'] & \pi_8^s \arrow[d, "0"] & \\
\pi_{12}(G/PL) \arrow[r] & \pi_{11}(PL) \arrow[r] & \pi_{11}^s \arrow[r] & 0,
\end{tikzcd}$$

where the isomorphism $\pi_n(PL) \cong \pi_n(G)$ holds for $n \equiv 0 \pmod 4$ by [59, Lemma 4.2], and $\nu_8^* : \pi_8^s \to \pi_{11}^s$ is a zero map from [125, Theorem 14.1]. Since $\pi_{11}(PL) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ [25] and the kernel of the map $\pi_{11}(PL) \to \pi_{11}^s$ is free abelian, $\nu_8^* : \pi_8(PL) \to \pi_{11}(PL)$ is a zero map.

Since $\pi_{13}(Top/O)_{(2)} = 0$, the map $\nu_l^* : \pi_l(Top/O) \to \pi_{l+3}(Top/O)$ is trivial for $l = 10$ and 13.

We observe that the exotic sphere in $\pi_{14}(Top/O)$ corresponds to either the generator κ or $\kappa + \sigma^2$ in π_{14}^s [61, Page 45]. Since $\nu \circ \sigma = 0$ by [125, Theorem 14.1], and $\nu\kappa$ lies in $\pi_{17}^s/Im(J)$ [103, Table A3.3], it follows that the image of the map

$$\nu_{14}^* : \pi_{14}(Top/O) \to \pi_{17}(Top/O)$$

is $\mathbb{Z}/2$.

For $l = 15$, we observe from the left portion of the following commutative diagram

$$\begin{tikzcd}
 & \pi_{15}(Top/O) \arrow[r, "\psi_*"] \arrow[d, "\nu_{15}^*"'] & \pi_{15}(G/O) \arrow[d, "\nu_{15}^*"] & \pi_{15}^s \arrow[l, "j_*"'] \arrow[d, "\nu_{15}^*"] \\
0 \arrow[r] & \pi_{18}(Top/O) \arrow[r, "\psi_*"'] & \pi_{18}(G/O) & \pi_{18}^s \arrow[l, "j_*"]
\end{tikzcd}$$

that if the map $\nu_{15}^* : \pi_{15}(G/O) \to \pi_{18}(G/O)$ is trivial, then the image of $\nu_{15}^* : \pi_{15}(Top/O) \to \pi_{18}(Top/O)$ must also be trivial. Since $j_*(\eta\kappa)$ is nontrivial in $\pi_{15}(G/O)$ [103, Table A3.3], and $(\nu_{15})_{(2)}^*(\eta\kappa) = 0$ by [125, Theorem 14.1], it follows that the image of the map

$$\nu_{15}^* : \pi_{15}(G/O) \to \pi_{18}(G/O)$$

is trivial.

We note that $\pi_{16}(Top/O) \cong \pi_{16}^s/Im(J) \cong \mathbb{Z}/2\{[\eta^*]\}$ [103, Table A3.3] and $\langle \eta, \sigma, 2\rangle = \eta^* \bmod \eta\rho$ [125, 91]. Using the properties of the Toda brackets [125, Page 33], we have $\nu_{16}^*\eta^* = 0$. This implies that $\nu_{16}^* : \pi_{16}(Top/O) \to \pi_{19}(Top/O)$ is a zero map.

For $l = 17$, we have

$$\pi_{17}(Top/O) \cong bP_{18} \oplus \pi_{17}^s/Im(J) \cong \pi_{18}(G/Top) \oplus \pi_{17}(G/O) \cong \bigoplus_{i=1}^{4} \mathbb{Z}/2.$$

Since the composite $\nu_{17}^* \circ \omega_* : \pi_{18}(G/Top) \to \pi_{20}(Top/O)$ is trivial, and $\omega_* : \pi_{18}(G/Top) \to \pi_{17}(Top/O)$ is injective, it follows that the restriction of $\nu_{17}^* : \pi_{17}(Top/O) \to \pi_{20}(Top/O)$ to $\omega_*(\pi_{18}(G/Top))$ is trivial. Moreover, $\pi_{17}^s/Im(J)$ decomposes as $\pi_{17}^s/Im(J) \cong \mathbb{Z}/2\{[\eta\eta^*]\} \oplus \mathbb{Z}/2\{[\nu\kappa]\} \oplus \mathbb{Z}/2\{[\bar{\mu}]\}$, where $\nu\kappa = \langle \eta\kappa, \eta, 2\rangle$ [91, Page 81] and $\bar{\mu} \in \langle \eta, 16\rho, 2\rangle$ [90]. The result then follows from the properties of Toda brackets and the relation $\eta \circ \nu = 0$ [125].

Since $\pi_{18}(Top/O) \cong \pi_{18}(G/O) \cong \pi_{18}^s \cong \mathbb{Z}/2\{\eta\bar{\mu}\} \oplus \mathbb{Z}/8\{\nu^*\}$ [125], and $\nu \circ \nu^* = \sigma^3$ with $j_*(\sigma^3) \neq 0$ in $\pi_{21}(G/O)$ [103, Table A3.3], together with the relation $(\bar{\mu} \circ \eta) \circ \nu = \bar{\mu} \circ (\eta \circ \nu) = 0$ [20, Proposition 1.3], the image of the map $\nu_{18}^* : \pi_{18}(Top/O) \to \pi_{21}(Top/O)$

is $\mathbb{Z}/2$.

From [25, 27], we obtain $\pi_{19}(Top/O) \cong bP_{20} \oplus \pi_{19}^s/Im(J)$. Now [104, Theorem B] indicates that the restriction of the Bredon pairing $\rho_{19,3} : \pi_{19}(Top/O) \times \pi_3^s \to \pi_{22}(Top/O)$ to bP_{20} is trivial.

By [20, Corollary 2.2], there exists a map $\mathcal{G} : \pi_{19}^s/Im(J) \times \pi_3^s \to \pi_{22}^s/Im(J)$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi_{19}(Top/O) \times \pi_3^s \arrow[r, "p' \times Id"] \arrow[d, "\rho_{19,3}"'] & \pi_{19}^s/Im(J) \times \pi_3^s \arrow[r] \arrow[d, "\mathcal{G}"] & 0 \\
\pi_{22}(Top/O) \arrow[r, "\cong", "p'"'] & \pi_{22}^s/Im(J), &
\end{tikzcd}$$

and $\mathcal{G}([\bar{\sigma}], \nu) = [\bar{\sigma} \circ \nu]$, which is nontrivial by [103, Table A3.3]. This implies that the image of the restriction of $\rho_{19,3} : \pi_{19}(Top/O) \times \pi_3^s \to \pi_{22}(Top/O)$ to $\pi_{19}^s/Im(J)$ is $\mathbb{Z}/2$.

Therefore, the image of $\rho_{19,3} : \pi_{19}(Top/O) \times \pi_3^s \to \pi_{22}(Top/O)$ is $\mathbb{Z}/2$, and so the map $\nu_{19}^* : \pi_{19}(Top/O) \to \pi_{22}(Top/O)$ has image $\mathbb{Z}/2$.

We note that $\pi_l(Top/O)_{(3)} = 0$ for all $7 \leq l \leq 19$, except when $l = 10$ or $l = 13$. Hence, the map $\alpha_1^* : \pi_l(Top/O) \to \pi_{l+3}(Top/O)$ is trivial for all such l, except when $l = 10$. Since $\pi_t(Top/O) \cong \pi_t(G/O) \cong \pi_t^s$ for $t = 10, 13$, and the image of $\alpha_1^* : \pi_{10}^s \to \pi_{13}^s$ is $\mathbb{Z}/3$ [125, Page 189], the result follows for $k = 10$. □

Fact 4.3.2. *[38, Remark 1.2] The non-triviality of the map $s\alpha_1$ is determined by the divisibility of the first Pontryagin class $p_1(M)$ by 3. Thus, if $3 \mid p_1(M)$, then s must be a multiple of 3.*

For convenience, we restate Lemma 4.1.1 and Lemma 4.3.1 in a form suited to the current context.

Lemma 4.3.3. *The image of $\eta_{5+k}^* : \pi_{5+k}(Top/O) \to \pi_{6+k}(Top/O)$ is*

(a) zero, if $k = 1, 2, 6, 7, 8$ or 10;

(b) $\mathbb{Z}/2$ if $k = 3, 4, 5$ or 9.

Lemma 4.3.4.

(a) The image of $(2\nu_{3+k})^* : \pi_{3+k}(Top/O) \to \pi_{6+k}(Top/O)$ *is zero for all* $1 \leq k \leq 10$.

(b) The map $\alpha_1^* : \pi_{3+k}(Top/O) \to \pi_{6+k}(Top/O)$ *has trivial image for* $1 \leq k \leq 10$ *with* $k \neq 7$ *and image* $\mathbb{Z}/3$ *for* $k = 7$.

Lemma 4.3.5. *The image of the map*

$$(\Sigma^{k+1}\pi)^* : [\Sigma^{k+1}\mathbb{C}P^2, Top/O] \to \pi_{6+k}(Top/O)$$

is zero for all $1 \leq k \leq 10$ *with* $k \neq 7$, *and is* $\mathbb{Z}/3$ *for* $k = 7$.

Using the Lemma 4.3.4, we derive the image of the attaching map corresponding to one of the Moore spectrum components.

Corollary 4.3.6. *Let* r *be a positive integer and* $1 \leq k \leq 10$.

(i) The image of $(\Sigma^{k+1}i' \circ 2\nu)^* : [\Sigma^{k+1}\mathbb{C}P^2, Top/O]_{(2)} \to \pi_{6+k}(Top/O)_{(2)}$ *is trivial for all* k.

(ii) The map $(\Sigma^{k+3}i_{2^r} \circ 2\nu)^* : [\Sigma^{3+k}M(\mathbb{Z}/2^r), Top/O] \to \pi_{6+k}(Top/O)$ *has image zero for all* k.

(iii) The map $(\Sigma^{k+1}i^r \circ \Sigma^{k+3}i_{2^r} \circ 2\nu)^* : [\Sigma^{k+1}C_{i_{2^r}\circ\eta}, Top/O] \to \pi_{6+k}(Top/O)$ *is trivial for all* k.

(iv) For all $1 \leq k \leq 10$, *the image of the map* $(\Sigma^{k+3}i_{3^r}\circ\alpha_1)^* : [\Sigma^{3+k}M(\mathbb{Z}/3^r), Top/O] \to \pi_{6+k}(Top/O)$ *is given by:*

  (a) zero, if $k \neq 7$;

  (b) $\mathbb{Z}/3$, *if* $k = 7$.

(v) The image of $(\Sigma^{k+1}i' \circ \alpha_1)^* : [\Sigma^{k+1}\mathbb{C}P^2, Top/O]_{(3)} \to \pi_{6+k}(Top/O)_{(3)}$ *is:*

  (a) trivial, if $k \neq 7$;

(b) $\mathbb{Z}/3$*, if* $k = 7$*.*

Proof. Statements (i), (ii), and (iii) follow from Lemma 4.3.4 (a).

Statements (iv)(a) and (v)(a) follow from the fact that the map

$$(\alpha_1)^* : \pi_{3+k}(Top/O) \to \pi_{6+k}(Top/O)$$

is trivial for all $1 \leq k \leq 10$, except when $k = 7$, as stated in Lemma 4.3.4 (b).

For $k = 7$, consider the long exact sequences induced by the cofiber sequences $\mathbb{S}^0 \xrightarrow{\times 3^r} \mathbb{S}^0 \overset{i_{3^r}}{\hookrightarrow} M(\mathbb{Z}/3^r)$ and $\mathbb{S}^3 \xrightarrow{\eta} \mathbb{S}^2 \overset{i'}{\hookrightarrow} \mathbb{C}P^2$, after applying $[\_, Top/O]$. From these, it follows that the images of both maps $(\Sigma^8 i_{3^r})^* : [\Sigma^{10} M(\mathbb{Z}/3^r), Top/O] \to \Theta_{10}$ and $(\Sigma^8 i')^* : [\Sigma^8 \mathbb{C}P^2, Top/O] \to \Theta_{10}$ are equal to $\mathbb{Z}/3 \subset \Theta_{10}$. Consequently, Lemma 4.3.4 (b) establishes statements (iv) (b) and (v) (b). □

Lemma 4.3.7. *For any positive integer* r*, the image of the map*

$$(\Sigma^{k+4} i_{2^r} \circ \eta^2)^* : [\Sigma^{k+4} M(\mathbb{Z}/2^r), Top/O] \to \pi_{6+k}(Top/O)$$

is

(a) zero, for $1 \leq k \leq 10, \;\; k \neq 5$*;*

(b) $\mathbb{Z}/2$*, for* $k = 5$*.*

Proof. By Corollary 4.1.3, the map $(\eta^2)^* : \pi_{4+k}(Top/O) \to \pi_{6+k}(Top/O)$ is trivial for $1 \leq k \leq 10$, $k \neq 5$, which proves (i) (a). For $k = 5$, the same corollary states that $(\eta^2)^* : \pi_9(Top/O) \to \pi_{11}(Top/O)$ has image $\mathbb{Z}/2$. Moreover, by (4.4), the map $(\Sigma^9 i_{2^r})^* : [\Sigma^9 M(\mathbb{Z}/2^r), Top/O] \to \pi_9(Top/O)$ is surjective, so the image of $(\Sigma^9 i_{2^r} \circ \eta^2)^* : [\Sigma^9 M(\mathbb{Z}/2^r), Top/O] \to \pi_{11}(Top/O)$ is also $\mathbb{Z}/2$. □

Lemma 4.3.8. *Let* $r \geq 1$ *be an integer. The image of the map*

$$(\widetilde{\eta^2_{2^r}})^* : [\Sigma^{k+3} M(\mathbb{Z}/2^r), Top/O] \to \pi_{6+k}(Top/O)$$

is as follows:

(i) zero, for all $1 \leq k \leq 10$*,* $k \neq 4, 5$*.*

(ii) For $k = 4$*:*

  (a) $\mathbb{Z}/2$*, if* $r = 1$ *or* 2*;*

  (b) zero, if $r \geq 3$*.*

(iii) $\mathbb{Z}/2$*, for* $k = 5$*.*

Proof. From Lemma 4.2.1, the map $(\widetilde{\eta}_{2^r})^* : [\Sigma^{k+3}M(\mathbb{Z}/2^r), Top/O] \to \pi_{5+k}(Top/O)$ is trivial for $k = 1, 2, 3, 7, 8, 9$. Since $\widetilde{\eta^2_{2^r}} = \widetilde{\eta}_{2^r} \circ \eta_{5+k}$, the map

$$(\widetilde{\eta^2_{2^r}})^* : [\Sigma^{k+3}M(\mathbb{Z}/2^r), Top/O] \to \pi_{6+k}(Top/O)$$

is also trivial for these values of k. By Lemma 4.3.3, the map $\eta^*_{5+k} : \pi_{5+k}(Top/O) \to \pi_{6+k}(Top/O)$ is trivial for $k = 6$ and 10. Therefore, $(\widetilde{\eta^2_{2^r}})^*$ is also trivial for these values of k.

Consider $k = 4$ and assume $r = 1$ or 2. Then $[\Sigma^7 M(\mathbb{Z}/2^r), Top/O]$ fits into the short exact sequence

$$0 \to \mathbb{Z}/2 \to [\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \to \mathbb{Z}/2^r \to 0. \tag{4.10}$$

Since $[\Sigma^7 M(\mathbb{Z}/2^r), G/O] = \mathbb{Z}/2^r \oplus \mathbb{Z}/2$ for any $r \geq 1$ and the map

$$\psi_* : [\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \to [\Sigma^7 M(\mathbb{Z}/2^r), G/O]$$

is injective, the short exact sequence (4.10) splits. Thus,

$$[\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \cong [\Sigma^7 M(\mathbb{Z}/2^r), G/O] = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2, & \text{if } r = 1; \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2, & \text{if } r = 2. \end{cases}$$

Since the map $\eta^* : \pi_8(G/O) \to \pi_9(G/O)$ is surjective and $q_8 \circ \widetilde{\eta}_2 = \eta$, the map $(\widetilde{\eta}_2)^* : [\Sigma^7 M(\mathbb{Z}/2), G/O] \to \pi_9(G/O)$ is also surjective. For $r = 1$ and 2, consider the commutative diagram:

$$\begin{tikzcd}
{[\Sigma^7 M(\mathbb{Z}/2^r), Top/O]} \arrow[r, "(\widetilde{\eta}_{2^r})^*"] \arrow[d, "\psi_*"', "\cong"] & \pi_9(Top/O) \arrow[d, two heads, "\psi_*"] \\
{[\Sigma^7 M(\mathbb{Z}/2^r), G/O]} \arrow[r, "(\widetilde{\eta}_{2^r})^*"'] & \pi_9(G/O).
\end{tikzcd}$$

Using Lemma 4.2.1 and the diagram above, it follows that the image of $(\widetilde{\eta}_{2^r})^* : [\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \to \pi_9(Top/O)$ is $\mathbb{Z}/2\{[\Sigma_{\nu^3}]\} \oplus \mathbb{Z}/2\{[\Sigma_\mu]\}$. Since $\eta \circ \nu = 0$ [125, Theorem 14.1], $\eta \circ \mu \in \pi^s_{10}/Im(J)$ [103, Theorem 1.1.14] and $(\widetilde{\eta^2_{2^r}})^* = \eta^*_9 \circ (\widetilde{\eta}_{2^r})^*$, the image of $(\widetilde{\eta^2_{2^r}})^* : [\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \to \pi_{10}(Top/O)$ is $\mathbb{Z}/2\{[\Sigma_{\eta\circ\mu}]\}$ for $r = 1$ and 2.

For $r \geq 3$, Lemma 4.2.1 (b) shows that the map $(\widetilde{\eta}_{2^r})^* : [\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \to \pi_8(Top/O)$ has image $\mathbb{Z}/2\{[\Sigma_{\nu^3}]\}$. Since $\eta \circ \nu = 0$ [125, Theorem 14.1] and $\widetilde{\eta^2_{2^r}} = \widetilde{\eta}_{2^r} \circ \eta$, the map $(\widetilde{\eta^2_{2^r}})^* : [\Sigma^7 M(\mathbb{Z}/2^r), Top/O] \to \pi_{10}(Top/O)$ is trivial for all $r \geq 3$.

For $k = 5$, Lemmas 4.2.1 and 4.3.3 imply that the images of both $\eta^* : \pi_{10}(Top/O) \to \pi_{11}(Top/O)$ and $(\widetilde{\eta}_{2^r})^* : [\Sigma^8 M(\mathbb{Z}/2^r), Top/O] \to \pi_{10}(Top/O)$ are $\mathbb{Z}/2$. The map $(\widetilde{\eta^2_{2^r}})^*$ is the composition $\eta^* \circ (\widetilde{\eta}_{2^r})^*$, and therefore its image $(\widetilde{\eta^2_{2^r}})^* : [\Sigma^8 M(\mathbb{Z}/2^r), Top/O] \to \pi_{11}(Top/O)$ is also $\mathbb{Z}/2$. $\square$

Using the above lemma, we get

Lemma 4.3.9. *Let r be a positive integer. The image of the map*

$$(\Sigma^{k+1} i^r \circ \widetilde{\eta^2_{2^r}})^* : [\Sigma^{k+1} C_{i_{2^r} \circ \eta}, Top/O] \to \pi_{6+k}(Top/O)$$

is given as follows:

(i) trivial, for all $1 \leq k \leq 10$, with $k \neq 4, 5$.

(ii) For $k = 4$:

(a) $\mathbb{Z}/2$, if $r = 1$ or 2;

    - *(b) trivial, if $r \geq 3$.*

- *(iii) $\mathbb{Z}/2$, for $k = 5$.*

Proof. Statement (i) is an immediate consequence of Lemma 4.3.8 (i).

For $k = 4$, the cofiber sequence $\mathbb{S}^3 \xrightarrow{\Sigma^2 i_{2^r} \circ \eta} \Sigma^2 M(\mathbb{Z}/2^r) \overset{i^r}{\hookrightarrow} C_{i_{2^r} \circ \eta}$ induces a long exact sequence, from which the surjectivity of $(\Sigma^5 i^r)^* : [\Sigma^5 C_{i_{2^r} \circ \eta}, Top/O] \to [\Sigma^7 M(\mathbb{Z}/2^r), Top/O]$ follows. Lemma 4.3.8 (ii) then determines the image of $(\Sigma^{k+1} i^r \circ \widetilde{\eta^2_{2^r}})^*$ for $k = 4$.

For $k = 5$, the image of $(\Sigma^6 i^r)^* : [\Sigma^6 C_{i_{2^r} \circ \eta}, Top/O] \to [\Sigma^8 M(\mathbb{Z}/2^r), Top/O]$ is contained in the image of $(q_9)^* : \pi_9(Top/O) \to [\Sigma^8 M(\mathbb{Z}/2^r), Top/O]$. By Lemma 4.3.8 (iii), the image of $(\widetilde{\eta^2_{2^r}})^* : [\Sigma^8 M(\mathbb{Z}/2^r), Top/O] \to \pi_{11}(Top/O)$ agrees with the image of the composition $(q_9 \circ \widetilde{\eta^2_{2^r}})^* = (\eta^2)^* : \pi_9(Top/O) \to \pi_{11}(Top/O)$. From these two observations, the result for $k = 5$ follows. ☐

The theorem below provides a complete description of the concordance inertia group of $M \times \mathbb{S}^k$ for $1 \leq k \leq 10$.

Theorem 4.3.10. *Let M be a simply connected, closed, smooth 6-manifold with homology of the form (3.10). Then:*

- *(i) $I_c(M \times \mathbb{S}^k) = 0$, for $k = 1, 2, 6, 8$.*
- *(ii) For $k = 3$:*
    - *(a) If Sq^2 acts trivially on $H^*(M; \mathbb{Z}/2)$, then $I_c(M \times \mathbb{S}^3) = 0$.*
    - *(b) If Sq^2 acts nontrivially on $H^*(M; \mathbb{Z}/2)$ and M satisfies condition C, then $I_c(M \times \mathbb{S}^3) = \mathbb{Z}/2$.*
    - *(c) Suppose Sq^2 acts nontrivially on $H^*(M; \mathbb{Z}/2)$ and M satisfies Condition D. Then*
        1. *$I_c(M \times \mathbb{S}^3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, if $r_{j_5} = 1$ or 2 in (3.22).*
        2. *$I_c(M \times \mathbb{S}^3) = \mathbb{Z}/2$, if $r_{j_5} \geq 3$ in (3.22).*

(iii) *For* $k = 4:$

- *(a)* *If* Sq^2 *is trivial on* $H^4(M;\mathbb{Z}/2)$ *and* Θ *is trivial on* $H^3(M;\mathbb{Z}/2)$*, then* $I_c(M \times \mathbb{S}^4) = 0$*.*
- *(b)* *Suppose* Sq^2 *be trivial on* $H^4(M;\mathbb{Z}/2)$ *and* Θ *is non-trivial on* $H^3(M;\mathbb{Z}/2)$*.*
  1. $I_c(M \times \mathbb{S}^4) = 0$*, if* M *satisfies condition A.*
  2. *If* M *satisfies Condition B, then*
     - *(I)* $I_c(M \times \mathbb{S}^4) = 0$*, if* $r \geq 3$ *in* (3.18)*.*
     - *(II)* $I_c(M \times \mathbb{S}^4) = \mathbb{Z}/2$*, if* r *is either* 1 *or* 2 *in* (3.18)*.*
- *(c)* *If* Sq^2 *acts nontrivially on* $H^4(M;\mathbb{Z}/2)$*, then* $I_c(M \times \mathbb{S}^4) = \mathbb{Z}/2$*.*

(iv) *For* $k = 5:$

- *(a)* *Suppose* $Sq^2 : H^4(M;\mathbb{Z}/2) \to H^6(M;\mathbb{Z}/2)$ *be trivial.*
  1. *If* Θ *is trivial on* $H^3(M;\mathbb{Z}/2)$*, then* $I_c(M \times \mathbb{S}^5) = 0$*.*
  2. *If* Θ *acts nontrivially on* $H^3(M;\mathbb{Z}/2)$*, then* $I_c(M \times \mathbb{S}^5) = \mathbb{Z}/2$*.*
- *(b)* *If* $Sq^2 : H^4(M;\mathbb{Z}/2) \to H^6(M;\mathbb{Z}/2)$ *is nontrivial and* M *satisfies Condition C, then* $I_c(M \times \mathbb{S}^5) = \mathbb{Z}/2$*.*
- *(c)* *If* $Sq^2 : H^4(M;\mathbb{Z}/2) \to H^6(M;\mathbb{Z}/2)$ *is nontrivial and* M *satisfies Condition D, then*
  - *(I)* $I_c(M \times \mathbb{S}^5) = \mathbb{Z}/4$*, if* $r_{j_5} = 1$ *in* (3.22)*.*
  - *(II)* $I_c(M \times \mathbb{S}^5) = \mathbb{Z}/2$*, if* $r_{j_5} \geq 2$ *in* (3.22)*.*

(v) *For* $k = 7:$

- *(a)* *If* $3 \mid p_1(M)$*, then* $I_c(M \times \mathbb{S}^7) = 0$*.*
- *(b)* *If* $3 \nmid p_1(M)$*, then* $I_c(M \times \mathbb{S}^7) = \mathbb{Z}/3$*.*

(vi) *For* $k = 9:$

- *(a)* *If* Sq^2 *acts trivially on* $H^4(M;\mathbb{Z}/2)$*, then* $I_c(M \times \mathbb{S}^9) = 0$*.*

    (b) *If* Sq^2 *acts nontrivially on* $H^4(M;\mathbb{Z}/2)$, *then* $I_c(M \times \mathbb{S}^9) = \mathbb{Z}/2$.

(vii) *For* $k = 10$:

    (a) *If* $Sq^2 : H^4(M;\mathbb{Z}/2) \to H^6(M;\mathbb{Z}/2)$ *is trivial, then* $I_c(M \times \mathbb{S}^{10}) = 0$.

    (b) *If* $Sq^2 : H^4(M;\mathbb{Z}/2) \to H^6(M;\mathbb{Z}/2)$ *is nontrivial and* M *satisfies Condition C, then* $I_c(M \times \mathbb{S}^{10}) = 0$.

    (c) *If* $Sq^2 : H^4(M;\mathbb{Z}/2) \to H^6(M;\mathbb{Z}/2)$ *is nontrivial and* M *satisfies Condition D, then*

        1. $I_c(M \times \mathbb{S}^{10}) = \mathbb{Z}/2$, *if* $r_{j_5} = 1$ *in* (3.22).
        2. $I_c(M \times \mathbb{S}^{10}) = 0$, *if* $r_{j_5} \geq 2$ *in* (3.22).

Proof. Using (3.14) and (4.9), together with Lemma 4.3.4 (a) and Corollary 4.3.6 (i), (ii), the computation of $I_c(M \times \mathbb{S}^k)$ for $1 \leq k \leq 10$ reduces to determining the image of the map

(4.11)

$$(\Sigma^{k+1}\phi_6)^* = \sum_{i=1}^{c} a'_i(\Sigma^{k+1}i' \circ \alpha_1)^* + \sum_{i=1}^{m-c} b'_i\alpha_1^* + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^{k+3}i_{3^{r_j}} \circ \alpha_1)^* + \sum_{j=1}^{s_2-l} \tilde{x}_j(\widetilde{\eta^2_{2^{r_j}}})^*$$
$$+ \sum_{j=1}^{s_2} \left(y_j(\widetilde{\eta}_{2^{r_j}})^* + y'_j(\Sigma^{k+4}i_{2^{r_j}} \circ \eta^2)^*\right) + \sum_{w=1}^{m} z_w\eta^* + \sum_{j=1}^{l} d_j(\Sigma^{k+1}i^{r_j} \circ \widetilde{\eta^2_{2^{r_j}}})^*,$$

induced on the Top/O-level.

For $k = 7$, Lemmas 4.3.3, 4.2.1, 4.3.7, 4.3.8, and 4.3.9 together imply that the map $(\Sigma^8\phi_6)^*$ in (4.11) further reduces to

$$(\Sigma^8\phi_6)^* = \sum_{i=1}^{c} a'_i(\Sigma^{k+1}i' \circ \alpha_1)^* + \sum_{i=1}^{m-c} b'_i\alpha_1^* + \sum_{j=1}^{\tilde{s}_1} c_j(\Sigma^{k+3}i_{3^{r_j}} \circ \alpha_1)^*.$$

We now analyze the two cases depending on the divisibility of $p_1(M)$ by 3:

- If $3 \mid p_1(M)$, then by Fact 4.3.2 all coefficients a'_i, b'_i, and c_j vanish. In this case, the map $(\Sigma^8\phi_6)^*$ is trivial, and hence $I_c(M \times \mathbb{S}^7) = 0$.

- If $3 \nmid p_1(M)$, then by [72, Lemma 4.4] and [52, §3], exactly one of a'_i, b'_i, c_j is nontrivial. Consequently, it follows from Lemma 4.3.4 (b) and Lemma 4.3.6 (iv), (v) that the image of $(\Sigma^8\phi_6)^*$ is $\mathbb{Z}/3$, and therefore $I_c(M \times \mathbb{S}^7) = \mathbb{Z}/3$.

For all $1 \leq k \leq 10$ with $k \neq 7$, Lemma 4.3.4(b) together with Corollary 4.3.6 (iv), (v) implies that the map in (4.11) simplifies to

$$\begin{aligned}(\Sigma^{k+1}\phi_6)^* = \sum_{j=1}^{s_2-l} \tilde{x}_j(\widetilde{\eta^2_{2^{r_j}}})^* + \sum_{j=1}^{s_2} \big(y_j(\widetilde{\eta}_{2^{r_j}})^* + y'_j(\Sigma^{k+4} i_{2^{r_j}} \circ \eta^2)^*\big) + \sum_{w=1}^{m} z_w \eta^* \\ + \sum_{j=1}^{l} d_j(\Sigma^{k+1} i^{r_j} \circ \widetilde{\eta^2_{2^{r_j}}})^*.\end{aligned} \tag{4.12}$$

For $k = 1, 2, 6$, and 8, the result follows from (4.12) using Lemmas 4.2.1, 4.3.3, 4.3.7, 4.3.8, and 4.3.9.

We proceed by distinguishing two cases, depending on whether the Steenrod square $Sq^2 : H^4(M; \mathbb{Z}/2) \to H^6(M; \mathbb{Z}/2)$ is trivial or nontrivial, and treat each case separately.

Assume that the Steenrod square $Sq^2 : H^4(M; \mathbb{Z}/2) \to H^6(M; \mathbb{Z}/2)$ is trivial. Then, combining (3.15) and (4.12), the map $(\Sigma^{k+1}\phi_6)^*$ takes the form

$$(\Sigma^{k+1}\phi_6)^* = \sum_{j=1}^{s_2-l} \tilde{x}_j(\widetilde{\eta^2_{2^{r_j}}})^* + \sum_{j=1}^{s_2} y'_j(\Sigma^{k+4} i_{2^{r_j}} \circ \eta^2)^* + \sum_{j=1}^{l} d_j(\Sigma^{k+1} i^{r_j} \circ \widetilde{\eta^2_{2^{r_j}}})^*. \tag{4.13}$$

- Since each component in (4.13) maps trivially for $k = 3, 9$, and 10, by Lemmas 4.3.7 (i), 4.3.8 (i), and 4.3.9 (i), respectively, it follows that the concordance inertia group of $M \times \mathbb{S}^k$ is trivial in these cases.

- Moreover, if the secondary cohomology operation Θ on $H^3(M; \mathbb{Z}/2)$ is trivial, then from (3.16), we observe that (4.13) further reduces to $(\Sigma^{k+1}\phi_6)^* = 0$. Consequently, $I_c(M \times \mathbb{S}^k)$ is trivial for all $1 \leq k \leq 10$.

- Suppose the secondary cohomology operation Θ on $H^3(M; \mathbb{Z}/2)$ is nontrivial and that M satisfies Condition A. Then, from (3.17) and (4.13), it follows that

$$I_c(M \times \mathbb{S}^k) = Im\left([\Sigma^{4+k} M(\mathbb{Z}/2^{r_{j_0}}), Top/O] \xrightarrow{(\Sigma^{k+4} i_{2^{r_{j_0}}} \circ \eta^2)^*} \pi_{6+k}(Top/O)\right).$$

Lemma 4.3.7 (i) then implies that $I_c(M \times \mathbb{S}^k)$ is trivial for $k = 4$ and is $\mathbb{Z}/2$ for $k = 5$.

- If Θ acts non-trivially on $H^3(M; \mathbb{Z}/2)$ and M satisfies Condition B, then combining (3.19), (3.20) and (4.13), we find that $I_c(M \times \mathbb{S}^k)$ is given by one of the following:

  $$Im\left([\Sigma^{k+1} C_{i_{2^r} \circ \eta}, Top/O] \xrightarrow{(\Sigma^{k+1} i^r \circ \widetilde{\eta^2_{2^r}})^*} \pi_{6+k}(Top/O)\right), \text{ if } r = r_{j_1} \text{ or } r_{j_3},$$

  or

  $$Im\left([\Sigma^{k+3} M(\mathbb{Z}/2^r), Top/O] \xrightarrow{(\widetilde{\eta^2_{2^r}})^*} \pi_{6+k}(Top/O)\right), \text{ if } r = r_{j_2} \text{ or } r_{j_4}.$$

  The conclusion for $k = 4$ and 5 now follows from Lemmas 4.3.8 and 4.3.9.

Suppose Sq^2 acts nontrivially on $H^4(M; \mathbb{Z}/2)$. Then at least one of the coefficients y_j or z_w in (4.12) must be nonzero.

- If M satisfies Condition C, then combining (3.21) with (4.12) yields

  $$I_c(M \times \mathbb{S}^k) = Im\left(\pi_{5+k}(Top/O) \xrightarrow{\eta^*} \pi_{6+k}(Top/O)\right).$$

  Thus, by Lemma 4.3.3, we obtain $I_c(M \times \mathbb{S}^k) = \mathbb{Z}/2$ for $k = 3, 4, 5, 9$ and $I_c(M \times \mathbb{S}^{10}) = 0$.

- If M satisfies Condition D, then (3.22) together with (4.12) implies that

  $$I_c(M \times \mathbb{S}^k) = Im\left([\Sigma^{4+k} M(\mathbb{Z}/2^{r_{j_5}}), Top/O] \xrightarrow{(\widetilde{\eta}_{2^{r_{j_5}}})^*} \pi_{6+k}(Top/O)\right).$$

  The result for $k = 3, 4, 5, 9$, and 10 then follows from Lemma 4.2.1.

This completes the proof. $\square$

Since $I_c(M\times\mathbb{S}^k)\subseteq I(M\times\mathbb{S}^k)\subseteq\Theta_{6+k}$, the preceding theorem implies the following:

Corollary 4.3.11. *Let M be a simply connected, closed, smooth 6-dimensional manifold with $3\nmid p_1(M)$. Then the inertia group of $M\times\mathbb{S}^7$ is Θ_{13}. In particular, $I(\mathbb{C}P^3\times\mathbb{S}^7)=\Theta_{13}$.*

We observe that for $k=17, 23, 33, 39, 43, 49, 59, 69, 78, 79, 87, 88, 89$, and 98 there is an element x in $(\pi^s_{3+k}/Im(J))_{(3)}$ such that $\alpha_1\circ x\neq 0$ in $(\pi^s_{6+k}/Im(J))_{(3)}$ [103, Table A3.4]. Thus, for these specified values of k, the image of $(\alpha_1)^*:\pi_{3+k}(Top/O)\to\pi_{6+k}(Top/O)$ is $\mathbb{Z}/3$. Now applying [72, Lemma 4.4] and [52, §3] in (3.14), we obtain from (4.9) that

Corollary 4.3.12. *Let M be a simply connected, closed, smooth 6-manifold such that $3\nmid p_1(M)$. Then $\mathbb{Z}/3\subseteq I(M\times\mathbb{S}^k)$ for $k=17, 23, 33, 39, 43, 49, 59, 69, 78, 79, 87,$
$88, 89, 98$.*

4.4 The Concordance Inertia Group of the Product of a 3-Connected 8-Manifold with a Sphere

Let M be a closed, 3-connected, 8-dimensional smooth manifold. Using the stable decomposition of M given in either (3.25) or (3.23), and applying Lemma 4.0.2, we obtain a formula for the concordance inertia group of $M\times\mathbb{S}^k$:

$$I_c(M\times\mathbb{S}^k)=Im\left(\pi_{5+k}(Top/O)\xrightarrow{(s\tau_{5+k})^*}\pi_{8+k}(Top/O)\right), \tag{4.14}$$

where $s\in\mathbb{Z}/24$ is either $\operatorname{ind}(M)$, as in (3.25), or t, as in (3.23). This description, together with Lemma 4.3.1 and Fact 4.3.2, allows us to compute $I_c(M\times\mathbb{S}^k)$ for specific values of k, as stated in the following theorem.

Theorem 4.4.1. *Let M be a closed, 3-connected, smooth, 8-manifold of rank $m\geq 1$. Then* *(i)* $I_c(M \times \mathbb{S}^k) = 0$, *for* $1 \leq k \leq 14, k \neq 5, 9, 13$ *and* 14.

(ii) *For* $k = 5$:

  (a) $I_c(M \times \mathbb{S}^5) = 0$, *if* $3 \mid p_1(M)$.

  (b) $I_c(M \times \mathbb{S}^5) = \mathbb{Z}/3$, *if* $3 \nmid p_1(M)$.

(iii) *For* $k = 9, 13, 14$:

  (a) *If* $m = 1$, *then* $I_c(M \times \mathbb{S}^k) = \mathbb{Z}/2$.

  (b) *If* $m \geq 2$, *then*

    1. $I_c(M \times \mathbb{S}^k) = 0$, *for even* $\mathrm{ind}(M)$.

    2. $I_c(M \times \mathbb{S}^k) = \mathbb{Z}/2$, *for odd* $\mathrm{ind}(M)$.

Proof. Assume first that the rank of M is one. In this case, the Steenrod operation $Sq^4 : H^4(M; \mathbb{Z}/2) \to H^8(M; \mathbb{Z}/2)$ is an isomorphism, and so $(\Sigma^{k+1} g)_{(2)}$ is an odd multiple of $(\nu_{5+k})_{(2)} \in \mathbb{Z}/8$. Combining this observation with Fact 4.3.2, Lemma 4.3.1, the results for $m = 1$ follow from (4.14).

For $m \geq 2$, the concordance inertia group of $M \times \mathbb{S}^k$ is determined from (4.14) using the Fact 4.3.2, and Lemma 4.3.1. □

From [103, Table A3.4], we observe that for $k = 15, 31, 67, 76, 86, 87$ and 96, there exists $x \in \pi^s_{5+k}/Im(J)$ such that $\alpha_1 \circ x \neq 0$ in $\pi^s_{8+k}/Im(J)$. From this, we deduce:

Corollary 4.4.2. *Let* M *be a closed,* 3*-connected, smooth,* 8*-manifold with* $3 \nmid p_1(M)$. *Then* $\mathbb{Z}/3 \subseteq I_c(M \times \mathbb{S}^k)$ *for* $k = 15, 31, 67, 76, 86, 87$ *and* 96.

Since $I_c(M \times \mathbb{S}^k) \subseteq I(M \times \mathbb{S}^k) \subseteq \Theta_{8+k}$, the following statement is an immediate consequence of Theorem 4.4.1:

Remark 4.4.3. *Let* M *be a closed, 3-connected, 8-dimensional smooth manifold with* $3 \nmid p_1(M)$. *Then the inertia group of* $M \times \mathbb{S}^5$ *is* $\mathbb{Z}/3$. *In particular,* $I(\mathbb{H}P^2 \times \mathbb{S}^5) = \mathbb{Z}/3$.

Chapter 5

The Concordance Structure Set of $M \times \mathbb{S}^k$

In this chapter, we compute the concordance structure set $\mathcal{C}(M \times \mathbb{S}^k)$ for certain closed smooth manifolds M and for some values of k with $1 \leq k \leq 10$. The emphasis is on understanding how taking products with spheres affects the concordance classification of smooth structures.

These computations rely on stable homotopy–theoretic methods, in particular on the identification

$$\mathcal{C}(X) \cong [X, Top/O],$$

established by Kirby and Siebenmann (see Theorem 2.3.5). This identification translates the problem into a homotopy–theoretic setting.

By combining this description with the split short exact sequence (4.1), the concordance structure set of $M \times \mathbb{S}^k$ admits a decomposition in terms of homotopy-theoretic data associated to M and Top/O. The precise formulation is given in the following proposition.

Proposition 5.0.1. *Let M be a closed, oriented, smooth manifold. If $\dim(M)+k \geq 5$, then*

$$\mathcal{C}(M \times \mathbb{S}^k) \cong [M, Top/O] \oplus \pi_k(Top/O) \oplus [\Sigma^k M, Top/O].$$

This expresses $\mathcal{C}(M \times \mathbb{S}^k)$ in terms of three separate components, each of which needs to be understood separately.

5.1 Concordance Classes of Smoothings of Product of a 4-Manifold with a Sphere

As a first step toward analyzing $\mathcal{C}(M \times \mathbb{S}^k)$ for a closed, oriented, smooth 4-manifold M, we focus on computing the summands $[\Sigma^k M, Top/O]$ and $[M, Top/O]$ in Proposition 5.0.1. These terms can be determined using the stable splittings given in (3.2) and (3.3), depending on whether M is spin or non-spin. In each case, it suffices to examine the individual components appearing in the respective splitting.

We begin with the computation of $[\Sigma^k \mathbb{C}P^2, Top/O]$, which serves as a model for other summands. Consider the cofiber sequence

$$\mathbb{S}^3 \xrightarrow{\eta_2} \mathbb{S}^2 \overset{i}{\hookrightarrow} \mathbb{C}P^2 \xrightarrow{f_{\mathbb{C}P^2}} \Sigma\mathbb{S}^3 \xrightarrow{\Sigma\eta_2} \Sigma\mathbb{S}^2 \hookrightarrow \cdots .$$

This induces the long exact sequence

(5.1)
$$\cdots \to \pi_{3+k}(X) \xrightarrow{\eta^*} \pi_{4+k}(X) \xrightarrow{(\Sigma^k f_{\mathbb{C}P^2})^*} [\Sigma^k \mathbb{C}P^2, X] \xrightarrow{(\Sigma^k i)^*} \pi_{2+k}(X) \xrightarrow{\eta^*} \pi_{3+k}(X) \to \cdots ,$$

where $\Sigma^k \eta_2 = \eta$ for $k \geq 1$ [125], and X is any infinite loop space.

Since $\pi_2(Top/O)$ and $\pi_4(Top/O)$ are both trivial by Theorem 2.2.32, it follows from the long exact sequence (5.1) that $[\mathbb{C}P^2, Top/O] \cong 0$. Using this, and combining (3.2), (3.3), and (4.4), we obtain

$$\begin{aligned}[M, Top/O] &\cong \bigoplus_{i=1}^{m} \pi_3(Top/O) \oplus \bigoplus_{j=1}^{l_2} [\Sigma^2 M(\mathbb{Z}/2^{r_{n_j}}), Top/O] \\ &\cong \bigoplus_{i=1}^{m} \pi_3(Top/O) \oplus \bigoplus_{j=1}^{l_2} \pi_3(Top/O) \cong H^3(M; \mathbb{Z}/2),\end{aligned} \tag{5.2}$$

where $l_2 \geq 0$ is the number of indices i for which the summands $\mathbb{Z}/2^{r_{n_i}}$ appear in the decomposition of $H_2(M;\mathbb{Z})$ in (3.1).

The following lemma will be useful in the computation of $[\Sigma^k\mathbb{C}P^2, Top/O]$.

Lemma 5.1.1. *The image of the map* $\eta^* : \pi_8(G/O) \to \pi_9(G/O)$ *is* $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proof. Since $\pi_9(G/O) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, it suffices to work locally at prime 2. We note from [77, Theorem 5.18] that

$$G/O_{(2)} \simeq BSO_{(2)} \times cok(J_{(2)}).$$

Here $\pi_8(cok(J_{(2)}))$ is the cokernel of the J-homomorphism $J : \pi_8(O) \to \pi_8^s$ and is equal to $\mathbb{Z}/2\{[\bar{\nu}]\}$ [103, Theorem 1.1.14]. Moreover, $\pi_9(cok(J_{(2)}))$ is isomorphic to an index two summand of the cokernel of the J-homorphism $J : \pi_9(O) \to \pi_9^s$ [74, Remark 11.43] and is given by $\mathbb{Z}/2\{[\nu^3]\}$ (see Lemma 5.1.2). Using the above decomposition, we obtain the following commutative diagram:

$$\begin{tikzcd}
\pi_8(G/O_{(2)}) \arrow[r, "\eta^*"] \arrow[d, "\cong"'] & \pi_9(G/O_{(2)}) \arrow[d, "\cong"] \\
\pi_8(BSO_{(2)}) \oplus \pi_8(cok(J_{(2)})) \arrow[r, "\eta^* \oplus \eta^*"'] & \pi_9(BSO_{(2)}) \oplus \pi_9(cok(J_{(2)})).
\end{tikzcd}$$

Since $[\bar{\nu}] \in \pi_8(cok(J_{(2)}))$, $[\nu^3] \in \pi_9(cok(J_{(2)}))$ and $\eta \circ \bar{\nu} = \nu^3$ [125, Theorem 14.1], the map $\eta^* : \pi_8(cok(J_{(2)})) \to \pi_9(cok(J_{(2)}))$ has image $\mathbb{Z}/2$. Moreover from [18, Page 29], the map $\eta^* : \pi_8(BSO_{(2)}) \to \pi_9(BSO_{(2)})$ is surjective. Hence, by the commutativity of the above diagram, the image $\eta^* : \pi_8(G/O_{(2)}) \to \pi_9(G/O_{(2)})$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. ☐

Lemma 5.1.2. *The generator of* $\pi_9(cok(J_{(2)}))$ *is* $[\nu^3]$.

Proof. We first show that the map

$$\eta^* : \pi_9(BSO_{(2)}) \to \pi_{10}(BSO_{(2)})$$

is an isomorphism. To show this, we consider the following commutative diagram:

$$\begin{tikzcd}
\pi_8^s \arrow[r, "\eta^*"] & \pi_9^s \\
\mathbb{Z}/2 \cong \pi_9(BSO_{(2)}) \arrow[u, "J"] \arrow[r, "\eta^*"'] & \pi_{10}(BSO_{(2)}) \cong \mathbb{Z}/2, \arrow[u, "J"']
\end{tikzcd}$$

where both maps $\eta^* : \pi_8^s \to \pi_9^s$ and $J : \pi_9(BSO_{(2)}) \to \pi_8^s$ are injective by [125, pp. 189–190] and [103, Theorem 1.1.13], respectively. Hence, it follows from the above diagram that $\eta^* : \pi_9(BSO_{(2)}) \to \pi_{10}(BSO_{(2)})$ is an isomorphism.

Next, we consider the map $\eta^* : \pi_9(G/O_{(2)}) \to \pi_{10}(G/O_{(2)})$. Since $\psi_* : \pi_{10}(Top/O) \to \pi_{10}(G/O)$ is an isomorphism, $\psi_* : \pi_9(Top/O) \to \pi_9(G/O)$ is surjective, and $\eta^* : \pi_9(Top/O_{(2)}) \to \pi_{10}(Top/O_{(2)})$ is surjective by Lemma 4.1.1, it follows from the diagram (4.2) that $\eta^* : \pi_9(G/O_{(2)}) \to \pi_{10}(G/O_{(2)})$ is surjective.
From [103, Theorem 1.1.14], we have

$$\pi_9(G/O_{(2)}) \cong \mathbb{Z}/2\{[\nu^3]\} \oplus \mathbb{Z}/2\{[\mu]\}, \quad \pi_{10}(G/O_{(2)}) \cong \mathbb{Z}/2\{[\eta \circ \mu]\}.$$

Also, by [74, Remark 11.43], we have

$$\pi_{10}(cok(J_{(2)})) = 0.$$

We now show that $[\nu^3] \in \pi_9(cok(J_{(2)})) \cong \mathbb{Z}/2$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}/2\{[\nu^3]\} \oplus \mathbb{Z}/2\{[\mu]\} \cong \pi_9(G/O_{(2)}) \arrow[r, "\eta^*"] \arrow[d, "\cong"'] & \pi_{10}(G/O_{(2)}) \cong \mathbb{Z}/2\{[\eta \circ \mu]\} \arrow[r] \arrow[d, "\cong"] & 0 \\
\pi_9(BSO_{(2)}) \oplus \pi_9(cok(J_{(2)})) \arrow[r, "\eta^* \oplus \eta^*"'] & \pi_{10}(BSO_{(2)}). &
\end{tikzcd}$$

Here the map $\eta_* : \pi_9(cok(J_{(2)})) \to \pi_{10}(cok(J_{(2)}))$ is trivial. Since $\eta^* : \pi_9(BSO_{(2)}) \to \pi_{10}(BSO_{(2)})$ is an isomorphism, the kernel of $\eta_* \oplus \eta_* : \pi_9(BSO_{(2)}) \oplus \pi_9(cok(J_{(2)})) \longrightarrow \pi_{10}(BSO_{(2)})$ is precisely the summand $\pi_9(cok(J_{(2)}))$.

On the other hand, since $\eta^* : \pi_9(G/O_{(2)}) \to \pi_{10}(G/O_{(2)})$ is surjective and its image is generated by $[\eta \circ \mu]$, both elements $[\mu]$ and $[\mu] \oplus [\nu^3]$ of $\pi_9(G/O_{(2)})$ map nontrivially under η^*. Therefore,

$$ker\big(\eta^* \colon \pi_9(G/O_{(2)}) \to \pi_{10}(G/O_{(2)})\big) = \mathbb{Z}/2\{[\nu^3]\}.$$

From these two observations and the commutative diagram, it follows that

$$[\nu^3] \in \pi_9(cok(J_{(2)})).$$

This completes the proof. □

Proposition 5.1.3.

(1) $[\Sigma\mathbb{C}P^2, Top/O] \cong \mathbb{Z}/2.$

(2) $[\Sigma^2\mathbb{C}P^2, Top/O] \cong 0.$

(3) $[\Sigma^3\mathbb{C}P^2, Top/O] \cong \Theta_7.$

(4) $[\Sigma^4\mathbb{C}P^2, Top/O] \cong \Theta_8.$

(5) There is a non-split short exact sequence

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to [\Sigma^5\mathbb{C}P^2, Top/O] \to \Theta_7 \to 0,$$

where $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \subset \Theta_9$.

(6) $[\Sigma^6\mathbb{C}P^2, Top/O] \cong \mathbb{Z}/3 \subset \Theta_{10}.$

(7) There is a short exact sequence

$$0 \to \mathbb{Z}/496 \to [\Sigma^7\mathbb{C}P^2, Top/O] \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0,$$

where $\mathbb{Z}/496 \subset \Theta_{11}$ *and* $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \subset \Theta_9$.

(8) $[\Sigma^8\mathbb{C}P^2, Top/O] \cong \mathbb{Z}/3 \subset \Theta_{10}$.

(9) $[\Sigma^9\mathbb{C}P^2, Top/O] \cong \Theta_{13} \oplus \Theta_{11}$.

(10) $[\Sigma^{10}\mathbb{C}P^2, Top/O] \cong \Theta_{14}$.

Proof. To begin, we consider the long exact sequence (5.1) with X taken to be Top/O, G/O, or G/Top. We also use the long exact sequence induced by the fiber sequence

$$\cdots \to \Omega(G/Top) \xrightarrow{\omega} Top/O \xrightarrow{\psi} G/O \xrightarrow{\phi} G/Top.$$

All statements from (1) to (10), except (5), follow directly from the long exact sequence (5.1), Lemma 4.1.1 and Theorem 2.2.32. In the case of statement (5), applying the same tools yields the following short exact sequence

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to [\Sigma^5\mathbb{C}P^2, Top/O] \to \Theta_7 \to 0. \tag{5.3}$$

To show that this short exact sequence does not split, we consider the following commutative diagram. The vertical maps arise from the fiber sequence $\cdots \to \Omega(G/Top) \xrightarrow{\omega} Top/O \xrightarrow{\psi} G/O \xrightarrow{\phi} G/Top$, and the horizontal maps form part of the long exact sequence (5.1).

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "(\Sigma^6 f_{\mathbb{C}P^2})^*"] \arrow[d, "\omega_*"'] & {[\Sigma^6\mathbb{C}P^2, G/Top]} \arrow[r, "(\Sigma^6 i)^*"] \arrow[d, "\omega_*"] & \mathbb{Z} \arrow[r] \arrow[d, "\omega_*"] & 0 \arrow[d] \\
\mathbb{Z}/2 \arrow[r, "\eta^*"'] \arrow[d] & \bigoplus_{i=1}^{3} \mathbb{Z}/2 \arrow[r, "(\Sigma^5 f_{\mathbb{C}P^2})^*"'] \arrow[d, "\psi_*"'] & {[\Sigma^5\mathbb{C}P^2, Top/O]} \arrow[r, "(\Sigma^5 i)^*"'] \arrow[d, "\psi_*"] & \mathbb{Z}/28 \arrow[r, "0"'] \arrow[d] & \mathbb{Z}/2 \\
\mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r, "\eta^*"'] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r, "(\Sigma^5 f_{\mathbb{C}P^2})^*"'] \arrow[d] & {[\Sigma^5\mathbb{C}P^2, G/O]} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0. & &
\end{tikzcd}$$

In the above diagram the top row splits at $[\Sigma^6\mathbb{C}P^2, G/Top]$ and $\psi_* : [\Sigma^5\mathbb{C}P^2, Top/O] \to [\Sigma^5\mathbb{C}P^2, G/O]$ is surjective as $[\Sigma^5\mathbb{C}P^2, G/Top] \cong 0$. Since $\pi_7(G/O) = 0$ and the map

$\eta^* : \pi_8(G/O) \to \pi_9(G/O)$ is surjective by Lemma 5.1.1, it follows from the bottom row of the diagram that $[\Sigma^5\mathbb{C}P^2, G/O] = 0$.Consequently, we deduce that the map $\omega_* : [\Sigma^6\mathbb{C}P^2, G/Top] \to [\Sigma^5\mathbb{C}P^2, Top/O]$ is also surjective. By combining this with the fact $[\Sigma^6\mathbb{C}P^2, G/Top] = \mathbb{Z} \oplus \mathbb{Z}_2$ and the commutativity of the top rectangle, we conclude that $[\Sigma^5\mathbb{C}P^2, Top/O] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/56$. This confirms that the short exact sequence (5.3) does not split, thereby completing the proof of Statement (5). This concludes the proof of the proposition.

□

Using the group structure of $\pi_m(Top/O)$ for $1 \leq m \leq 18$, localized at odd primes, together with the long exact sequence induced by the cofiber sequence $\mathbb{S}^0 \xrightarrow{\times p^r} \mathbb{S}^0 \xhookrightarrow{i_{p^r}} M(\mathbb{Z}/p^r)$ along Top/O, we obtain the following lemma:

Lemma 5.1.4. *Let r be a positive integer. Then the following hold:*

(a) $[\Sigma^k M(\mathbb{Z}/3^r), Top/O] \cong 0$, *for* $1 \leq k \leq 17$ *with* $k \neq 9, 10, 12,$ *and* 13.

(b) $[\Sigma^k M(\mathbb{Z}/3^r), Top/O] \cong \mathbb{Z}/3$, *for* $k = 9, 10, 12,$ *and* 13.

(c) $[\Sigma^k M(\mathbb{Z}/7^r), Top/O] \cong 0$, *for* $1 \leq k \leq 17$ *with* $k \neq 6, 7$.

(d) $[\Sigma^k M(\mathbb{Z}/7^r), Top/O] \cong \mathbb{Z}/7$, *for* $k = 6$ *and* 7.

(e) $[\Sigma^{11} M(\mathbb{Z}/31^r), Top/O] \cong \mathbb{Z}/31$.

Let p be 3, or 7 and let l_p represent the number of p-torsion summands of $H_2(M;\mathbb{Z})$ in (3.1).

Proposition 5.1.5. *Let M be a 4-dimensional closed, oriented, smooth manifold with homology as in (3.1). Then*

(i) $[\Sigma M, Top/O] \cong \bigoplus_{j=1}^{2l_2} \pi_3(Top/O) \oplus \bigoplus_{l=1}^{d} \pi_3(Top/O)$.

(ii) $[\Sigma^2 M, Top/O] \cong \bigoplus_{i=1}^{m} \pi_3(Top/O) \oplus \bigoplus_{j=1}^{l_2} \pi_3(Top/O)$.

(iii) $[\Sigma^3 M, Top/O] \cong \Theta_7$.

(iv) $[\Sigma^4 M, Top/O] \cong \Theta_8 \oplus \bigoplus_{i=1}^{m} \Theta_7 \oplus \bigoplus_{j=1}^{l_2} [\Sigma^6 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{j=1}^{l_7} \mathbb{Z}/7$.

(v) $[\Sigma^9 M, Top/O] \cong \Theta_{13} \oplus \bigoplus_{i=1}^{m} \Theta_{10} \oplus \bigoplus_{j=1}^{n} [\Sigma^{10} M(\mathbb{Z}/b_j), Top/O] \oplus$
$\bigoplus_{i=1}^{n} [\Sigma^{11} M(\mathbb{Z}/b_j), Top/O] \oplus \bigoplus_{l=1}^{d} \Theta_{11}$.

(vi) $[\Sigma^{10} M, Top/O] \cong \Theta_{14} \oplus \bigoplus_{i=1}^{m} (\Theta_{11} \oplus \Theta_{13}) \oplus \bigoplus_{j=1}^{n} [\Sigma^{11} M(\mathbb{Z}/b_j), Top/O] \oplus \bigoplus_{j=1}^{l_3} \Theta_{13}$.

(vii) *(a)* $[\Sigma^5 M, Top/O] \cong \Theta_9 \oplus \bigoplus_{i=1}^{m} \Theta_8 \oplus \bigoplus_{j=1}^{l_2} [\Sigma^6 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{j=1}^{2l_7} \mathbb{Z}/7$
$\oplus \bigoplus_{j=1}^{l_2} [\Sigma^7 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{l=1}^{d} \Theta_7$, *if* M *is a spin manifold.*

(a) $[\Sigma^5 M, Top/O] \cong \mathbb{Z}/56 \oplus \mathbb{Z}/2 \oplus \bigoplus_{i=1}^{m} \Theta_8 \oplus \bigoplus_{j=1}^{l_2} [\Sigma^6 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{j=1}^{2l_7} \mathbb{Z}/7$
$\oplus \bigoplus_{j=1}^{l_2} [\Sigma^7 M(\mathbb{Z}/b_j), Top/O] \oplus \bigoplus_{l=1}^{d-1} \Theta_7$, *if* M *is a non-spin manifold.*

(viii) *(a) if* M *is a spin manifold, then* $[\Sigma^6 M, Top/O] \cong \Theta_{10} \oplus \bigoplus_{i=1}^{m} (\Theta_7 \oplus \Theta_9) \oplus$
$\bigoplus_{j=1}^{l_2} [\Sigma^7 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{j=1}^{l_7} \mathbb{Z}/7 \oplus \bigoplus_{j=1}^{l_2} [\Sigma^8 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{l=1}^{d} \Theta_8$.

(b) if M *is a non-spin manifold, then* $[\Sigma^6 M, Top/O] \cong \mathbb{Z}/3 \oplus \bigoplus_{i=1}^{m} (\Theta_7 \oplus \Theta_9) \oplus$
$\bigoplus_{j=1}^{l_2} [\Sigma^7 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{j=1}^{l_7} \mathbb{Z}/7 \oplus \bigoplus_{j=1}^{l_2} [\Sigma^8 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{l=1}^{d-1} \Theta_8$.

(ix) *(a)* $[\Sigma^7 M, Top/O] \cong \Theta_{11} \oplus \bigoplus_{i=1}^{m} (\Theta_8 \oplus \Theta_{10}) \oplus \bigoplus_{j=1}^{l_2} [\Sigma^8 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus$
$\bigoplus_{j=1}^{l_2} [\Sigma^9 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{j=1}^{l_3} \mathbb{Z}/3 \oplus \bigoplus_{l=1}^{d} \Theta_9$ *for a spin manifold* M.

(b) $[\Sigma^7 M, Top/O] \cong [\Sigma^7 \mathbb{C}P^2, Top/O] \oplus \bigoplus_{i=1}^{m} (\Theta_8 \oplus \Theta_{10}) \oplus \bigoplus_{j=1}^{l_2} [\Sigma^8 M(\mathbb{Z}/2^{r_j}), Top/O]$
$\oplus \bigoplus_{j=1}^{l_2} [\Sigma^9 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{j=1}^{l_3} \mathbb{Z}/3 \oplus \bigoplus_{l=1}^{d-1} \Theta_9$ *for a non-spin manifold* M.

(x) *(a) if* M *is a spin manifold, then* $[\Sigma^8 M, Top/O] \cong \bigoplus_{i=1}^{m} (\Theta_9 \oplus \Theta_{11}) \oplus$
$\bigoplus_{j=1}^{l_2} [\Sigma^9 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{j=1}^{l_3} \mathbb{Z}/3 \oplus \bigoplus_{j=1}^{n} [\Sigma^{10} M(\mathbb{Z}/b_j), Top/O] \oplus \bigoplus_{l=1}^{d} \Theta_{10}$.

(b) if M is a non-spin manifold, then $[\Sigma^8 M, Top/O] \cong \mathbb{Z}/3 \oplus \bigoplus_{i=1}^{m} (\Theta_9 \oplus \Theta_{11}) \oplus \bigoplus_{j=1}^{l_2} [\Sigma^9 M(\mathbb{Z}/2^{r_j}), Top/O] \oplus \bigoplus_{j=1}^{l_3} \mathbb{Z}/3 \oplus \bigoplus_{j=1}^{n} [\Sigma^{10} M(\mathbb{Z}/b_j), Top/O] \oplus \bigoplus_{l=1}^{d-1} \Theta_{10}.$

Proof. The computations of $[\Sigma^k M, Top/O]$ for $1 \leq k \leq 10$ follow directly from the stable splittings (3.2) and (3.3), together with Proposition 5.1.3 and Lemma 5.1.4. □

Combining (5.2), Propositions 5.0.1, and 5.1.5, we obtain $\mathcal{C}(M \times \mathbb{S}^k)$ for $1 \leq k \leq 10$.

5.2 Concordance Structure Set of Product of a Simply Connected 5-Manifold with Sphere

Let M be a simply connected, closed, smooth 5-manifold. Applying $[\_, Top/O]$ to the cofiber sequence (3.6), together with the description of the stable homotopy of the 3-skeleton of M in (3.5), yields

$$[M, Top/O] \cong \bigoplus_{i=1}^{d+t_1} \pi_3(Top/O) \cong H^3(M; \mathbb{Z}/2), \tag{5.4}$$

where t_1 denotes the number of torsion summands of the form $\mathbb{Z}/2^{r_i}$ in $H_2(M;\mathbb{Z})$, for some $r_i \geq 1$. Hence, by Proposition 5.0.1, determining $\mathcal{C}(M \times \mathbb{S}^k)$ reduces to computing $[\Sigma^k M, Top/O]$. As a direct consequence of the stable homotopy type of M, described in Proposition 3.7 and in (3.8), (3.9), we obtain the following:

Proposition 5.2.1. *Let M be a closed, simply connected, smooth 5-manifold whose homology is as in (3.4). Then the following holds:*

(a) If M is a spin manifold, then $[\Sigma^k M, Top/O]$ is isomorphic to

$$\pi_{5+k}(Top/O) \oplus \bigoplus_{i=1}^{d} (\pi_{2+k}(Top/O) \oplus \pi_{3+k}(Top/O)) \oplus \bigoplus_{j=1}^{t} [\Sigma^{2+k} M(\mathbb{Z}/p_j^{r_j}), Top/O].$$

(b) If M is a non-spin manifold, then $[\Sigma^k M, Top/O]$ is isomorphic to either

$$[\Sigma^{k+1}\mathbb{C}P^2, Top/O] \oplus \bigoplus_{i=1}^{d-1} (\pi_{2+k}(Top/O) \oplus \pi_{3+k}(Top/O)) \\ \oplus \bigoplus_{j=1}^{t} [\Sigma^{2+k} M(\mathbb{Z}/{p_j}^{r_j}), Top/O]$$

or

$$[\Sigma^k Cone(\widetilde{\eta}_{2^r}), Top/O] \oplus \bigoplus_{i=1}^{d} (\pi_{2+k}(Top/O) \oplus \pi_{3+k}(Top/O)) \\ \oplus \bigoplus_{j=2}^{t} [\Sigma^{2+k} M(\mathbb{Z}/{p_j}^{r_j}), Top/O]$$

Proof. For $1 \leq k \leq 10$, the computation of $[\Sigma^k M, Top/O]$ follows directly from the stable splittings (3.2) and (3.3), in combination with Proposition 5.1.3 and Lemma 5.1.4.

□

Now we determine $\mathcal{C}(M \times \mathbb{S}^k)$ for all $1 \leq k \leq 10$ using Propositions 5.0.1, 5.2.1, and equation (5.4).

5.3 Concordance Structure Set of Product of a Simply Connected 6-Manifold with a Sphere

Let M be a simply connected, closed, smooth manifold of dimension 6. Then from (3.11), (3.13) and (4.8), we get

$$[M, Top/O] \cong \bigoplus_{i=1}^{2s_2} \pi_3(Top/O) \oplus \bigoplus_{i=1}^{2d} \pi_3(Top/O) \cong H^3(M; \mathbb{Z}/2). \tag{5.5}$$

By Proposition 5.0.1, it is then enough to compute $[\Sigma^k M, Top/O]$ in order to determine the concordance structure set $\mathcal{C}(M \times \mathbb{S}^k)$.

Theorem 5.3.1. *Let M be a simply connected, closed, smooth 6-manifold whose ho-*

mology is given by (3.10)*. Then there is a split short exact sequence*

$$0 \to \Theta_7 \to [\Sigma M, Top/O] \to [\Sigma M^{(5)}, Top/O] \cong \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2 \to 0.$$

In particular,

$$\mathcal{C}(M \times \mathbb{S}^1) \cong \mathbb{Z}/28 \oplus \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2 \oplus H^3(M;\mathbb{Z}/2).$$

Combining (3.11), (3.13), and (4.8), we obtain the following short exact sequence:

$$0 \to \Theta_7 \to [\Sigma M, Top/O] \to [\Sigma M^{(5)}, Top/O] \cong \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2 \to 0. \tag{5.6}$$

We note that proving the short exact sequence splits 2-locally establishes the theorem. Recall that Top/O is the infinite loop space $\Omega^\infty(top/o)$, where top/o is the corresponding spectrum. Let $\mathcal{P}_7(Top/O)$ denote the 7th Postnikov section of Top/O, defined by

$$\pi_k(\mathcal{P}_7(Top/O)) \cong \begin{cases} \pi_k(Top/O) & \text{for } k \leq 7, \\ 0 & \text{otherwise.} \end{cases}$$

Since ΣM is 7-dimensional, we have an isomorphism

$$[\Sigma M, Top/O] \cong [\Sigma M, \mathcal{P}_7(Top/O)],$$

where $\mathcal{P}_7(Top/O) \simeq \Omega^\infty(\tau_{\leq 7} top/o)$, the infinite loop space of the 7th Postnikov section of the spectrum top/o. Let $to_{\leq 7} = (\tau_{\leq 7} top/o)_{(2)}$ denote its 2-localization. Then

$$\pi_k(to_{\leq 7}) \cong \begin{cases} \mathbb{Z}/2 & \text{if } k = 3, \\ \mathbb{Z}/4 & \text{if } k = 7, \\ 0 & \text{otherwise.} \end{cases}$$

We prove that $\{\Sigma M, to_{\leq 7}\} \cong \mathbb{Z}/4 \oplus \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2$. Otherwise, from the short exact sequence

(5.6), we get $\{\Sigma M, to_{\leq 7}\} \cong \mathbb{Z}/8 \oplus \bigoplus_{i=2}^{s_2+m} \mathbb{Z}/2$. As a spectrum, we have the following cofiber sequence

$$to_{\leq 7} \to \Sigma^3 H\mathbb{Z}/2 \xrightarrow{\delta_{to}} \Sigma^8 H\mathbb{Z}/4.$$

Define the following

$$\delta_{\tilde{t}} : \Sigma^3 H\mathbb{Z}/2 \xrightarrow{\delta_{to}} \Sigma^8 H\mathbb{Z}/4 \to \Sigma^8 H\mathbb{Z}/2 \quad \text{and} \quad \tilde{t} = \text{homotopy fiber of } \delta_{\tilde{t}}.$$

Lemma 5.3.2. *Let M be a simply connected, closed, smooth 6-manifold with homology* (3.10). *Then*

(a) $\{\Sigma M, to_{\leq 7}\} = \mathbb{Z}/8 \oplus \bigoplus_{i=1}^{s_2+m-1} \mathbb{Z}/2$ *if and only if* $\{\Sigma M, \tilde{t}\} \cong \mathbb{Z}/4 \oplus \bigoplus_{i=1}^{s_2+m-1} \mathbb{Z}/2.$

(b) $\{\Sigma M, to_{\leq 7}\} = \mathbb{Z}/4 \oplus \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2$ *if and only if* $\{\Sigma M, \tilde{t}\} \cong \mathbb{Z}/2 \oplus \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2.$

Proof. We now examine the following commutative diagram of homotopy cofibrations, which defines a map of spectra $\tilde{\phi} : to_{\leq 7} \to \tilde{t}$:

$$\begin{tikzcd}
to_{\leq 7} \arrow[r] \arrow[d, dashed, "\tilde{\phi}"'] & \Sigma^3 H\mathbb{Z}/2 \arrow[r, "\delta_{to}"] \arrow[d, equal] & \Sigma^8 H\mathbb{Z}/4 \arrow[d] \\
\tilde{t} \arrow[r] & \Sigma^3 H\mathbb{Z}/2 \arrow[r, "\delta_{\tilde{t}}"'] & \Sigma^8 H\mathbb{Z}/2.
\end{tikzcd}$$

This diagram induces the following commutative diagram of long exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \{\Sigma M, \Sigma^7 H\mathbb{Z}/4\} \cong \mathbb{Z}/4 \arrow[r] \arrow[d] & \{\Sigma M, to_{\leq 7}\} \arrow[r] \arrow[d, "(\tilde{\phi})_*"] & \{\Sigma M, \Sigma^3 H\mathbb{Z}/2\} \cong \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \{\Sigma M, \Sigma^7 H\mathbb{Z}/2\} \cong \mathbb{Z}/2 \arrow[r] & \{\Sigma M, \tilde{t}\} \arrow[r] & \{\Sigma M, \Sigma^3 H\mathbb{Z}/2\} \cong \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2 \arrow[r] & 0.
\end{tikzcd}$$

This diagram shows that $(\tilde{\phi})_*$ is surjective and $\{\Sigma M, \tilde{t}\} \cong \{\Sigma M, to_{\leq 7}\}/A$, where A is generated by the image of $\bar{2} \in \mathbb{Z}/4$ under the map $\{\Sigma M, \Sigma^7 H\mathbb{Z}/4\} \to \{\Sigma M, to_{\leq 7}\}$.

(i) If $\{\Sigma M, to_{\leq 7}\} \cong \mathbb{Z}/8 \oplus \bigoplus_{i=1}^{s_2+m-1} \mathbb{Z}/2$, then the image of $\bar{2} \in \{\Sigma M, \Sigma^7 H\mathbb{Z}/4\}$ must be the class $(\bar{4}, 0, 0, ..., 0)$ in $\{\Sigma M, to_{\leq 7}\}$. Hence $\{\Sigma M, \tilde{t}\} \cong \mathbb{Z}/4 \oplus \bigoplus_{i=1}^{s_2+m-1} \mathbb{Z}/2.$

(ii) If $\{\Sigma M, to_{\leq 7}\} \cong \mathbb{Z}/4 \oplus \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2$, then $\bar{2} \in \mathbb{Z}/4 \cong \{\Sigma M, \Sigma^7 H\mathbb{Z}/4\}$ maps to $(\bar{2}, 0, 0, ..., 0) \in \{\Sigma M, to_{\leq 7}\}$. Hence $\{\Sigma M, \tilde{t}\} \cong \mathbb{Z}/2 \oplus \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2$.

This completes the proof. □

Lemma 5.3.3. *The map $\delta_{\tilde{t}} : \Sigma^3 H\mathbb{Z}/2 \to \Sigma^8 H\mathbb{Z}/2$ is either the Steenrod square Sq^5 or zero map, up to homotopy.*

Proof. From the definition of $\delta_{\tilde{t}}$ we have $\Sigma^3 H\mathbb{Z}/2 \xrightarrow{\delta_{\tilde{t}}} \Sigma^8 H\mathbb{Z}/2 \xrightarrow{Sq^1} \Sigma^9 H\mathbb{Z}/2$ zero. Now $\{\Sigma^3 H\mathbb{Z}/2, \Sigma^8 H\mathbb{Z}/2\} \cong \mathcal{A}_2^{(5)}$, the degree 5 homogeneous part of the mod 2 Steenrod algebra $\mathcal{A}_2$. As a basis of $\mathcal{A}_2$ is given by admissible monomials,

$$\mathcal{A}_2^{(5)} = \mathbb{Z}/2\{\text{admissible monomials in degree } 5\}.$$

Note that Sq^4Sq^1 and Sq^5 are the only admissible monomials of degree 5. Of these $Sq^1Sq^4Sq^1 = Sq^5Sq^1 \neq 0$ and $Sq^1Sq^5 = 0$. This proves the lemma. □

Proof of Theorem 5.3.1. Our goal is to prove $\{\Sigma M, \tilde{t}\} \cong \mathbb{Z}/2 \oplus \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2$ by using the fact that Sq^4 acts trivially on $H^*(\Sigma M; \mathbb{Z}/2)$.

If $\delta_{\tilde{t}} = 0$, then there is nothing to prove, as the equivalence $\tilde{t} \simeq \Sigma^3 H\mathbb{Z}/2 \vee \Sigma^7 H\mathbb{Z}/2$ implies $\{\Sigma M, \tilde{t}\} \cong H^3(\Sigma M; \mathbb{Z}/2) \oplus H^7(\Sigma M; \mathbb{Z}/2)$.

If $\delta_{\tilde{t}} = Sq^5 = Sq^1Sq^4$, then we consider the following diagram:

(5.7)
$$\begin{tikzcd}
\tilde{t} \arrow[r] \arrow[d, "\tilde{\psi}"'] & \Sigma^3 H\mathbb{Z}/2 \arrow[r, "\delta_{\tilde{t}}"] \arrow[d, "Sq^4"] & \Sigma^8 H\mathbb{Z}/2 \arrow[d, equal] \\
\Sigma^7 H\mathbb{Z}/4 \arrow[r] & \Sigma^7 H\mathbb{Z}/2 \arrow[r, "Sq^1"'] & \Sigma^8 H\mathbb{Z}/2.
\end{tikzcd}$$

Note that Sq^1 =Bockstein homomorphism β, so we get a cofiber sequence

$$H\mathbb{Z}/2 \to H\mathbb{Z}/4 \to H\mathbb{Z}/2 \xrightarrow{Sq^1} \Sigma H\mathbb{Z}/2 \to \cdots .$$

The diagram (5.7) induces the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \{\Sigma M, \Sigma^7 H\mathbb{Z}/2\} \cong \mathbb{Z}/2 \arrow[r] \arrow[d, "\cong"'] & \{\Sigma M, \tilde{t}\} \arrow[r] \arrow[d, "(\tilde{\psi})_*"] & \{\Sigma M, \Sigma^3 H\mathbb{Z}/2\} \cong \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2 \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \{\Sigma M, \Sigma^7 H\mathbb{Z}/2\} \cong \mathbb{Z}/2 \arrow[r] & \{\Sigma M, \Sigma^7 H\mathbb{Z}/4\} \cong \mathbb{Z}/4 \arrow[r] & \{\Sigma M, \Sigma^7 H\mathbb{Z}/2\} \cong \mathbb{Z}/2 \arrow[r] & 0,
\end{tikzcd}$$

where the map $\{\Sigma M, \Sigma^3 H\mathbb{Z}/2\} \to \{\Sigma M, \Sigma^7 H\mathbb{Z}/2\}$ is zero as Sq^4 acts trivially on M.

Suppose $\{\Sigma M, \tilde{t}\} \cong \mathbb{Z}/4 \oplus \bigoplus_{i=1}^{s_2+m-1} \mathbb{Z}/2$. Then the top left arrow sends $\bar{1}$ to the class $(\bar{2}, 0, 0, ..., 0)$ and the bottom left arrow sends $\bar{1}$ to $\bar{2}$ in $\mathbb{Z}/4$. Hence $\psi^*(\bar{2}, 0, 0, ..., 0) = \bar{2}$ and so $\psi^*(\bar{1}, 0, ..., 0) = \bar{1}$. But the top right arrow carries $(\bar{1}, 0, 0, ..., 0)$ to a non-zero class and the bottom right arrow carries $\bar{1}$ to $\bar{1}$. This contradicts the fact that the rightmost vertical arrow is zero. Thus, we conclude $\{\Sigma M, \tilde{t}\} \cong \mathbb{Z}/2 \oplus \bigoplus_{i=1}^{s_2+m} \mathbb{Z}/2$. The theorem is now derived from Lemma 5.3.2.

Now, $\mathcal{C}(M \times \mathbb{S}^1)$ follows from Proposition 5.0.1 and (5.5). ☐

Corollary 5.3.4. *Let M be a simply connected, closed, smooth 6-manifold. Then the short exact sequence*

$$0 \to [M \times \mathbb{S}^1, PL/O] \to [M \times \mathbb{S}^1, Top/O] \to [M \times \mathbb{S}^1, Top/PL] \to 0$$

splits.

In particular,

$$\begin{aligned} [M \times \mathbb{S}^1, Top/O] &\cong \Theta_7 \oplus H^2(M; \mathbb{Z}/2) \oplus H^3(M; \mathbb{Z}/2) \\ &\cong H^7(M \times \mathbb{S}^1; \Theta_7) \oplus H^3(M \times \mathbb{S}^1; \mathbb{Z}/2). \end{aligned}$$

Proof. We note that the fiber sequence $PL/O \to Top/O \to Top/PL$ induces the following long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & {[M \times \mathbb{S}^1, \Omega(Top/PL)]} \arrow[r] & {[M \times \mathbb{S}^1, PL/O]} \arrow[r] & {[M \times \mathbb{S}^1, Top/O]} \arrow[dlll, rounded corners] \\
{[M \times \mathbb{S}^1, Top/PL]} \arrow[r] & {[M \times \mathbb{S}^1, B(PL/O)]} \arrow[r] & {[M \times \mathbb{S}^1, B(Top/O)]} \arrow[r] & \cdots .
\end{tikzcd}$$

To show that the sequence splits at $[M \times \mathbb{S}^1, Top/O]$, it suffices to verify that each summand in the decomposition of $[M \times \mathbb{S}^1, Top/O]$, as given in Proposition 5.0.1, splits individually. Since (5.5) gives an isomorphism $[M, Top/O] \cong [M, Top/PL]$, the problem reduces to checking whether the following portion of the long exact sequence, induced by the fiber sequence $PL/O \to Top/O \to Top/PL$, splits at $[\Sigma M, Top/O]$:

$$\begin{tikzcd}
0 \cong H^1(M;\mathbb{Z}/2) \arrow[r] & {[\Sigma M, PL/O] \cong \Theta_7} \arrow[r] & {[\Sigma M, Top/O]} \arrow[dll, rounded corners] \\
{[\Sigma M, Top/PL] \cong H^2(M;\mathbb{Z}/2)} \arrow[r] & {[M, \Omega B(PL/O)] \cong [M, PL/O]} \arrow[r] & {[M, Top/O]}.
\end{tikzcd}$$

This splitting at $[\Sigma M, Top/O]$ follows from Theorem 5.3.1. This proves the result. □

Proposition 5.3.5. *Let M be a simply connected, closed, smooth manifold of dimension 6. Then, for any $k \geq 2$, the short exact sequence*

$$0 \to [M \times \mathbb{S}^k, PL/O] \to [M \times \mathbb{S}^k, Top/O] \to [M \times \mathbb{S}^k, Top/PL] \to 0$$

splits.

Proof. By incorporating the split short exact sequence (4.1) for $Y = PL/O, Top/O$, and Top/PL, we obtain the following commutative diagram.

$$\begin{tikzcd}
 & {\widetilde{H}^{2-k}(M;\mathbb{Z}/2) \cong [\Sigma^k M, \Omega(Top/PL)]} \arrow[r] \arrow[d, "0"'] & {[M \times \mathbb{S}^k, \Omega(Top/PL)]} \arrow[r] \arrow[d] & {[M, \Omega(Top/PL)] \oplus \pi_{k+1}(Top/PL)} \arrow[d, "0"] & \\
0 \arrow[r] & {[\Sigma^k M, PL/O]} \arrow[r] \arrow[d] & {[M \times \mathbb{S}^k, PL/O]} \arrow[r] \arrow[d] & {[M, PL/O] \oplus \pi_k(PL/O)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {[\Sigma^k M, Top/O]} \arrow[r] \arrow[d] & {[M \times \mathbb{S}^k, Top/O]} \arrow[r] \arrow[d] & {[M, Top/O] \oplus \pi_k(Top/O)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {[\Sigma^k M, Top/PL] \cong \widetilde{H}^{3-k}(M;\mathbb{Z}/2)} \arrow[r] \arrow[d] & {[M \times \mathbb{S}^k, Top/PL]} \arrow[r] \arrow[d] & {[M, Top/PL] \oplus \pi_k(Top/PL)} \arrow[r] \arrow[d] & 0 \\
 & {[\Sigma^k M, B(PL/O)]} \arrow[r] & {[M \times \mathbb{S}^k, B(PL/O)]} \arrow[r] & {[M, B(PL/O)] \oplus \pi_{k-1}(PL/O),} &
\end{tikzcd}$$

where each column is induced from the fiber sequence $\cdots \to \Omega(Top/PL) \to PL/O \to Top/O \to Top/PL$.

Since $Top/PL \simeq \Sigma^3 H\mathbb{Z}/2$ and PL/O is 6-connected, we have $[\Sigma^k M, PL/O] \cong [\Sigma^k M, Top/O]$ for $k \geq 2$ and $\pi_k(Top/O) \cong \pi_k(PL/O)$ for $k \geq 6$. Also from (5.5), we have $[M, Top/O] \cong [M, Top/PL]$.

Applying the Four Lemma [75, Page 14] to the diagram above, we obtain the following short exact sequence:

$$0 \to [M \times \mathbb{S}^k, PL/O] \to [M \times \mathbb{S}^k, Top/O] \to [M \times \mathbb{S}^k, Top/PL] \to 0.$$

The splitting of this short exact sequence follows from the above diagram upon examining the splitting of the following short exact sequences:

$$0 \to [M \vee \mathbb{S}^k, PL/O] \to [M \vee \mathbb{S}^k, Top/O] \to [M \vee \mathbb{S}^k, Top/PL] \to 0$$

and

$$0 \to [\Sigma^k M, PL/O] \to [M \times \mathbb{S}^k, PL/O] \to [M, PL/O] \oplus \pi_k(PL/O) \to 0.$$

□

Now instead of finding $\mathcal{C}(M \times \mathbb{S}^k)$ for any simply connected, closed, smooth 6-manifold, we focus on finding $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^k)$ and thereby calculating $[\Sigma^k \mathbb{C}P^3, Top/O]$. To do so we use the following long exact sequence

(5.8)

$$\begin{tikzcd}
\cdots \arrow[r] & {[\Sigma^{k+1}\mathbb{C}P^2, Top/O]} \arrow[r, "(\Sigma^{k+1}\pi)^*"] & \pi_{6+k}(Top/O) \arrow[r, "(\Sigma^k f_{\mathbb{C}P^3})^*"] & {[\Sigma^k \mathbb{C}P^3, Top/O]} \arrow[dlll, "(\Sigma^k i')^*"'] & \\
{[\Sigma^k \mathbb{C}P^2, Top/O]} \arrow[r, "(\Sigma^k \pi)^*"'] & \pi_{5+k}(Top/O) \arrow[r, "(\Sigma^{k-1} f_{\mathbb{C}P^3})^*"'] & {[\Sigma^{k-1}\mathbb{C}P^3, Top/O]} \arrow[r] & \cdots,
\end{tikzcd}$$

induced from $\mathbb{S}^5 \xrightarrow{\pi} \mathbb{C}P^2 \overset{i'}{\hookrightarrow} \mathbb{C}P^3$, where $\Sigma^k \pi_{(2)} = i \circ 2\nu$ and $\Sigma^k \pi_{(3)} = i \circ \alpha_1$ for $k \geq 8$ [92].

Proposition 5.3.6.

(i) $[\Sigma^k \mathbb{C}P^3, Top/O] \cong \Theta_{6+k}$, *for* $k = 2$ *and* 8.

(ii) $[\Sigma^k \mathbb{C}P^3, Top/O] \cong [\Sigma^k \mathbb{C}P^2, Top/O]$, *if* $k = 6$ *and* 7.

(iii) $[\Sigma^k\mathbb{C}P^3, Top/O] \cong \Theta_{6+k} \oplus [\Sigma^k\mathbb{C}P^2, Top/O]$ *for* $k = 1, 3, 4, 5, 9$ *and* 10.

Proof. The result for $k = 1$ directly follows from Theorem 5.3.1 and Proposition 5.1.3 (i).

We observe that the cases $k = 2, 6, 7,$ and 8 follow from the long exact sequence (5.8), together with Proposition 5.1.3 and Lemma 4.3.5.

Since $\mathbb{C}P^3/\mathbb{C}P^1 \simeq \mathbb{S}^4 \vee \mathbb{S}^6$, we have another cofiber sequence involving the complex $\mathbb{C}P^3$,

$$\mathbb{C}P^1 \simeq \mathbb{S}^2 \overset{\tilde{\tilde{i}}}{\hookrightarrow} \mathbb{C}P^3 \xrightarrow{q} \mathbb{S}^4 \vee \mathbb{S}^6 \xrightarrow{\eta+\phi} \mathbb{S}^3 \hookrightarrow \cdots, \tag{5.9}$$

where $(\phi)_{(2)}$ is stably 2ν and $(\phi)_{(3)}$ is stably α_1.

For $k = 3$ and 10, we consider the long exact sequence associated to the cofiber sequence (5.9). This yields

$$[\Sigma^k\mathbb{C}P^3, Top/O] \cong \Theta_{6+k} \oplus \Theta_{4+k},$$

which, by Proposition 5.1.3 (3) and (10), is isomorphic to $\Theta_{6+k} \oplus [\Sigma^k\mathbb{C}P^2, Top/O]$.

For $k = 4$, we consider the following ladder of exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & {[\Sigma^5\mathbb{C}P^3, G/Top]} \arrow[r] \arrow[d, "\omega_*"] & 0 \arrow[d] \\
\Theta_7 \arrow[r] \arrow[d, "\psi_*"'] & \Theta_8 \oplus \Theta_{10} \arrow[r, "(\Sigma^4 q)^*"] \arrow[d, "\psi_*"'] & {[\Sigma^4\mathbb{C}P^3, Top/O]} \arrow[r] \arrow[d, "\psi_*"] & \Theta_6 \cong 0 \\
0 \cong \pi_7(G/O) \arrow[r] & \pi_8(G/O) \oplus \pi_{10}(G/O) \arrow[r, "(\Sigma^4 q)^*"'] & {[\Sigma^4\mathbb{C}P^3, G/O],} &
\end{tikzcd}$$

where the rows are induced from the cofiber sequence (5.9) and columns are induced from the fiber sequence $\Omega(G/Top) \xrightarrow{\omega} Top/O \xrightarrow{\psi} G/O$. Now, from the above commutative diagram, we conclude that

$$[\Sigma^4\mathbb{C}P^3, TOP/O] \cong \Theta_{10} \oplus \Theta_8,$$

which, by Proposition 5.1.3 (4), is isomorphic to $\Theta_{10} \oplus [\Sigma^4\mathbb{C}P^2, Top/O]$.

For $k = 5$, the cofiber sequence (5.9) applied to Top/O, together with Lemmas 4.3.3

and 4.3.1, gives the following short exact sequence:

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/992 \to [\Sigma^5\mathbb{C}P^3, Top/O] \to \mathbb{Z}/28 \to 0. \tag{5.10}$$

Examining (5.10), it is enough to determine the 2-primary component of $[\Sigma^5\mathbb{C}P^3, Top/O]$. Now the above short exact sequence narrows down the possibilities for $[\Sigma^5\mathbb{C}P^3, Top/O]_{(2)}$ to three abelian groups:

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^5 \oplus \mathbb{Z}/4, \quad \mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2^5, \quad \text{or} \quad \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^7.$$

On the other hand, applying Proposition 5.1.3 (5) and Theorem 4.3.10 to the long exact sequence arising from (5.8), we obtain another short exact sequence:

$$0 \to \Theta_{11} \to [\Sigma^5\mathbb{C}P^3, Top/O] \to \mathbb{Z}/56 \oplus \mathbb{Z}/2 \to 0.$$

This further restricts $[\Sigma^5\mathbb{C}P^3, Top/O]_{(2)}$ to one of the following three groups:

$$\mathbb{Z}/2^5 \oplus \mathbb{Z}/2^3 \oplus \mathbb{Z}/2, \quad \mathbb{Z}/2^8 \oplus \mathbb{Z}/2, \quad \text{or} \quad \mathbb{Z}/2^3 \oplus \mathbb{Z}/2^6.$$

Combining the constraints from both short exact sequences, we conclude that

$$[\Sigma^5\mathbb{C}P^3, Top/O] \cong \Theta_{11} \oplus [\Sigma^5\mathbb{C}P^2, Top/O].$$

For $k = 9$, applying the functor $[\_, Top/O]$ to the cofiber sequence (5.9) yields the short exact sequence

$$0 \to \Theta_{15} \oplus \Theta_{13} \xrightarrow{(\Sigma^9 q)^*} [\Sigma^9\mathbb{C}P^3, Top/O] \xrightarrow{(\Sigma^9\widetilde{\tilde{i}})^*} \Theta_{11} \to 0,$$

as $\Theta_{12} = 0$, and any map from Θ_{11} to Θ_{14} is trivial. Based on this short exact sequence, the 2-primary component of $[\Sigma^9\mathbb{C}P^3, Top/O]$ can be one of the following:

$$\mathbb{Z}/2^6 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^5, \quad \mathbb{Z}/2^{11} \oplus \mathbb{Z}/2, \quad \text{or} \quad \mathbb{Z}/2^6 \oplus \mathbb{Z}/2^6.$$

To further refine the possibilities, note that the map $\pi_{12}(G/O) \to \pi_{15}(G/O)$ has trivial image, so $[\Sigma^9\mathbb{C}P^3, G/O]_{(2)} \cong \mathbb{Z}/2$. Moreover, the long exact sequence induced from the cofiber sequence (5.9) over G/O gives

$$[\Sigma^{10}\mathbb{C}P^3, G/O] \cong \pi_{16}(G/O) \oplus \pi_{14}(G/O) \oplus \pi_{12}(G/O),$$

and the same sequence over G/Top yields

$$[\Sigma^{10}\mathbb{C}P^3, G/Top] \cong \pi_{16}(G/Top) \oplus \pi_{14}(G/Top) \oplus \pi_{12}(G/Top).$$

From the splitting, it follows that the image of

$$\phi_* : [\Sigma^{10}\mathbb{C}P^3, G/O] \to [\Sigma^{10}\mathbb{C}P^3, G/Top]$$

is $8128\mathbb{Z} \oplus \mathbb{Z}/2 \oplus 992\mathbb{Z}$. Therefore, the map

$$(\omega_*)_{(2)} : [\Sigma^{10}\mathbb{C}P^3, G/Top]_{(2)} \to [\Sigma^9\mathbb{C}P^3, Top/O]_{(2)}$$

has image $\mathbb{Z}/2^6 \oplus \mathbb{Z}/2^5$, ruling out the possibility that $[\Sigma^9\mathbb{C}P^3, Top/O]_{(2)} \cong \mathbb{Z}/2^{11} \oplus \mathbb{Z}/2$.

Suppose instead that $[\Sigma^9\mathbb{C}P^3, Top/O]_{(2)} \cong \mathbb{Z}/2^6 \oplus \mathbb{Z}/2^6$. To show this cannot occur, consider the ladder of exact sequences induced from the cofiber sequence (5.9) and the fiber sequence

$$\cdots \to \Omega(G/Top) \xrightarrow{\omega} Top/O \xrightarrow{\psi} G/O \xrightarrow{\phi} G/Top.$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_{16}(G/O)_{(2)} \oplus \pi_{14}(G/O)_{(2)} \arrow[r, "(\Sigma^{10}q)^*_{(2)}"] \arrow[d, "(\phi_*)_{(2)}"'] & {[\Sigma^{10}\mathbb{C}P^3, G/O]_{(2)}} \arrow[r, "(\Sigma^{10}\tilde{i})^*_{(2)}"] \arrow[d, "(\phi_*)_{(2)}"] & \pi_{12}(G/O)_{(2)} \arrow[r, "0"] \arrow[d, "(\phi_*)_{(2)}"] & \pi_{15}(G/O)_{(2)} \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \arrow[r, "(\Sigma^{10}q)^*_{(2)}"] \arrow[d, "(\omega_*)_{(2)}"'] & {[\Sigma^{10}\mathbb{C}P^3, G/Top]_{(2)}} \arrow[r, "(\Sigma^{10}\tilde{i})^*_{(2)}"] \arrow[d, "(\omega_*)_{(2)}"] & \mathbb{Z}_{(2)} \arrow[r] \arrow[d, "(\omega_*)_{(2)}"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z}/2^6 \oplus \mathbb{Z}/2 \arrow[r, "(\Sigma^9 q)^*_{(2)}"] \arrow[d, two heads, "(\psi_*)_{(2)}"'] & \mathbb{Z}/2^6 \oplus \mathbb{Z}/2^6 \arrow[r, "(\Sigma^9\tilde{i})^*_{(2)}"] \arrow[d, two heads, "(\psi_*)_{(2)}"] & \mathbb{Z}/2^5 \arrow[r, "0"] \arrow[d] & \mathbb{Z}/2 \\
\mathbb{Z}_{(2)} \arrow[r, "0"'] & \mathbb{Z}/2 \arrow[r, "\cong"'] & {[\Sigma^9\mathbb{C}P^3, G/O]_{(2)}} \arrow[r] & 0.
\end{tikzcd}$$

From this diagram, we observe that the composite

$$(\Sigma^9\widetilde{\widetilde{i}})^*_{(2)} \circ (\omega_*)_{(2)} : [\Sigma^{10}\mathbb{C}P^3, G/Top]_{(2)} \to (\Theta_{11})_{(2)}$$

has image of order 2^4. This contradicts the fact that the map

$$\omega_* \circ (\Sigma^{10}\widetilde{\widetilde{i}})^* : [\Sigma^{10}\mathbb{C}P^3, G/Top] \to \Theta_{11}$$

is surjective. Therefore, the case $[\Sigma^9\mathbb{C}P^3, Top/O]_{(2)} \cong \mathbb{Z}/2^6 \oplus \mathbb{Z}/2^6$ is not possible.

We thus conclude that

$$[\Sigma^9\mathbb{C}P^3, Top/O] \cong \Theta_{15} \oplus \Theta_{13} \oplus \Theta_{11},$$

which is equal to $\Theta_{15} \oplus [\Sigma^9\mathbb{C}P^2, Top/O]$ by Proposition 5.1.3 (9). $\square$

Alternatively, the case $k = 1$ can be established independently using the quaternionic quasi-projective space. Recall that the quaternionic quasi-projective space $\mathbb{H}Q_n$ is defined by

$$\mathbb{H}Q_n := \mathbb{S}^{4n-1} \times \mathbb{S}^3/\sim,$$

where the equivalence relation $\sim$ is given by

$$(x, w) \sim (x\lambda, \lambda^{-1}w\lambda), \text{ for } \lambda \in \mathbb{S}^3, \quad \text{and} \quad (x, 1) \sim (y, 1).$$

We note from [94, Corollary 1.2] that the space $\mathbb{H}Q_2$ is the cofiber of $\mathbb{C}P^3 \to \mathbb{H}P^1 \simeq \mathbb{S}^4$. Applying $[\_, Top/O]$ to this cofiber sequence yields the isomorphism

$$[\Sigma\mathbb{C}P^3, Top/O] \cong [\mathbb{H}Q_2, Top/O].$$

According to [65], $\mathbb{H}Q_2$ admits a CW decomposition of the form $\mathbb{S}^3 \cup_\hbar \mathbb{D}^7$, where the attaching map $\hbar \in \pi_6(\mathbb{S}^3)$ is of order 12 [93]. Now, the cofiber sequence $\mathbb{S}^6 \xrightarrow{\hbar} \mathbb{S}^3 \overset{\imath}{\hookrightarrow} \mathbb{H}Q_2$ induces long exact sequences along G/O and G/Top, from which we obtain the identifications

$$[\Sigma\mathbb{H}Q_2, G/O] \cong \pi_8(G/O) \oplus \pi_4(G/O), \text{ and } [\Sigma\mathbb{H}Q_2, G/Top] \cong \pi_8(G/Top) \oplus \pi_4(G/Top).$$

Moreover, $[\mathbb{H}Q_2, Top/O]$ fits into the following short exact sequence

$$0 \to \mathbb{Z}/28 \xrightarrow{(f_{\mathbb{H}Q_2})^*} [\mathbb{H}Q_2, Top/O] \xrightarrow{\imath^*} \mathbb{Z}/2 \to 0.$$

We now show that this short exact sequence splits. Suppose, for contradiction, that it does not. Then $[\mathbb{H}Q_2, Top/O] \cong \mathbb{Z}/56$. Now, consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\cong"] \arrow[d, "\psi_*"'] & {[\Sigma\mathbb{H}Q_2, Top/O]} \arrow[r] \arrow[d, "\psi_*"] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r, "(\Sigma f_{\mathbb{H}Q_2})^*"] \arrow[d, "\phi_*"'] & {[\Sigma\mathbb{H}Q_2, G/O]} \arrow[r, "(\Sigma\imath)^*"] \arrow[d, "\phi_*"] & \mathbb{Z} \arrow[r] \arrow[d, "\phi_*"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "(\Sigma f_{\mathbb{H}Q_2})^*"'] \arrow[d, two heads, "\omega_*"'] & {[\Sigma\mathbb{H}Q_2, G/Top]} \arrow[r, "(\Sigma\imath)^*"'] \arrow[d, two heads, "\omega_*"] & \mathbb{Z} \arrow[r] \arrow[d, two heads, "\omega_*"] & 0 \\
0 \arrow[r] & \mathbb{Z}/28 \arrow[r, "(f_{\mathbb{H}Q_2})^*"'] & {[\mathbb{H}Q_2, Top/O]} \arrow[r, "\imath^*"'] & \mathbb{Z}/2 \arrow[r] & 0,
\end{tikzcd}$$

where the rows are induced from the cofiber sequence $\mathbb{S}^6 \xrightarrow{\hbar} \mathbb{S}^3 \overset{\imath}{\hookrightarrow} \mathbb{H}Q_2$ and columns are induced from the fiber sequence $\cdots \to \Omega(G/Top) \xrightarrow{\omega} Top/O \xrightarrow{\psi} G/O \xrightarrow{\phi} G/Top$.

We note from the above diagram that the image of $\phi_* : [\Sigma\mathbb{H}Q_2, G/O] \to [\Sigma\mathbb{H}Q_2, G/Top]$ that $28\mathbb{Z} \oplus 2\mathbb{Z}$. Therefore, the induced map $\omega_* : [\Sigma\mathbb{H}Q_2, G/Top] \to [\mathbb{H}Q_2, Top/O] \cong \mathbb{Z}/56$ cannot be surjective, leading to a contradiction. Hence, the short exact sequence must split, and we conclude that

$$[\mathbb{H}Q_2, Top/O] \cong \mathbb{Z}/28 \oplus \mathbb{Z}/2.$$

This concludes the proof of Proposition 5.3.6 (i).

5.4 Concordance Structure Set of the Product of a 3-Connected 8-Manifold with a Sphere

Let M be a closed, 3-connected, smooth 8-dimensional manifold of rank $m \geq 1$. Since $M \simeq \left(\bigvee_{i=1}^{m} \mathbb{S}^4\right) \cup_g \mathbb{D}^8$ and $\pi_4(Top/O)$ is trivial, it follows that $[M, Top/O] = \pi_8(Top/O)$. Thus, by Proposition 5.0.1, computing $[\Sigma^k M, Top/O]$ is sufficient to determine $\mathcal{C}(M \times \mathbb{S}^k)$. Rather than analyzing $\mathcal{C}(M \times \mathbb{S}^k)$ directly, we begin by studying $\mathcal{C}(\mathbb{H}P^2 \times \mathbb{S}^k)$, and in particular, we compute $[\Sigma^k\mathbb{H}P^2, Top/O]$.

To compute $[\Sigma^k\mathbb{H}P^2, Top/O]$, we repeatedly use the following long exact sequence (5.11)

$$\cdots \to \Theta_{5+k} \xrightarrow{\tau^*_{5+k}} \Theta_{8+k} \xrightarrow{(\Sigma^k f_{\mathbb{H}P^2})} [\Sigma^k\mathbb{H}P^2, Top/O] \xrightarrow{(\widehat{\Sigma^k i})^*} \Theta_{4+k} \xrightarrow{\tau^*_{4+k}} \Theta_{7+k} \to \cdots,$$

induced from the cofiber sequence

$$\cdots \to \mathbb{S}^{7+k} \xrightarrow{\tau_{4+k}} \mathbb{S}^{4+k} \xhookrightarrow{\Sigma^k \widehat{i}} \Sigma^k\mathbb{H}P^2 \xrightarrow{\Sigma^k f_{\mathbb{H}P^2}} \mathbb{S}^{8+k} \xrightarrow{\tau_{5+k}} \mathbb{S}^{5+k} \to \cdots. \tag{5.12}$$

In the following proposition, we formulate $[\Sigma^k\mathbb{H}P^2, Top/O]$ for $1 \leq k \leq 10$.

Proposition 5.4.1.

(i) $[\Sigma^k\mathbb{H}P^2, Top/O] \cong \Theta_{8+k}$ *for* $k = 1, 2, 8, 10$.

(ii) $[\Sigma^k \mathbb{H}P^2, Top/O] \cong \Theta_{4+k}$ *for* $k = 4, 5$.

(iii) There exists a split short exact sequence $0 \to \Theta_{11} \xrightarrow{(\Sigma^3 f_{\mathbb{H}P^2})^*} [\Sigma^3 \mathbb{H}P^2, Top/O] \xrightarrow{(\Sigma^3 \widehat{i})^*} \Theta_7 \to 0$.

(iv) There exists a short exact sequence $0 \to \mathbb{Z}/2 \xrightarrow{(\Sigma^6 f_{\mathbb{H}P^2})^*} [\Sigma^6 \mathbb{H}P^2, Top/O] \xrightarrow{(\Sigma^6 \widehat{i})^*} \mathbb{Z}/2 \to 0$, *where* $\mathbb{Z}/2 = \Theta_{14}$ *on the left side is the cokernel of* τ_{11}^* *and* $\mathbb{Z}/2 \subset \Theta_{10}$ *on the right side is the kernel of* τ_{10}^*.

(v) There exists a split short exact sequence $0 \to \Theta_{15} \xrightarrow{(\Sigma^7 f_{\mathbb{H}P^2})^*} [\Sigma^7 \mathbb{H}P^2, Top/O] \xrightarrow{(\Sigma^7 \widehat{i})^*} \Theta_{11} \to 0$, *where* Θ_{11} *is the kernel of* τ_{11}^*.

(vi) There exists a split short exact sequence $0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to [\Sigma^9 \mathbb{H}P^2, Top/O] \to \mathbb{Z}/3 \to 0$ *where* $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \subset \Theta_{17}$ *is the cokernel of* τ_{14}^* *and* $\mathbb{Z}/3$ *is the kernel of* τ_{13}^*.

Proof. The proofs of (i) and (ii) follow from the long exact sequence (5.11) and Lemma 4.3.1.

Let us consider the case $k = 3$. Since both $\tau_8^* : \Theta_8 \to \Theta_{11}$ and $\tau_7^* : \Theta_7 \to \Theta_{10}$ are zero by Lemma 4.3.1, the long exact sequence (5.11) reduces to the short exact sequence

$$0 \to \Theta_{11} \xrightarrow{(\Sigma^3 f_{\mathbb{H}P^2})^*} [\Sigma^3 \mathbb{H}P^2, Top/O] \xrightarrow{(\Sigma^3 \widehat{i})^*} \Theta_7 \to 0. \tag{5.13}$$

Our goal now is to prove that this sequence splits. Before proving that we note that the following short exact sequence induced from (5.12)

$$0 \to \pi_{12}(G/Top) \xrightarrow{(\Sigma^4 f_{\mathbb{H}P^2})^*} [\Sigma^4 \mathbb{H}P^2, G/Top] \xrightarrow{(\Sigma^4 \widehat{i})^*} \pi_8(G/Top) \to 0$$

splits as $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$.

Now consider the following ladder of exact sequences with the columns induced from the fiber sequence $\cdots \to \Omega(G/Top) \xrightarrow{\omega} Top/O \xrightarrow{\psi} G/O \xrightarrow{\phi} G/Top$, the rows arise from the cofiber sequence (5.12).

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & {[\Sigma^4\mathbb{H}P^2, Top/O]} \arrow[r, "\cong", "(\Sigma^4\widehat{i})^*"'] \arrow[d, "\psi_*"] & \mathbb{Z}/2 \arrow[r, "0"] \arrow[d, "\psi_*"] & \mathbb{Z}/992 \arrow[d] \\
\mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[r, "0"] \arrow[d] & \mathbb{Z} \arrow[r, "(\Sigma^4 f_{\mathbb{H}P^2})^*"] \arrow[d, "\phi_*"'] & {[\Sigma^4\mathbb{H}P^2, G/O]} \arrow[r, "(\Sigma^4\widehat{i})^*"] \arrow[d, "\phi_*"] & \mathbb{Z}\oplus\mathbb{Z}/2 \arrow[r] \arrow[d, "\phi_*"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "(\Sigma^4 f_{\mathbb{H}P^2})^*"] \arrow[d, "\omega_*"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(\Sigma^4\widehat{i})^*"] \arrow[d, "\omega_*"] & \mathbb{Z} \arrow[r] \arrow[d, "\omega_*"] & 0 \arrow[d] \\
\mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/992 \arrow[r, "(\Sigma^3 f_{\mathbb{H}P^2})^*"'] \arrow[d] & {[\Sigma^3\mathbb{H}P^2, Top/O]} \arrow[r, "(\Sigma^3\widehat{i})^*"'] \arrow[d] & \mathbb{Z}/28 \arrow[r, "0"] \arrow[d] & \mathbb{Z}/6 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

From the top part of the diagram, we observe that $[\Sigma^4\mathbb{H}P^2, G/O]$ is either $\mathbb{Z}\oplus\mathbb{Z}$ or $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2$. Since the map $\psi_* : [\Sigma^4\mathbb{H}P^2, Top/O] \to [\Sigma^4\mathbb{H}P^2, G/O]$ is injective and $[\Sigma^4\mathbb{H}P^2, Top/O] \cong \mathbb{Z}/2$, we conclude that

$$[\Sigma^4\mathbb{H}P^2, G/O] \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2.$$

Moreover, the image of the composition

$$(\Sigma^4 f_{\mathbb{H}P^2})^* \circ \phi_* : \pi_{12}(G/O) \to [\Sigma^4\mathbb{H}P^2, G/Top]$$

is $992\mathbb{Z}$, and the image of

$$\phi_* \circ (\Sigma^4\widehat{i})^* : [\Sigma^4\mathbb{H}P^2, G/O] \to \pi_8(G/Top)$$

is $28\mathbb{Z}$. It follows from the diagram that the image of

$$\omega_* : [\Sigma^4\mathbb{H}P^2, G/Top] \to [\Sigma^3\mathbb{H}P^2, Top/O]$$

is $\mathbb{Z}/992\oplus\mathbb{Z}/28$. This implies that the short exact sequence (5.13) splits.

The case $k = 6$ follows directly from the long exact sequence (5.11) and Theorem 4.4.1 (i).

We now consider the case $k = 7$. From the long exact sequence (5.11) and Theorem 4.4.1 (i), we obtain the following short exact sequence:

$$0 \to \Theta_{15} \xrightarrow{(\Sigma^7 f_{\mathbb{H}P^2})^*} [\Sigma^7 \mathbb{H}P^2, Top/O] \xrightarrow{(\Sigma^7 \widehat{i})^*} \Theta_{11} \to 0. \tag{5.14}$$

Note that it is enough to work 2-locally to determine whether this sequence splits. Assume, for contradiction, that the short exact sequence (5.14) does not split. Then $[\Sigma^7 \mathbb{H}P^2, Top/O]_{(2)}$ must be either $\mathbb{Z}/2 \oplus \mathbb{Z}/2^{11}$ or $\mathbb{Z}/2^6 \oplus \mathbb{Z}/2^6$. To analyze this, consider the commutative diagram where each row and column is an exact sequence induced from the cofiber sequence (5.12) and the fiber sequence $\cdots \to \Omega(G/Top) \xrightarrow{\omega} Top/O \xrightarrow{\psi} G/O \xrightarrow{\phi} G/Top$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "(\Sigma^8 f_{\mathbb{H}P^2})^*_{(2)}", "\cong"'] \arrow[d, "(\psi_*)_{(2)}"'] & {[\Sigma^8 \mathbb{H}P^2, Top/O]_{(2)}} \arrow[r, "(\Sigma^8 \widehat{i})^*_{(2)}"] \arrow[d, "(\psi_*)_{(2)}"] & 0 \arrow[r] \arrow[d, "(\psi_*)_{(2)}"] & \Theta_{15} \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \arrow[r, "(\Sigma^8 f_{\mathbb{H}P^2})^*_{(2)}"] \arrow[d, "(\phi_*)_{(2)}"'] & {[\Sigma^8 \mathbb{H}P^2, G/O]_{(2)}} \arrow[r, "(\Sigma^8 \widehat{i})^*_{(2)}"] \arrow[d, "(\phi_*)_{(2)}"] & \mathbb{Z}_{(2)} \arrow[r, "(\tau_{12})^*_{(2)}"] \arrow[d] & \mathbb{Z}/2 \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z}_{(2)} \arrow[r, "(\Sigma^8 f_{\mathbb{H}P^2})^*_{(2)}"] \arrow[d, "(\omega_*)_{(2)}"'] & \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)} \arrow[r, "(\Sigma^8 \widehat{i})^*_{(2)}"] \arrow[d, "(\omega_*)_{(2)}"] & \mathbb{Z}_{(2)} \arrow[r] \arrow[d, "(\omega_*)_{(2)}"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \oplus \mathbb{Z}/2^6 \arrow[r, "(\Sigma^8 f_{\mathbb{H}P^2})^*_{(2)}"'] \arrow[d, "(\psi_*)_{(2)}"'] & {[\Sigma^7 \mathbb{H}P^2, Top/O]_{(2)}} \arrow[r, "(\Sigma^7 \widehat{i})^*_{(2)}"'] \arrow[d, "(\psi_*)_{(2)}"] & \mathbb{Z}/2^5 \arrow[r, "0"] \arrow[d] & \mathbb{Z}/2 \\
\mathbb{Z}_{(2)} \arrow[r, "(\tau_{12})^*_{(2)}"'] & \mathbb{Z}/2 \arrow[r, "(\Sigma^8 f_{\mathbb{H}P^2})^*_{(2)}"'] \arrow[d] & {[\Sigma^7 \mathbb{H}P^2, G/O]_{(2)}} \arrow[r, "(\Sigma^7 \widehat{i})^*_{(2)}"'] \arrow[d] & 0 & \\
 & 0 & 0. & &
\end{tikzcd}$$

Since $\tau_{12}^* : \pi_{12}(G/O) \to \pi_{15}(G/O)$ is trivial and $\pi_{2t+1}(G/Top) = 0$ for all $t \geq 0$, the long exact sequences induced by the cofiber sequence (5.12) along G/O and G/Top, respectively, yield the following split short exact sequences:

$$0 \to \mathbb{Z} \to [\Sigma^8 \mathbb{H}P^2, G/Top] \to \mathbb{Z} \to 0, \tag{5.15}$$

and

$$0 \to \mathbb{Z} \oplus \mathbb{Z}/2 \to [\Sigma^8 \mathbb{H}P^2, G/O] \to \mathbb{Z} \to 0. \tag{5.16}$$

Since the short exact sequences (5.15) and (5.16) split, it follows from the above diagram that the image of

$$(\phi_*)_{(2)} : [\Sigma^8 \mathbb{H}P^2, G/O] \to [\Sigma^8 \mathbb{H}P^2, G/Top]$$

is isomorphic to $2^6\mathbb{Z}_{(2)} \oplus 2^5\mathbb{Z}_{(2)}$. Assume that

$$[\Sigma^7 \mathbb{H}P^2, Top/O]_{(2)} \cong \mathbb{Z}/2^6 \oplus \mathbb{Z}/2^6.$$

Then the image of the composition

$$(\widehat{\Sigma^7 i})^*_{(2)} \circ (\omega_*)_{(2)} : [\Sigma^8 \mathbb{H}P^2, G/Top]_{(2)} \to (\Theta_{11})_{(2)}$$

would be of order 2^4, contradicting the surjectivity of

$$(\omega_*)_{(2)} \circ (\widehat{\Sigma^8 i})^*_{(2)} : [\Sigma^8 \mathbb{H}P^2, G/Top]_{(2)} \to \pi_{11}(Top/O)_{(2)}.$$

Alternatively, suppose

$$[\Sigma^7 \mathbb{H}P^2, Top/O]_{(2)} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2^{11}.$$

Then the image of

$$(\omega_*)_{(2)} : [\Sigma^8 \mathbb{H}P^2, G/Top]_{(2)} \to [\Sigma^7 \mathbb{H}P^2, Top/O]$$

would have order 2^6, which is incompatible with the surjectivity of

$$(\psi_*)_{(2)} : [\Sigma^7 \mathbb{H}P^2, Top/O]_{(2)} \to [\Sigma^7 \mathbb{H}P^2, G/O]_{(2)} \cong \mathbb{Z}/2.$$

Therefore, the short exact sequence (5.14) must split.

For the case $k = 9$, Lemma 4.3.1 shows that the map $\tau_{14}^* : \Theta_{14} \to \Theta_{17}$ has image $\mathbb{Z}/2$, while $\tau_{13}^* : \Theta_{13} \to \Theta_{16}$ is trivial. It then follows from the long exact sequence (5.11) that

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to [\Sigma^9 \mathbb{H}P^2, Top/O] \to \mathbb{Z}/3 \to 0$$

is a short exact sequence, which clearly splits. □

For any closed, 3-connected, smooth 8-manifold M, it follows from the stable decompositions (3.25) and (3.23) that computing $[\Sigma^k M, Top/O]$ reduces to computing $[Cone(s\tau_{4+k}), Top/O]$, where $s \in \mathbb{Z}/24$ is determined by $\mathrm{ind}(M)$ in (3.25) or by t in (3.23). Note that $[Cone(s\tau_{4+k}), Top/O]$ fits into the long exact sequence:

(5.17)
$$\cdots \to \Theta_{5+k} \xrightarrow{(s\tau_{5+k})^*} \Theta_{8+k} \to [Cone(s\tau_{4+k}), Top/O] \to \Theta_{4+k} \xrightarrow{(s\tau_{4+k})^*} \Theta_{7+k} \to \cdots .$$

Following the same method as in Proposition 5.4.1, we compute $[Cone(s\tau_{4+k}), Top/O]$ using the long exact sequence (5.17) and Theorem 4.4.1. Combining these computations for $1 \leq k \leq 10$ with the stable decompositions (3.25) for rank $m \geq 2$ and (3.23) for rank $m = 1$, we then apply Proposition 5.0.1 to obtain the following result:

Theorem 5.4.2. *Let M be a closed, smooth, 3-connected, 8-dimensional manifold of rank $m \geq 1$. Then:*

(i) $\mathcal{C}(M \times \mathbb{S}^k) = \Theta_{8+k} \oplus \Theta_8$ *for* $k = 1$ *and* 2.

(ii) $\mathcal{C}(M \times \mathbb{S}^3) = \Theta_{11} \oplus \bigoplus_{i=1}^{m} \Theta_7 \oplus \Theta_8 \oplus \mathbb{Z}/2.$

(iii) $\mathcal{C}(M \times \mathbb{S}^4) = \bigoplus_{i=1}^{m+1} \Theta_8.$

(iv) $\mathcal{C}(M\times\mathbb{S}^k)=\Theta_{8+k}\oplus\bigoplus_{i=1}^{m}\Theta_{4+k}\oplus\Theta_8\oplus\Theta_k$ *for* $k=7$ *and* 8.

(v) *For* $k=5$:

- *(a)* $\mathcal{C}(M\times\mathbb{S}^5)=\Theta_{13}\oplus\bigoplus_{i=1}^{m}\Theta_9\oplus\Theta_8$, *if* $3\mid p_1(M)$.
- *(b)* $\mathcal{C}(M\times\mathbb{S}^5)=\bigoplus_{i=1}^{m}\Theta_9\oplus\Theta_8$, *if* $3\nmid p_1(M)$.

(vi) *For* $k=6$:

- *(a)* $\mathcal{C}(M\times\mathbb{S}^6)=G_1\oplus\mathbb{Z}/3\oplus\bigoplus_{i=1}^{m-1}\Theta_{10}\oplus\Theta_8$, *when* $3\mid p_1(M)$.
- *(b)* $\mathcal{C}(M\times\mathbb{S}^6)=G_2\oplus\bigoplus_{i=1}^{m-1}\Theta_{10}\oplus\Theta_8$, *when* $3\nmid p_1(M)$.

where the groups G_1 *and* G_2 *are extensions of the group* $\mathbb{Z}/2\subset\Theta_{10}$ *by the group* Θ_{14}.

(vii) *For* $k=9$:

- *(1)* *If* $m=1$, *then* $\mathcal{C}(M\times\mathbb{S}^9)=\bigoplus_{i=1}^{3}\mathbb{Z}/2\oplus\Theta_{13}\oplus\Theta_8\oplus\Theta_9$.
- *(2)* *If* $m\geq 2$:
  - *(a)* $\mathcal{C}(M\times\mathbb{S}^9)=\Theta_{17}\oplus\bigoplus_{i=1}^{m}\Theta_{13}\oplus\Theta_8\oplus\Theta_9$, *for even* $\operatorname{ind}(M)$.
  - *(b)* $\mathcal{C}(M\times\mathbb{S}^9)=\bigoplus_{i=1}^{3}\mathbb{Z}/2\oplus\bigoplus_{i=1}^{m}\Theta_{13}\oplus\Theta_8\oplus\Theta_9$, *for odd* $\operatorname{ind}(M)$.

(viii) *For* $k=10$:

- *(1)* *If* $m=1$, *then* $\mathcal{C}(M\times\mathbb{S}^{10})=\Theta_{18}\oplus\Theta_8\oplus\Theta_{10}$.
- *(2)* *If* $m\geq 2$, *then:*
  - *(a)* *when* $\operatorname{ind}(M)$ *is even, then* $\mathcal{C}(M\times\mathbb{S}^{10})=H\oplus\bigoplus_{i=1}^{m-1}\Theta_{14}\oplus\Theta_8\oplus\Theta_{10}$, *where the group* H *is an extension of the group* Θ_{14} *by the group* Θ_{18}.
  - *(b)* *When* $\operatorname{ind}(M)$ *is odd, then* $\mathcal{C}(M\times\mathbb{S}^{10})=\Theta_{18}\oplus\bigoplus_{i=1}^{m-1}\Theta_{14}\oplus\Theta_8\oplus\Theta_{10}$.

Chapter 6

Classification Results

In this chapter, we investigate the smooth classification problem for manifolds of the form $\mathbb{C}P^2 \times \mathbb{S}^k$, $\mathbb{C}P^3 \times \mathbb{S}^k$, and $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$, where $\widetilde{\mathbb{H}P}^2$ is a closed, 8-dimensional, smooth projective plane–like manifold, and k takes certain values between 1 and 10. We study the action of the group of self-homotopy equivalences on the concordance structure sets of these manifolds, as computed in Chapter 5, using normal invariants to distinguish homeomorphic manifolds that admit distinct smooth structures.

To proceed, we recall several results from [54, 16, 107, 18] concerning smooth and tangential structure sets, which will be used later in this chapter. Let $E_1(\mathbb{C}P^q)$ denote the path component of the space of self-maps of $\mathbb{C}P^q$ containing the identity. Let $F_{\mathbb{S}^1}(\mathbb{C}^{q+1})$ be the space of $\mathbb{S}^1$-equivariant self-maps of $\mathbb{S}^{2q+1}$. There exists a stabilization map

$$s_{q+1} : F_{\mathbb{S}^1}(\mathbb{C}^{q+1}) \to F_{\mathbb{S}^1}(\mathbb{C}^{q+2}),$$

induced by taking the equivariant join of an equivariant self-map of $\mathbb{S}^{2q+1}$ with the identity $\mathbb{S}^1$ [16, Page 2]. Now we define $F_{\mathbb{S}^1}$ as the colimit of this sequence of stabilization maps.

Fact 6.0.1.

(a) There is a group homomorphism $\alpha_{\mathbb{S}^k} : \pi_k(E_1(\mathbb{C}P^q)) \to \mathcal{E}(\mathbb{C}P^q \times \mathbb{S}^k)$ for all $k, q \geq 1$, where $\mathcal{E}(X)$ denotes the group of all homotopy classes of based self-

homotopy equivalences of X [18, Page 15].

(b) For $q \geq 1$ and $k \geq 2$, there is a group homomorphism $\Psi : \pi_k(E_1(\mathbb{C}P^q)) \to \mathcal{S}^{Diff}(\mathbb{C}P^q \times \mathbb{D}^k \text{ rel } \mathbb{C}P^q \times \mathbb{S}^{k-1})$, and a canonical base-point preserving map $\Gamma : \mathcal{S}^{Diff}(\mathbb{C}P^q \times \mathbb{D}^k \text{ rel } \mathbb{C}P^q \times \mathbb{S}^{k-1}) \to \mathcal{S}^{Diff}(\mathbb{C}P^q \times \mathbb{S}^k)$ [18, Page 24]. Moreover, the composition $\Gamma \circ \Psi$ sends the element $\alpha \in \pi_k(E_1(\mathbb{C}P^q))$ to the element $[(\mathbb{C}P^q \times \mathbb{S}^k, f)]$ in $\mathcal{S}^{Diff}(\mathbb{C}P^q \times \mathbb{S}^k)$, where f represents a tangential self-homotopy equivalence of $\mathbb{C}P^q \times \mathbb{S}^k$ that comes from α [18, Proposition 7.3].

(c) The homotopy groups $\pi_k(F_{\mathbb{S}^1}(\mathbb{C}^{q+1}))$ and $\pi_k(E_1(\mathbb{C}P^q))$ are isomorphic for all $q \geq 1$ and $k \geq 2$ (see [54, Theorems 2.1-2.2] and [16, Theorem 11.1]).

(d) For all $q \geq 1$, $\pi_k(F_{\mathbb{S}^1}(\mathbb{C}^{q+1}))$ is isomorphic to $\pi_k(F_{\mathbb{S}^1})$ if $1 \leq k \leq 2q$ [18, Proposition 6.1].

(e) $\pi_k(F_{\mathbb{S}^1}) \cong \pi_k^s(\Sigma\mathbb{C}P^\infty) \oplus \pi_{k-1}^s$ for $k \geq 1$ (see [16, Theorem 6.6] and [18, Page 18]).

(f) Let $\alpha \in \pi_{k-1}^s$ correspond to the element $f \in \pi_k(F_{\mathbb{S}^1})$. Then the normal invariant of f is the image of the composition $\mathbb{S}^k \wedge \mathbb{C}P^\infty_+ \xrightarrow{\alpha \wedge Id_{\mathbb{C}P^\infty_+}} \mathbb{S}^1 \wedge \mathbb{C}P^\infty_+ \xrightarrow{t} \mathbb{S}^0$ under the canonical map $j_* : [\Sigma^k\mathbb{C}P^\infty_+, G] \to [\Sigma^k\mathbb{C}P^\infty_+, G/O]$, [107, Page 193] where $t : \mathbb{S}^1 \wedge \mathbb{C}P^\infty_+ \to \mathbb{S}^0$ is the Umkehr or tranfer map [16]. Further, we note that $t \circ (\alpha \wedge Id_{\mathbb{C}P^\infty_+})$ is stably homotopic to $\alpha \circ \Sigma^{k-1}t$ for $k \geq 2$.

Note 6.0.2. *We note from [18, Page 33] that the restriction of the Umkehr map t to $\Sigma\mathbb{C}P^1$ is a generator of $\pi_3^s \cong \mathbb{Z}/24$. Using this fact, it follows from the following* *diagram that*

$$\begin{tikzcd}
t \arrow[r, mapsto] \arrow[ddd, mapsto, bend right=60] & t|_{\Sigma\mathbb{C}P^1} \arrow[ddd, mapsto, bend left=60] \\
\{\Sigma\mathbb{C}P^\infty, \mathbb{S}^0\} \arrow[r] \arrow[d, "\Sigma^3"'] & \{\mathbb{S}^3, \mathbb{S}^0\} \cong \pi_3^s \arrow[d, "\cong"', "\Sigma^3"] \\
\{\Sigma^4\mathbb{C}P^\infty, \mathbb{S}^3\} \arrow[r] & \{\mathbb{S}^6, \mathbb{S}^3\} \cong \pi_3^s \\
\Sigma^3 t \arrow[r, mapsto] & \Sigma^3 t|_{\Sigma^4\mathbb{C}P^1}
\end{tikzcd}$$

$\Sigma^3 t : \Sigma^4\mathbb{C}P^\infty_+ \to \mathbb{S}^3$ *is non-trivial.*

6.1 Smooth Classification of $\mathbb{C}P^2 \times \mathbb{S}^k$ for $4 \leq k \leq 6$

We begin with the product manifold $\mathbb{C}P^2 \times \mathbb{S}^4$. Using the previously recalled facts and methods from surgery theory, we establish the following classification result.

Lemma 6.1.1. *There exists a tangential homotopy equivalence* $f_{2\nu} : \mathbb{C}P^2 \times \mathbb{S}^4 \to \mathbb{C}P^2 \times \mathbb{S}^4$ *whose normal invariant is nontrivial. In particular, the normal invariant of* $f_{2\nu}$ *is given by* $f^*_{\mathbb{C}P^2\times\mathbb{S}^4}([\epsilon])$*, where* $[\epsilon]$ *denotes the generator of the 2-torsion subgroup of* $\pi_8(G/O) \subseteq [\mathbb{C}P^2 \times \mathbb{S}^4, G/O]$.

Proof. By Fact 6.0.1 (d) and (e), we have $\pi_4(F_{\mathbb{S}^1}(\mathbb{C}^3)) \cong \pi_4(F_{\mathbb{S}^1}) \cong \mathbb{Z}/8\{\nu\} \oplus \mathbb{Z}/3\{\alpha_1\}$. Let $f_{2\nu} : \mathbb{C}P^2 \times \mathbb{S}^4 \to \mathbb{C}P^2 \times \mathbb{S}^4$ be the self-homotopy equivalence induced by the element $2\nu \in \pi_4(F_{\mathbb{S}^1}(\mathbb{C}^3))$.

Now it follows from Fact 6.0.1 (f) together with $\pi_4^s = 0$ that the normal invariant $\eta^{Diff}(f_{2\nu})$ is the image of the composition map $(2\nu) \circ \Sigma^3 t : \Sigma^4\mathbb{C}P^2 \to \mathbb{S}^0$ in $[\Sigma^4\mathbb{C}P^2, G/O] \subset [\mathbb{C}P^2 \times \mathbb{S}^4, G/O]$. Since the restriction $((2\nu) \circ \Sigma^3 t)|_{\Sigma^4\mathbb{C}P^1}$ is null-homotopic, it suffices to analyze the Toda bracket $\langle \eta, \nu, 2\nu \rangle$ to determine whether the composition is trivial.

By [125, Page 189], we have $\langle \eta, \nu, 2\nu \rangle = \{\bar{\nu}, \epsilon\}$, which projects nontrivially to the class $[\epsilon] \in \pi_8(G/O)$. This shows that $\eta^{Diff}(f_{2\nu})$ is the image of $[\epsilon]$ under the map

$(\Sigma^4 f_{\mathbb{C}P^2})^* : \pi_8(G/O) \to [\Sigma^4\mathbb{C}P^2, G/O]$.

The conclusion follows from the commutative diagram:

$$\begin{tikzcd}
 & {[\Sigma^4\mathbb{C}P^2, G/O]} \arrow[dr, "p^*"] & \\
\pi_8(G/O) \arrow[ur, "(\Sigma^4 f_{\mathbb{C}P^2})^*"] \arrow[rr, "(f_{\mathbb{C}P^2\times\mathbb{S}^4})^*"'] & & {[\mathbb{C}P^2\times\mathbb{S}^4, G/O],}
\end{tikzcd}$$

where the map $p^* : [\Sigma^4\mathbb{C}P^2, G/O] \to [\mathbb{C}P^2\times\mathbb{S}^4, G/O]$ is injective. □

Theorem 6.1.2. *Let N be a closed, oriented, smooth manifold that is homeomorphic to $\mathbb{C}P^2\times\mathbb{S}^4$. Then N is oriented diffeomorphic to $\mathbb{C}P^2\times\mathbb{S}^4$.*

Proof. Suppose we have a homeomorphism $g : N \to \mathbb{C}P^2\times\mathbb{S}^4$. This yields an element $[(N, g)] \in \mathcal{C}(\mathbb{C}P^2\times\mathbb{S}^4)$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\Theta_8 \arrow[rr, "(f_{\mathbb{C}P^2\times\mathbb{S}^4})^*", "\cong"'] \arrow[d, "\psi_*"'] & & \mathcal{C}(\mathbb{C}P^2\times\mathbb{S}^4) \cong [\Sigma^4\mathbb{C}P^2, Top/O] \arrow[d, "\psi_*"] \\
\pi_8(G/O) \arrow[rr, "(f_{\mathbb{C}P^2\times\mathbb{S}^4})^*"'] \arrow[dr, "(\Sigma^4 f_{\mathbb{C}P^2})^*"'] & & {[\mathbb{C}P^2\times\mathbb{S}^4, G/O]} \\
 & {[\Sigma^4\mathbb{C}P^2, G/O],} \arrow[ur, "p^*"'] &
\end{tikzcd}$$

where the induced degree one map $f^*_{\mathbb{C}P^2\times\mathbb{S}^4} : \Theta_8 \to \mathcal{C}(\mathbb{C}P^2\times\mathbb{S}^4)$ is an isomorphism by Proposition 5.0.1 and Proposition 5.1.3 (4), and the map $(\Sigma^4 f_{\mathbb{C}P^2})^* : \pi_8(G/O) \to [\Sigma^4\mathbb{C}P^2, G/O]$ is injective as $\pi_7(G/O) = 0$.

From this diagram, it follows that the normal invariant $\eta^{Diff}(g)$ is either trivial or equal to $(f_{\mathbb{C}P^2\times\mathbb{S}^4})^*([\epsilon])$. In the latter case, applying Lemma 6.1.1 and the composition formula for normal invariants yields

$$\begin{aligned}\eta^{Diff}(f_{2\nu}\circ g) &= \eta^{Diff}(f_{2\nu}) + (f_{2\nu}^{-1})^*\eta^{Diff}(g)\\ &= (f_{\mathbb{C}P^2\times\mathbb{S}^4})^*([\epsilon]) \pm (f_{\mathbb{C}P^2\times\mathbb{S}^4})^*([\epsilon]) = 0.\end{aligned}$$

This implies that $(N, f_{2\nu}\circ g)$ and $(\mathbb{C}P^2\times\mathbb{S}^4, Id)$ represent the same element in $\mathcal{S}_h^{Diff}(\mathbb{C}P^2\times\mathbb{S}^4)$ from the smooth surgery exact sequence (2.8). This completes the proof of the theorem. $\square$

From the result established in the preceding theorem, we determine the inertia group of $\mathbb{C}P^2 \times \mathbb{S}^4$.

Corollary 6.1.3. *The inertia group of the product manifold $\mathbb{C}P^2 \times \mathbb{S}^4$ is*

$$I(\mathbb{C}P^2 \times \mathbb{S}^4) = \Theta_8.$$

We now proceed to analyze the smooth structures on $\mathbb{C}P^2 \times \mathbb{S}^5$, beginning with an investigation of the normal invariants of elements in $\mathcal{C}(\mathbb{C}P^2 \times \mathbb{S}^5)$. The first step in this direction is the following result.

Lemma 6.1.4. *The map $\psi_* : [\mathbb{C}P^2 \times \mathbb{S}^5, Top/O] \to [\mathbb{C}P^2 \times \mathbb{S}^5, G/O]$ is trivial.*

Proof. Since the short exact sequence (4.1) splits for both Top/O and G/O, we obtain a commutative diagram in which each summand of $[\mathbb{C}P^2 \times \mathbb{S}^5, Top/O]$ maps to the corresponding summand of $[\mathbb{C}P^2 \times \mathbb{S}^5, G/O]$ under the induced homomorphism from the canonical map $\psi : Top/O \to G/O$. Examining this diagram, and using the facts that both $[\Sigma^5\mathbb{C}P^2, G/O]$ and $[\mathbb{C}P^2, Top/O]$ vanish, we conclude that the map

$$\psi_* : [\mathbb{C}P^2 \times \mathbb{S}^5, Top/O] \to [\mathbb{C}P^2 \times \mathbb{S}^5, G/O]$$

is trivial, as claimed. $\square$

Theorem 6.1.5. *Let N be a closed, smooth, oriented manifold that is homeomorphic to $\mathbb{C}P^2 \times \mathbb{S}^5$. Then N is oriented diffeomorphic to $\mathbb{C}P^2 \times \mathbb{S}^5$.*

Proof. Let $g : N \to \mathbb{C}P^2 \times \mathbb{S}^5$ be a homeomorphism. We show that the pairs (N, g) and $(\mathbb{C}P^2 \times \mathbb{S}^5, Id)$ represent the same element in $\mathcal{S}_h^{Diff}(\mathbb{C}P^2 \times \mathbb{S}^5)$. Since the normal invariant $\eta^{Diff}(g)$ vanishes by Lemma 6.1.4, the smooth surgery exact sequence (2.8) for $\mathbb{C}P^2 \times \mathbb{S}^5$ implies that there exists $[\Sigma] \in bP_{10} = \mathbb{Z}/2$ such that (N, g) is equivalent to $(\mathbb{C}P^2 \times \mathbb{S}^5 \# \Sigma, h_\Sigma)$ in $\mathcal{S}_h^{Diff}(\mathbb{C}P^2 \times \mathbb{S}^5)$, where $h_\Sigma : (\mathbb{C}P^2 \times \mathbb{S}^5) \# \Sigma \to \mathbb{C}P^2 \times \mathbb{S}^5$

is the canonical homeomorphism. We claim that Σ is (oriented) diffeomorphic to the standard 9-sphere $\mathbb{S}^9$, which is equivalent to showing that the surgery obstruction map $\sigma_{10}^{Diff} : [\Sigma(\mathbb{C}P^2 \times \mathbb{S}^5), G/O] \to L_{10}(\mathbb{Z})$ is nonzero. Now consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}/2\{[\nu^2]\} \cong \pi_6(G/O) \arrow[r, "(Pr)^*"] & {[\Sigma(\mathbb{C}P^2 \times \mathbb{S}^5), G/O]} \arrow[r, "p^*"] \arrow[d, "\sigma_{10}^{Diff}"'] & {[\mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1, G/O]} \arrow[d, "\sigma_{10}^{Diff}"] \\
& L_{10}(\mathbb{Z}) \arrow[r, "\cong"'] \arrow[d, "\omega^{diff}"'] & L_{10}(\mathbb{Z}[\mathbb{Z}]) \\
& \mathcal{S}_h^{Diff}(\mathbb{C}P^2 \times \mathbb{S}^5), &
\end{tikzcd}$$

where the map $p^* : [\Sigma(\mathbb{C}P^2 \times \mathbb{S}^5), G/O] \to [\mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1, G/O]$ is injective by (4.1), the induced map $(Pr)^* : \pi_6(G/O) \to [\Sigma(\mathbb{C}P^2 \times \mathbb{S}^5), G/O]$, coming from the projection $Pr : \Sigma(\mathbb{C}P^2 \times \mathbb{S}^5) \simeq \Sigma\mathbb{C}P^2 \vee \Sigma\mathbb{S}^5 \vee \Sigma^6\mathbb{C}P^2 \to \Sigma\mathbb{S}^5$, is injective, and the inclusion $L_{10}(\mathbb{Z}) \hookrightarrow L_{10}(\mathbb{Z}[\mathbb{Z}])$ is the isomorphism given in [113].

Using the above commutative diagram, the nontriviality of the map $\sigma_{10}^{Diff} : [\Sigma(\mathbb{C}P^2 \times \mathbb{S}^5), G/O] \to L_{10}(\mathbb{Z})$ follows from the nontriviality of the composition $\sigma_{10}^{Diff} \circ p^* \circ (Pr)^* : \pi_6(G/O) \to L_{10}(\mathbb{Z}[\mathbb{Z}])$. Let $f : \mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1 \to G/O$ be the image of $[\nu^2] \in \pi_6(G/O)$ under the induced map $p^* \circ (Pr)^* : \pi_6(G/O) \to [\mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1, G/O]$. Then, the surgery obstruction of f is given by [128]:

$$\begin{aligned}
\sigma_{10}^{Diff}(f) &= \langle V(\mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1)^2 f^* \left(\bigoplus_j \mathcal{K}_{2^{j+1}-2} \right), \ [\mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1] \rangle \\
&= \langle (1 + x^2) f^* \left(\bigoplus_j \mathcal{K}_{2^{j+1}-2} \right), \ [\mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1] \rangle \\
&= \langle x^2 f^*(\mathcal{K}_6), \ [\mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1] \rangle,
\end{aligned} \tag{6.1}$$

where $(V(\mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1))^2 = 1 + x^2$ is the square of the total Wu class of the manifold $\mathbb{C}P^2 \times \mathbb{S}^5 \times \mathbb{S}^1$, x is the generator of $H^2(\mathbb{C}P^2; \mathbb{Z}/2)$, and $\bigoplus_j \mathcal{K}_{2^{j+1}-2} \in H^*(G/O; \mathbb{Z}/2)$ is the total smooth Kervaire class.

Observe that $f^*(\mathcal{K}_6) = p^* \circ (Pr)^* \circ ([\nu^2])^*(\mathcal{K}_6) \neq 0$, since $([\nu^2])^*(\mathcal{K}_6) \neq 0$. It then follows from (6.1) that $\sigma_{10}^{Diff}(f) \neq 0$ in $L_{10}(\mathbb{Z})$. This concludes the proof. □

The following corollary is an immediate consequence of the preceding theorem:

Corollary 6.1.6. *The inertia group of* $\mathbb{C}P^2 \times \mathbb{S}^5$ *is equal to* Θ_9*; that is,*

$$I(\mathbb{C}P^2 \times \mathbb{S}^5) = \Theta_9.$$

We now proceed to the classification of smooth structures on $\mathbb{C}P^2 \times \mathbb{S}^6$. The initial step in this direction is the following lemma.

Lemma 6.1.7. *Every self-homotopy equivalence of* $\mathbb{C}P^2 \times \mathbb{S}^6$ *arising from an element of* $\pi_6(F_{\mathbb{S}^1}(\mathbb{C}^3))$ *is homotopic to a diffeomorphism.*

Proof. To prove the lemma, it is enough to show that the homomorphism

$$\Psi : \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^3)) \to \mathcal{S}_6^{Diff}(\mathbb{C}P^2)$$

is trivial, by Fact 6.0.1 (b) and (c). We now consider the commutative diagram (2.11) for $M = \mathbb{C}P^2$ and $k = 6$:

$$\begin{tikzcd}
0 \arrow[r] & {[\Sigma^6\mathbb{C}P^2_+, PL]} \arrow[r] \arrow[d, "\mathcal{F}\circ\mathcal{H}"'] & {[\Sigma^6\mathbb{C}P^2_+, G]} \arrow[r, "\beta"] \arrow[d, "j_*"] & {[\Sigma^6\mathbb{C}P^2_+, G/PL]} \arrow[d, "\sigma_{10}^{PL}"] \\
0 \arrow[r] & \mathcal{S}_6^{Diff}(\mathbb{C}P^2) \arrow[r, "\eta_{\text{rel}}^{Diff}"'] & {[\Sigma^6\mathbb{C}P^2_+, G/O]} \arrow[r, "\sigma_{\text{rel}}^{Diff}"'] & L_{10}(\mathbb{Z}),
\end{tikzcd}$$

where the top row is induced from the fiber sequence $\cdots \to \Omega(G/PL) \to PL \to G \to G/PL$, while the bottom row corresponds to the relative smooth surgery exact sequence (2.9).

Note that the restricted maps $\sigma_{10}^{PL}|_{\pi_6(G/PL)} : \pi_6(G/PL) \to L_{10}(\mathbb{Z})$, $j_*|_{\pi_6^s} : \pi_6^s \to \pi_6(G/O)$, and $\beta|_{\pi_6^s} : \pi_6^s \to \pi_6(G/PL) \subset [\Sigma^6\mathbb{C}P^2_+, G/PL]$ are all isomorphisms. It follows from the diagram that the surgery obstruction map

$$\sigma_{\text{rel}}^{Diff}|_{\pi_6(G/O)} : \pi_6(G/O) \to L_{10}(\mathbb{Z})$$

is non-zero, and the image of the map

$$\eta_{\text{rel}}^{Diff} : \mathcal{S}_6^{Diff}(\mathbb{C}P^2) \to [\Sigma^6\mathbb{C}P^2_+, G/O]$$

lies in the component $[\Sigma^6\mathbb{C}P^2, G/O]$. Since $\pi_6(F_{\mathbb{S}^1}(\mathbb{C}^3))$ is a finite group [18, Page 29], it follows that the image of the composition

$$\eta_{\text{rel}}^{Diff} \circ \Psi : \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^3)) \to [\Sigma^6\mathbb{C}P^2_+, G/O]$$

is contained in the torsion subgroup of $[\Sigma^6\mathbb{C}P^2, G/O]$, which is isomorphic to $\mathbb{Z}/3$. However, as shown in [110], this composition fits into the following commutative diagram:

$$\begin{tikzcd}
\pi_6(F_{\mathbb{S}^1}(\mathbb{C}^3)) \arrow[r, "\eta_{\text{rel}}^{Diff}\circ\Psi"] \arrow[d, "s_3"'] & {[\Sigma^6\mathbb{C}P^2_+, G/O]} \\
\pi_6(F_{\mathbb{S}^1}(\mathbb{C}^4)) \cong \pi_6(F_{\mathbb{S}^1}) \arrow[r, "\eta_{\text{rel}}^{Diff}\circ\Psi"'] & {[\Sigma^6\mathbb{C}P^3_+, G/O],} \arrow[u, "(\Sigma^6 i' \vee Id)^*"']
\end{tikzcd}$$

where $s_3 : \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^3)) \to \pi_6(F_{\mathbb{S}^1})$ is the stabilization map and the right vertical arrow is induced from the inculsion $\Sigma^6 i' : \Sigma^6\mathbb{C}P^2 \hookrightarrow \Sigma^6\mathbb{C}P^3$ along G/O.

Since $\pi_6(F_{\mathbb{S}^1}) \cong \mathbb{Z}/2$ [92], the diagram shows that the image of the composition

$$\eta_{\text{rel}}^{Diff} \circ \Psi : \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^3)) \to [\Sigma^6\mathbb{C}P^2_+, G/O]$$

is trivial. Using the injectivity of the map $\eta_{\text{rel}}^{Diff} : \mathcal{S}_6^{Diff}(\mathbb{C}P^2) \to [\Sigma^6\mathbb{C}P^2_+, G/O]$, we conclude that the homomorphism

$$\Psi : \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^3)) \to \mathcal{S}_6^{Diff}(\mathbb{C}P^2)$$

is trivial. This completes the proof. $\square$

We now examine whether every element of $\mathcal{E}(\mathbb{C}P^2 \times \mathbb{S}^6)$ is represented by a diffeomorphism:

Theorem 6.1.8. *Let $f \in \mathcal{E}(\mathbb{C}P^2 \times \mathbb{S}^6)$. Then f is homotopic to a diffeomorphism.*

Proof. We note from the long exact sequence (5.1) that $[\mathbb{C}P^2, \mathbb{S}^6] = 0$. Therefore, by Proposition 2.6.21, every self-homotopy equivalence of $\mathbb{C}P^2 \times \mathbb{S}^6$ is diagonalizable. By Proposition 2.6.22 (a), the group of self-homotopy equivalences then admits the decomposition:

$$\mathcal{E}(\mathbb{C}P^2 \times \mathbb{S}^6) = \mathcal{E}_{\mathbb{C}P^2}(\mathbb{C}P^2 \times \mathbb{S}^6) \cdot \mathcal{E}_{\mathbb{S}^6}(\mathbb{C}P^2 \times \mathbb{S}^6). \tag{6.2}$$

Moreover, according to Proposition 2.6.22 (b), $\mathcal{E}_{\mathbb{C}P^2}(\mathbb{C}P^2 \times \mathbb{S}^6)$ and $\mathcal{E}_{\mathbb{S}^6}(\mathbb{C}P^2 \times \mathbb{S}^6)$ fit into the following two split short exact sequences:

$$0 \to [\mathbb{C}P^2, E_1(\mathbb{S}^6)] \to \mathcal{E}_{\mathbb{C}P^2}(\mathbb{C}P^2 \times \mathbb{S}^6) \to \mathcal{E}(\mathbb{S}^6) \to 0$$

and

$$0 \to \pi_6(E_1(\mathbb{C}P^2)) \to \mathcal{E}_{\mathbb{S}^6}(\mathbb{C}P^2 \times \mathbb{S}^6) \to \mathcal{E}(\mathcal{C}P^2) \to 0,$$

where $[\mathbb{C}P^2, E_1(\mathbb{S}^6)] \cong [\mathbb{C}P^2, SG_7] \cong 0$.

Observe that any self-homotopy equivalence of $\mathbb{C}P^2 \times \mathbb{S}^6$ arising from either factor $\mathcal{E}(\mathbb{S}^6)$ or $\mathcal{E}(\mathbb{C}P^2)$ is homotopic to a diffeomorphism. In addition, Lemma 6.1.7 guarantees the same for self-homotopy equivalences corresponding to elements of $\pi_6(E_1(\mathbb{C}P^2)) \cong \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^3))$. Combining these observations with the decomposition (6.2), and using the composition formula for normal invariants, we conclude that the normal invariant of any self-homotopy equivalence of $\mathbb{C}P^2 \times \mathbb{S}^6$ vanishes. Hence, any self-homotopy equivalence of $\mathbb{C}P^2 \times \mathbb{S}^6$ is homotopic to a diffeomorphism. □

It is known that $I_c(M) \subseteq I_h(M)$ for any closed, oriented smooth manifold M. The following lemma shows that equality holds in the case of the product manifold $\mathbb{C}P^q \times \mathbb{S}^{2k}$.

Lemma 6.1.9. *For all integers $q, k \geq 1$, we have*

$$I_h(\mathbb{C}P^q \times \mathbb{S}^{2k}) = I_c(\mathbb{C}P^q \times \mathbb{S}^{2k}).$$

Proof. This follows directly from [59, Corollary 3.2]. ☐

Theorem 6.1.10. *The inertia group of $\mathbb{C}P^2 \times \mathbb{S}^6$ is $\mathbb{Z}/2$.*

Proof. Theorem 4.1.5, together with Lemma 6.1.9, implies that $I_h(\mathbb{C}P^2 \times \mathbb{S}^6) = \mathbb{Z}/2$. The result then follows from Theorem 6.1.8. ☐

Furthermore, we obtain

Theorem 6.1.11. *Let N be a closed, oriented, smooth manifold that is homeomorphic to $\mathbb{C}P^2 \times \mathbb{S}^6$. Then N is oriented diffeomorphic to exactly one of the following manifolds:*

$$\mathbb{C}P^2 \times \mathbb{S}^6, \quad (\mathbb{C}P^2 \times \mathbb{S}^6)\#\Sigma_{\beta_1}, \quad \textit{or} \quad (\mathbb{C}P^2 \times \mathbb{S}^6)\#\Sigma_{\beta_1}^{-1},$$

where $[\Sigma_{\beta_1}] \in \Theta_{10}$ is the exotic 10-sphere of order 3.

Proof. It follows from the decomposition (4.1) and the proof of Proposition 5.1.3 (6) that

$$[\mathbb{C}P^2 \times \mathbb{S}^6, Top/O] \cong [\Sigma^6\mathbb{C}P^2, Top/O] \cong \mathbb{Z}/3$$

and the induced degree one map $f^*_{\mathbb{C}P^2\times\mathbb{S}^6} : \Theta_{10} \to [\mathbb{C}P^2 \times \mathbb{S}^6, Top/O]$ is onto. This implies that the concordance structure set $\mathcal{C}(\mathbb{C}P^2 \times \mathbb{S}^6)$ is equal to

$$\{[(\mathbb{C}P^2 \times \mathbb{S}^6)\#\Sigma_{\beta_1}] \ : \ [\Sigma_{\beta_1}] \in \mathbb{Z}/3 \subset \Theta_{10}\}.$$

The theorem now follows from Theorem 6.1.10. ☐

6.2 Smooth Classification of $\mathbb{C}P^3 \times \mathbb{S}^k$ for $1 \leq k \leq 7$

This section is devoted to computing the structure set $\mathcal{S}(\mathbb{C}P^3 \times \mathbb{S}^k)$ for each k with $1 \leq k \leq 7$, using the results established in Proposition 5.3.6. Additionally, we briefly recall the following facts.

Fact 6.2.1. *[18, Proposition 10.2] Let M be a closed, connected, smooth n-dimensional manifold. Suppose $k \geq 2$ and $n+k \geq 5$. Then the normal invariant of any self-homotopy equivalence $f : M \times \mathbb{S}^k \to M \times \mathbb{S}^k$ arising from an element of $[M, SG_{k+1}]$ lies in the image of the inclusion*

$$[M, G/O] \hookrightarrow [M \times \mathbb{S}^k, G/O].$$

Fact 6.2.2. *[109, Page-144] Let M be a closed, oriented, smooth manifold of dimension $n \geq 5$. Let $\sigma^{Top}_{n+1} : [\Sigma M, G/Top] \to L_{n+1}(\pi_1(M))$ be the topological surgery obstruction map, and $\omega : \Omega(G/Top) \to Top/O$ be the canonical map in the extended fiber sequence $Top/O \xrightarrow{\psi} G/O \xrightarrow{\phi} G/Top$. Then the concordance classes of smoothing on M that are equivalent to (M, Id) in $\mathcal{S}^{Diff}_h(M)$ are the elements of the set $\omega_*(Ker(\sigma^{Top}_{n+1})) \subset [M, Top/O]$.*

The following lemma is a central ingredient in the proofs of the classification results.

Lemma 6.2.3.

(i) The map $\psi_* : [\mathbb{C}P^3 \times \mathbb{S}^1, Top/O] \to [\mathbb{C}P^3 \times \mathbb{S}^1, G/O]$ is trivial.

(ii) The image of $\psi_* : [\mathbb{C}P^3 \times \mathbb{S}^2, Top/O] \to [\mathbb{C}P^3 \times \mathbb{S}^2, G/O]$ is $\mathbb{Z}/2\{(f_{\mathbb{C}P^3 \times \mathbb{S}^2})^*([\epsilon])\}$.

(iii) The image of $\psi_* : [\mathbb{C}P^3 \times \mathbb{S}^3, Top/O] \to [\mathbb{C}P^3 \times \mathbb{S}^3, G/O]$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Moreover, the image is generated by $(f_{\mathbb{C}P^3 \times \mathbb{S}^3})^*([\epsilon\eta])$ and $(f_{\mathbb{C}P^3 \times \mathbb{S}^3})^*([\mu])$.

(iv) The image of $\psi_* : [\mathbb{C}P^3 \times \mathbb{S}^4, Top/O] \to [\mathbb{C}P^3 \times \mathbb{S}^4, G/O]$ is $\mathbb{Z}/2\{(\Sigma^4 Pr \circ \Sigma^4 q)^*[\epsilon]\} \oplus \mathbb{Z}/2\{(f_{\mathbb{C}P^3 \times \mathbb{S}^4})^*[\eta\mu]\} \oplus \mathbb{Z}/3\{(f_{\mathbb{C}P^3 \times \mathbb{S}^4})^*[\beta_1]\}$.

(v) $\psi_* : [\mathbb{C}P^3 \times \mathbb{S}^5, Top/O] \to [\mathbb{C}P^3 \times \mathbb{S}^5, G/O]$ is a trivial map.

(vi) The image of $\psi_* : [\mathbb{C}P^3 \times \mathbb{S}^6, Top/O] \to [\mathbb{C}P^3 \times \mathbb{S}^6, G/O]$ is $\mathbb{Z}/3$, generated by $(\Sigma^6 Pr \circ \Sigma^6 q)^*[\beta_1]$.

(vii) The image of $\psi_* : [\mathbb{C}P^3 \times \mathbb{S}^7, Top/O] \to [\mathbb{C}P^3 \times \mathbb{S}^7, G/O]$ is $\mathbb{Z}/2$ and is generated by $[\nu^3]$.

Here $Pr : \mathbb{C}P^3/\mathbb{C}P^1 \simeq \mathbb{S}^4 \vee \mathbb{S}^6 \to \mathbb{S}^4$ denotes the projection map.

Proof. Since the short exact sequence (4.1) splits both along Top/O and G/O, we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & {[\Sigma^k\mathbb{C}P^3, Top/O]} \arrow[r, "p^*"] \arrow[d, "\psi_*"'] & {[\mathbb{C}P^3 \times \mathbb{S}^k, Top/O]} \arrow[r, "i^*"] \arrow[d, "\psi_*"] & {[\mathbb{C}P^3 \vee \mathbb{S}^k, Top/O]} \arrow[r] \arrow[d, "\psi_*"] & 0 \\
0 \arrow[r] & {[\Sigma^k\mathbb{C}P^3, G/O]} \arrow[r, "p^*"'] & {[\mathbb{C}P^3 \times \mathbb{S}^k, G/O]} \arrow[r, "i^*"'] & {[\mathbb{C}P^3 \vee \mathbb{S}^k, G/O]} \arrow[r] & 0.
\end{tikzcd}$$

Note that $[\mathbb{C}P^3, Top/O]$, $\pi_\ell(Top/O)$ for $\ell = 1, 2, 4, 6$, and $\pi_j(G/O)$ for $j = 3, 5, 7$ all vanish. It follows that for $1 \leq k \leq 7$, the image of $\psi_* : [\mathbb{C}P^3 \times \mathbb{S}^k, Top/O] \to [\mathbb{C}P^3 \times \mathbb{S}^k, G/O]$ coincides with the image of $\psi_* : [\Sigma^k\mathbb{C}P^3, Top/O] \to [\Sigma^k\mathbb{C}P^3, G/O]$.

(i) Since $[\Sigma\mathbb{C}P^3, G/O] = 0$, it follows that the map

$$\psi_* : [\mathbb{C}P^3 \times \mathbb{S}^1, Top/O] \to [\mathbb{C}P^3 \times \mathbb{S}^1, G/O]$$

is trivial.

(ii) From Proposition 5.3.6 (i), we have $[\Sigma^2\mathbb{C}P^3, Top/O] = \Theta_8$. Hence, the image of the map

$$\psi_* : [\Sigma^2\mathbb{C}P^3, Top/O] \to [\Sigma^2\mathbb{C}P^3, G/O]$$

coincides with the image of the composition

$$(\Sigma^2 f_{\mathbb{C}P^3})^* \circ \psi_* : \Theta_8 \to [\Sigma^2\mathbb{C}P^3, G/O],$$

which is the subgroup $\mathbb{Z}/2\{(\Sigma^2 f_{\mathbb{C}P^3})^*([\epsilon])\}$, where $[\epsilon]$ is the generator of $\pi_8(\mathrm{Cok}J_2)$ [103]. Since $(f_{\mathbb{C}P^3 \times \mathbb{S}^2})^* = p^* \circ (\Sigma^2 f_{\mathbb{C}P^3})^*$, the result follows.

(iii) From the long exact sequence (5.8) with $X = G/O$, we observe that the map

$$(\Sigma^3 f_{\mathbb{C}P^3})^* : \pi_9(G/O) \to [\Sigma^3\mathbb{C}P^3, G/O]$$

is an isomorphism, since $(\Sigma^4\pi)^* : [\Sigma^4\mathbb{C}P^2, G/O] \to \pi_9(G/O)$ is the zero map and $[\Sigma^3\mathbb{C}P^2, G/O] = 0$.

Therefore, Proposition 5.3.6 (iii) with $k = 3$ implies that the image of

$$\psi_* : [\Sigma^3\mathbb{C}P^3, Top/O] \to [\Sigma^3\mathbb{C}P^3, G/O]$$

coincides with the image of the composition

$$(f_{\mathbb{C}P^3\times\mathbb{S}^3})^* \circ \psi_* : \Theta_9 \to [\Sigma^3\mathbb{C}P^3, G/O],$$

which is given by the subgroup

$$\mathbb{Z}/2\{f^*_{\mathbb{C}P^3\times\mathbb{S}^3}([\epsilon\eta])\} \oplus \mathbb{Z}/2\{f^*_{\mathbb{C}P^3\times\mathbb{S}^3}([\mu])\},$$

where $[\epsilon\eta]$ and $[\mu]$ are the generators of $\pi_9(\mathrm{cok}J_{(2)})$ [103].

(iv) By Proposition 5.1.3 (4), we have $[\Sigma^4\mathbb{C}P^2, Top/O] \cong \Theta_8$, and hence the image of

$$\psi_* : [\Sigma^4\mathbb{C}P^2, Top/O] \to [\Sigma^4\mathbb{C}P^2, G/O]$$

is $\mathbb{Z}/2\{(\Sigma^4 f_{\mathbb{C}P^2})^*([\epsilon])\}$, $[\epsilon]$ being the generator of $\pi_8(\mathrm{cok}J_{(2)})$ [103]. Combining this with the splitting of $[\Sigma^4\mathbb{C}P^3, Top/O]$ given in Proposition 5.3.6, the result follows.

(v) As the map $\eta^* : \pi_8(G/O) \to \pi_9(G/O)$ is surjective, the long exact sequence associated to (5.9) implies that $[\Sigma^5\mathbb{C}P^3, G/O] = 0$, which establishes the result.

(vi) Since $\psi_* : \Theta_{10} \to \pi_{10}(G/O)$ is an isomorphism and $[\Sigma^6\mathbb{C}P^3, Top/O] \cong \mathbb{Z}/3 \subset \Theta_{10}$ by Proposition 5.3.6 (ii), the result follows.

(vii) To complete the argument using Proposition 5.3.6 (ii), it suffices to determine the image of

$$\psi_* : [\Sigma^7\mathbb{C}P^2, Top/O] \to [\Sigma^7\mathbb{C}P^2, G/O].$$

Since $[\Sigma^7\mathbb{C}P^2, G/O] \cong \mathbb{Z}/2$, the conclusion follows from the splitting established in Proposition 5.1.3 (7).

This completes the proof. □

We now begin the study of smooth manifolds, up to diffeomorphism, that are homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^1$.

Lemma 6.2.4. *The image of the restricted map*

$$\omega_* : Ker(\sigma_8^{Top}) \to \mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^1)$$

is isomorphic to $\mathbb{Z}/7 \oplus \mathbb{Z}/2$.

Proof. Using Sullivan's identification, we obtain

$$\begin{aligned}[][\mathbb{C}P^3 \times \mathbb{S}^1,\, \Omega(G/Top)]_{(2)} \xrightarrow{\cong} & H^7(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}_{(2)}) \oplus H^5(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}/2) \\ & \oplus H^3(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}_{(2)}) \oplus H^1(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}/2), \\ f \longmapsto & \big(f^*(l_8),\, f^*(\kappa_6),\, f^*(l_4),\, f^*(\kappa_2)\big).\end{aligned}$$

According to [128], the surgery obstruction map

$$(\sigma_8^{Top})_{(2)} : \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \longrightarrow L_8(\mathbb{Z})_{(2)}$$

is determined by the following expressions:

$$(\sigma_8^{Top})_{(2)}(0,0,n,0) = \langle (1+\frac{4}{3}z^2)2nz\sigma, [\mathbb{C}P^3 \times \mathbb{S}^1 \times \mathbb{S}^1] \rangle = \frac{8}{3}n,$$

$$(\sigma_8^{Top})_{(2)}(m,0,0,0) = m,\ (\sigma_8^{Top})_{(2)}(0,\bar{1},0,0) = 0 \text{ and } (\sigma_8^{Top})_{(2)}(0,0,0,\bar{1}) = 0,$$

where $z \in H^2(\mathbb{C}P^3;\mathbb{Z})$ and $\sigma \in H^1(\mathbb{S}^1;\mathbb{Z})$ are the generators. Hence, the kernel of the surgery obstruction map $(\sigma_8^{Top})_{(2)}$ is given by

$$Ker((\sigma_8^{Top})_{(2)}) \cong \tfrac{8}{3}\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2.$$

Now, consider the following commutative diagram:

$$\begin{tikzcd}
 & {[\mathbb{C}P^3 \times \mathbb{S}^1, \Omega(G/Top)]} \arrow[r, "(P_{(2)}^{G/Top})_*"] \arrow[d, "\sigma_8^{Top}"'] & {[\mathbb{C}P^3 \times \mathbb{S}^1, \Omega(G/Top)]_{(2)}} \arrow[d, "(\sigma_8^{Top})_{(2)}"] \\
0 \arrow[r] & L_8(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Z} \arrow[r] & L_8(\mathbb{Z}[\mathbb{Z}])_{(2)} \cong \mathbb{Z}_{(2)}.
\end{tikzcd}$$

Since $\mathbb{C}P^3 \times \mathbb{S}^1$ has no odd torsion in cohomology, $(P_{(2)}^{G/Top})_* : [\mathbb{C}P^3 \times \mathbb{S}^1, \Omega(G/Top)] \to [\mathbb{C}P^3 \times \mathbb{S}^1, \Omega(G/Top)]_{(2)}$ is a monomorphism [106] and $[\mathbb{C}P^3 \times \mathbb{S}^1, \Omega(G/Top)] \cong \bigoplus_{i=1}^{2} H^{4i-1}(\mathbb{C}P^3 \times \mathbb{S}^1;\mathbb{Z}) \oplus \bigoplus_{j=0}^{1} H^{4j+1}(\mathbb{C}P^3 \times \mathbb{S}^1;\mathbb{Z}/2)$.

As $Ker((\sigma_8^{Top})_{(2)}) \cong \frac{8}{3}\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$ and the map $(P_{(2)}^{G/Top})_*$ is injective, the kernel of the composition

$$(\sigma_8^{Top})_{(2)} \circ (P_{(2)}^{G/Top})_* : [\mathbb{C}P^3 \times \mathbb{S}^1, \Omega(G/Top)] \to L_8(\mathbb{Z}[\mathbb{Z}])_{(2)}$$

is $8\mathbb{Z} \oplus \mathbb{Z}/2 \oplus 3\mathbb{Z} \oplus \mathbb{Z}/2$. Noting that $L_8(\mathbb{Z}[\mathbb{Z}]) \to L_8(\mathbb{Z}[\mathbb{Z}])_{(2)}$ is injective, it follows from the commutative diagram that the kernel of $\sigma_8^{Top} : [\mathbb{C}P^3 \times \mathbb{S}^1, \Omega(G/Top)] \to L_8(\mathbb{Z}[\mathbb{Z}])$ coincides with the kernel of the composition $(\sigma_8^{Top})_{(2)} \circ (P_{(2)}^{G/Top})_*$.

Using the identifications

$$[\mathbb{C}P^3 \times \mathbb{S}^1, \Omega(G/Top)] \cong \bigoplus_{i=1}^{2} H^{4i-1}(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}) \oplus \bigoplus_{j=0}^{1} H^{4j+1}(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}/2)$$

and

$$[\mathbb{C}P^3 \times \mathbb{S}^1, Top/O] \cong H^7(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}/28) \oplus H^3(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}/2),$$

together with $\{\Sigma^3 H\mathbb{Z}, \Sigma^7 H\mathbb{Z}/28\} = 0$, it follows that the image of $\omega_*(Ker(\sigma_8^{Top}))$ is generated by $\{(\bar{8}, \bar{3})\} \in H^7(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}/28) \oplus H^3(\mathbb{C}P^3 \times \mathbb{S}^1; \mathbb{Z}/2)$.

This completes the proof. □

Theorem 6.2.5.

(i) $\mathbb{Z}/4 \subset \Theta_7$ *is mapped nontrivially under the map* $\mathcal{H} : \mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^1) \to \mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^1)$.

(ii) Any closed, oriented, smooth manifold homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^1$ *is diffeomorphic to exactly one of the manifolds*

$$\mathbb{C}P^3 \times \mathbb{S}^1 \text{ or } (\mathbb{C}P^3 \times \mathbb{S}^1)\#\Sigma,$$

where $[\Sigma]$ *is a nontrivial element of* $\mathbb{Z}/4 \subset \Theta_7$.

Proof. Statement (i) follows by combining Lemma 6.2.4 with Fact 6.2.2. Statement (ii) then follows directly from (i) and Lemma 6.2.3 (i). □

To proceed with the classification of smooth manifolds homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^2$, we first examine the group of self-homotopy equivalences of $\mathbb{C}P^3 \times \mathbb{S}^2$.

Lemma 6.2.6. *Any self-homotopy equivalence of* $\mathbb{C}P^3 \times \mathbb{S}^2$ *is diagonalizable.*

Proof. Let $f : \mathbb{C}P^3 \times \mathbb{S}^2 \to \mathbb{C}P^3 \times \mathbb{S}^2$ be a self-homotopy equivalence. To prove that f is diagonalizable, we examine the induced map on cohomology.

The cohomology ring of $\mathbb{C}P^3 \times \mathbb{S}^2$ is:

$$H^*(\mathbb{C}P^3 \times \mathbb{S}^2; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^4) \otimes \mathbb{Z}[y]/(y^2),$$

where x generates $H^2(\mathbb{C}P^3; \mathbb{Z})$ and y generates $H^2(\mathbb{S}^2; \mathbb{Z})$. Thus

$$H^2(\mathbb{C}P^3 \times \mathbb{S}^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},$$

with generators $\alpha = x \otimes 1$ and $\beta = 1 \otimes y$. The induced map f^* on cohomology is determined by its action on these generators:

$$f^*(\alpha) = a\alpha + b\beta$$

$$f^*(\beta) = c\alpha + d\beta$$

which can be represented by the matrix:

$$f^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since f is a self-homotopy equivalence, f^* must preserve the cup product structure. Specifically, we examine the image of $\alpha^3\beta$ under f^*:

$$f^*(\alpha^3\beta) = f^*(\alpha)^3 f^*(\beta).$$

Using the relations $\alpha^4 = 0$ and $\beta^2 = 0$, we have:

$$f^*(\alpha)^3 = (a\alpha + b\beta)^3 = a^3\alpha^3 + 3a^2b\alpha^2\beta$$

and

$$f^*(\alpha^3\beta) = f^*(\alpha)^3 \cdot f^*(\beta) = (a^3\alpha^3 + 3a^2b\alpha^2\beta)(c\alpha + d\beta) = (a^3d + 3a^2bc)\alpha^3\beta.$$

For f^* to correspond to a self-homotopy equivalence, the condition $f^*(\alpha^3\beta) = \pm\alpha^3\beta$ must hold, which leads to the following two cases:

- If $f^*(\alpha^3\beta) = \alpha^3\beta$, then
$$a^2(3bc + ad) = 1.$$
  This condition is satisfied in exactly two cases:
$$(a, b) = (1, k_1) \text{ and } (c, d) = (k_2, 1 - 3k_1k_2),$$
  or
$$(a, b) = (-1, k_1) \text{ and } (c, d) = (k_2, -1 + 3k_1k_2).$$
- If $f^*(\alpha^3\beta) = -\alpha^3\beta$, then
$$a^2(3bc + ad) = -1.$$
  This holds in either of the following cases:
$$(a, b) = (1, k_1) \text{ and } (c, d) = (k_2, -1 - 3k_1k_2),$$
  or
$$(a, b) = (-1, k_1) \text{ and } (c, d) = (k_2, 1 + 3k_1k_2).$$

Since any map $\mathbb{C}P^3 \to \mathbb{S}^2$ can be written as $g \circ f_{\mathbb{C}P^3}$, where $g \in \pi_6(\mathbb{S}^2)$, it follows that $c = 0$. Therefore,
$$d = \pm 1.$$

With $c = 0$, the matrix representing f^* simplifies to:
$$\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}.$$

Consequently, both compositions $p_{\mathbb{C}P^3} \circ f \circ i_{\mathbb{C}P^3} : \mathbb{C}P^3 \to \mathbb{C}P^3$ and $p_{\mathbb{S}^2} \circ f \circ i_{\mathbb{S}^2} : \mathbb{S}^2 \to \mathbb{S}^2$

are self-homotopy equivalences.

Hence, f is diagonalizable, completing the proof. □

Proposition 6.2.7. *There is no self-homotopy equivalence f of $\mathbb{C}P^3 \times \mathbb{S}^2$ whose normal invariant is given by*

$$\eta^{Diff}(f) = (f_{\mathbb{C}P^3\times\mathbb{S}^2})^*([\epsilon]),$$

where $[\epsilon]$ is the generator of $\pi_8(\mathrm{cok}J_{(2)})$.

Proof. To prove the lemma, we use Proposition 2.6.22, Lemma 6.2.6, and Fact 6.0.1 (d) and (e) to observe that it suffices to compute the normal invariant of each self-homotopy equivalence arising from the components of the following two split short exact sequences:

$$0 \to [\mathbb{C}P^3, SG_3] \to \mathcal{E}_{\mathbb{C}P^3}(\mathbb{C}P^3 \times \mathbb{S}^2) \to \mathcal{E}(\mathbb{S}^2) \to 0,$$

and

$$0 \to \pi_2(F_{\mathbb{S}^1}) \cong \mathbb{Z}/2 \to \mathcal{E}_{\mathbb{S}^2}(\mathbb{C}P^3 \times \mathbb{S}^2) \to \mathcal{E}(\mathbb{C}P^3) \to 0.$$

The self-homotopy equivalences of $\mathbb{C}P^3 \times \mathbb{S}^2$ arising from $\mathcal{E}(\mathbb{S}^2)$ and $\mathcal{E}(\mathbb{C}P^3)$ are homotopic to diffeomorphisms, so their normal invariants vanish. Moreover, by Fact 6.2.1, the normal invariants of the self-homotopy equivalences coming from $[\mathbb{C}P^3, SG_2]$ do not lie in the image of $\pi_8(G/O) \subset [\mathbb{C}P^3 \times \mathbb{S}^2, G/O]$. Hence, by the composition formula for the normal invariant, it suffices to determine the normal invariants of the self-homotopy equivalences arising from $\pi_2(F_{\mathbb{S}^1})$.

By Fact 6.0.1 (f), the normal invariant of $f_\eta \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^2)$, corresponding to the generator η of $\pi_2(F_{\mathbb{S}^1})$, is given by the image of the composition

$$\Sigma^2\mathbb{C}P^3_+ \xrightarrow{\Sigma t} \mathbb{S}^1 \xrightarrow{\eta} \mathbb{S}^0$$

in $[\mathbb{C}P^3 \times \mathbb{S}^2, G/O]$, under the canonical map $j : G \to G/O$.

As $\{\Sigma^2\mathbb{C}P^3_+, \mathbb{S}^0\} \cong \pi_6^s \oplus \pi_8^s \oplus \pi_2^s$, we work locally at prime 2. Since $\eta \circ \nu = 0$

by [125, Theorem 14.1], it follows that $\eta \circ \Sigma t|_{\mathbb{S}^4} = 0$, and hence the computation of $\eta \circ \Sigma t|_{\Sigma^2 \mathbb{C}P^2}$ reduces to evaluating the Toda bracket $\langle \eta, \nu, \eta \rangle$, which equals ν^2 by [125, Lemma 5.12]. Hence, $(\Sigma^2 i')^*(\eta^{Diff}(f_\eta))$ is nontrivial and is equal to $(\Sigma^2 f_{\mathbb{C}P^2})^*([\nu^2])$, where $i' : \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3$ is the inclusion.

Now since $\eta \circ \Sigma t|_{\mathbb{S}^4} = 0$, there exists a map $\mathcal{T} : \mathbb{S}^6 \vee \mathbb{S}^8 \to \mathbb{S}^0$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{S}^5 \vee \mathbb{S}^7 \arrow[r] & \mathbb{S}^4 \arrow[r, hook, "\Sigma^2 \tilde{\tilde{i}}"] \arrow[dr, "\nu"'] & \Sigma^2 \mathbb{C}P^3 \arrow[r, "\Sigma^2 q"] \arrow[d, "\Sigma t"'] & \mathbb{S}^6 \vee \mathbb{S}^8 \arrow[r, "{(\eta, 2\nu)}"] \arrow[ddl, dashed, "\mathcal{T}"] & \mathbb{S}^5 \\
 & & \mathbb{S}^1 \arrow[d, "\eta"'] & & \\
 & & \mathbb{S}^0, & &
\end{tikzcd}$$

where $\mathcal{T}|_{\mathbb{S}^6} : \mathbb{S}^6 \to \mathbb{S}^0$ is given by ν^2, while $\mathcal{T}|_{\mathbb{S}^8}$ is determined by the Toda bracket $\langle 2\nu, \nu, \eta \rangle$.

By [125], the Toda bracket $\langle 2\nu, \nu, \eta \rangle = \{\bar{\nu}, \epsilon\}$, and it maps nontrivially to $[\epsilon] \in \pi_8(G/O)$. Hence, the required map $\eta \circ \Sigma t : \Sigma^2 \mathbb{C}P^3 \to \mathbb{S}^0$ corresponds to the image of one of the elements $\nu^2, \bar{\nu} + \nu^2, \epsilon + \nu^2$ under the map $(\Sigma^2 q)^* : \pi_6^s \oplus \pi_8^s \to [\Sigma^2 \mathbb{C}P^3, G]$. Consequently, $j_*(\eta \circ \Sigma t) \neq (f_{\mathbb{C}P^3 \times \mathbb{S}^2})^*([\epsilon])$. This completes the proof. ☐

Theorem 6.2.8. *Let N be a closed, oriented, smooth manifold homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^2$. Then, it is (oriented) diffeomorphic to exactly one of the manifolds*

$$\mathbb{C}P^3 \times \mathbb{S}^2 \ \textit{or} \ (\mathbb{C}P^3 \times \mathbb{S}^2) \# \Sigma,$$

where $[\Sigma]$ is the 8-dimensional exotic sphere.

Proof. Let $[(N, g)]$ be an element of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^2)$. Then, by Lemma 6.2.3 (ii), we have $\eta^{Diff}(g)$ is either 0 or $(f_{\mathbb{C}P^3 \times \mathbb{S}^2})^*([\epsilon])$.

If $\eta^{Diff}(g)$ is trivial, then from the smooth surgery exact sequence (2.8) with $X = \mathbb{C}P^3 \times \mathbb{S}^2$, the pairs (N, g) and $(\mathbb{C}P^3 \times \mathbb{S}^2, Id)$ represent the same element in $\mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^2)$.

If $\eta^{Diff}(g) = (f_{\mathbb{C}P^3\times\mathbb{S}^2})^*([\epsilon])$, then from Lemma 6.2.7 we see that for any $f \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^2)$, the normal invariant $\eta^{Diff}(f \circ g) \neq 0$. Hence again by the smooth surgery exact sequence for $\mathbb{C}P^3 \times \mathbb{S}^2$, the element $[(N, g)] = [(\mathbb{C}P^3 \times \mathbb{S}^2 \# \Sigma, h_\Sigma)]$ in $\mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^2)$, where $[\Sigma]$ is the exotic 8-sphere.

Thus, N is diffeomorphic to either $\mathbb{C}P^3 \times \mathbb{S}^2$ or $(\mathbb{C}P^3 \times \mathbb{S}^2)\#\Sigma$. □

As a direct consequence of Theorem 6.2.8, we conclude:

Corollary 6.2.9. *The inertia group of* $\mathbb{C}P^3 \times \mathbb{S}^2$ *is zero.*

We now consider the classification of smooth structures on manifolds that are homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^3$.

Lemma 6.2.10.

(1) There exists a self-homotopy equivalence $f_{\eta^2} : \mathbb{C}P^3 \times \mathbb{S}^3 \to \mathbb{C}P^3 \times \mathbb{S}^3$ *arising from the torsion summand of* $\pi_3(F_{\mathbb{S}^1}(\mathbb{C}^4)) \cong \pi_3(U_4) \oplus \pi_2^s \cong \mathbb{Z} \oplus \mathbb{Z}/2\{\eta^2\}$ *such that* $\eta^{Diff}(f_{\eta^2}) = (f_{\mathbb{C}P^3\times\mathbb{S}^3})^*([\epsilon\eta])$, *where* $\epsilon\eta$ *is a generator of* π_9^s.

(2) Suppose f *be a self-homotopy equivalence of* $\mathbb{C}P^3 \times \mathbb{S}^3$ *arising from an element of* $\pi_3(F_{\mathbb{S}^1}(\mathbb{C}^4))$, *and that its normal invariant is nontrivial. Then* $\eta^{Diff}(f)$ *has odd filtration.*

Proof. Statement (1) follows from [107, Proposition 3.5].

Since $\pi_3(G/O) = 0$ and $[\mathbb{C}P^3, Top/O] = 0$, it follows from (4.1) that $\eta^{Diff}(f)$ lies in $[\Sigma^3\mathbb{C}P^3, G/O]$. The suspension $\Sigma\mathbb{C}P^3$ has a CW decomposition with cells in dimensions 5, 7, and 9. Now for any $f \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^3)$ induced from the finite abelian summand of $\pi_3(F_{\mathbb{S}^1}(\mathbb{C}^4))$, we conclude from the discussion in [107, pp. 193–194] that $\eta^{Diff}(f)$ has odd filtration. □

Using the preceding lemma and applying the techniques of [18, Propositions 10.1 and 10.5], we obtain the following:

Lemma 6.2.11.

(a) *Let* $f \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^3)$ *arise from an element of* $\pi_3(F_{\mathbb{S}^1}(\mathbb{C}^4)) \cong \pi_3(F_{\mathbb{S}^1})$. *Then* $\eta^{Diff}(f) = 0$ *if and only if* f *is homotopic to a diffeomorphism.*

(b) *Let* $f \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^3)$ *be induced by an element of* $[\mathbb{C}P^3, SG_4]$. *If the normal invariant of* f *is nontrivial, then it has even filtration.*

Lemma 6.2.12. *Any self-homotopy equivalence* f *of* $\mathbb{C}P^3 \times \mathbb{S}^3$ *is homotopic to a diffeomorphism if and only if its normal invariant is zero, i.e.,* $\eta^{Diff}(f) = 0$.

Proof. This proof follows from [18, Proposition 5.1 and 5.2], together with Lemma 6.2.10 and 6.2.11. □

Lemma 6.2.13. $bP_{10} \cap I_h(\mathbb{C}P^3 \times \mathbb{S}^3) = \emptyset$.

Proof. From the smooth surgery exact sequence of $\mathbb{C}P^3 \times \mathbb{S}^3$, it is enough to show that the surgery obstruction map

$$\sigma_{10}^{Diff} : [\Sigma(\mathbb{C}P^3 \times \mathbb{S}^3), G/O] \to L_{10}(\mathbb{Z})$$

is zero map. This map fits into the following commutative diagram

$$\begin{tikzcd}
{[\Sigma(\mathbb{C}P^3 \times \mathbb{S}^3), G/O]} \arrow[r, "\sigma_{10}^{Diff}"] \arrow[d, "p^*"'] & L_{10}(\mathbb{Z}) \arrow[r, "\omega^{Diff}"] \arrow[d, "\cong"] & \mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^3) \\
{[\mathbb{C}P^3 \times \mathbb{S}^3 \times \mathbb{S}^1, G/O]} \arrow[r, "\sigma_{10}^{Diff}"'] & L_{10}(\mathbb{Z}[\mathbb{Z}]),
\end{tikzcd}$$

where the map $p^* : [\Sigma(\mathbb{C}P^3 \times \mathbb{S}^3), G/O] \to [\mathbb{C}P^3 \times \mathbb{S}^3 \times \mathbb{S}^1, G/O]$ is injective by (4.1), and the isomorphism $L_{10}(\mathbb{Z}) \to L_{10}(\mathbb{Z}[\mathbb{Z}])$ is established in [113].

Now for any $f : \Sigma(\mathbb{C}P^3 \times \mathbb{S}^3) \to G/O$, the surgery obstruction is given by [28] as

$$\sigma_{10}^{Diff}(f) = \sigma_{10}^{Diff}(p^*(f)) = \langle v_4^2(\mathbb{C}P^3 \times \mathbb{S}^3 \times \mathbb{S}^1)(p^*(f))^*(\mathcal{K}_2),\ [\mathbb{C}P^3 \times \mathbb{S}^3 \times \mathbb{S}^1] \rangle = 0,$$

where the 4th Wu class $v_4(\mathbb{C}P^3 \times \mathbb{S}^3 \times \mathbb{S}^1)$ of the manifold $\mathbb{C}P^3 \times \mathbb{S}^3 \times \mathbb{S}^1$ is equal to zero [82] and $\mathcal{K}_2 \in H^2(G/O; \mathbb{Z}/2)$ is a suitable class. This completes the proof. □

Lemma 6.2.14. $bP_{10} \cap I(\mathbb{C}P^3 \times \mathbb{S}^3) = \emptyset$.

Proof. Suppose $[\Sigma]$ is the non-trivial element of bP_{10} contained in the inertia group $I(\mathbb{C}P^3 \times \mathbb{S}^3)$, and let $h_\Sigma : (\mathbb{C}P^3 \times \mathbb{S}^3)\#\Sigma \to \mathbb{C}P^3 \times \mathbb{S}^3$ be the canonical homeomorphism corresponding to $[\Sigma]$. Then $\eta^{Diff}(h_\Sigma) = 0$, and there exists an oriented diffeomorphism $\varphi : (\mathbb{C}P^3 \times \mathbb{S}^3)\#\Sigma \to \mathbb{C}P^3 \times \mathbb{S}^3$. Hence, the composition $\varphi \circ h_\Sigma^{-1}$ is a self-homotopy equivalence of $\mathbb{C}P^3 \times \mathbb{S}^3$. Using the composition formula for normal invariants, we obtain:

$$\eta^{Diff}(\varphi \circ h_\Sigma^{-1}) = \eta^{Diff}(\varphi) + (\varphi^{-1})^* \eta^{Diff}(h_\Sigma) = 0.$$

It then follows from Lemma 6.2.12 that h_Σ is homotopic to a diffeomorphism, contradicting Lemma 6.2.13. □

Theorem 6.2.15. *The homotopy inertia group $I_h(\mathbb{C}P^3 \times \mathbb{S}^3)$ is zero.*

Proof. By Lemma 6.2.13, the proof reduces to showing that the restriction map

$$(f_{\mathbb{C}P^3 \times \mathbb{S}^3})^* : \Theta_9 \setminus bP_{10} \to \mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^3)$$

is injective. The map $(f_{\mathbb{C}P^3 \times \mathbb{S}^3})^* : \Theta_9 \to \mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^3)$ fits into the following commutative diagram whose rows correspond to the smooth surgery exact sequences for $\mathbb{S}^9$ and $\mathbb{C}P^3 \times \mathbb{S}^3$, respectively.

$$\begin{tikzcd}
L_{10}(\mathbb{Z}) \arrow[r] \arrow[d, equal] & \Theta_9 \arrow[r, "\eta^{Diff}"] \arrow[d, "(f_{\mathbb{C}P^3 \times \mathbb{S}^3})^*"'] & \pi_9(G/O) \arrow[r] \arrow[d, "(f_{\mathbb{C}P^3 \times \mathbb{S}^3})^*"] & 0 \\
L_{10}(\mathbb{Z}) \arrow[r, "\omega^{Diff}"'] & \mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^3) \arrow[r, "\eta^{Diff}"'] & {[\mathbb{C}P^3 \times \mathbb{S}^3, G/O]} \arrow[r] & 0.
\end{tikzcd}$$

We know that the restriction map $\eta^{Diff} : \Theta_9 \setminus bP_{10} \to \pi_9(G/O)$ is injective. Hence the injectivity of $(f_{\mathbb{C}P^3 \times \mathbb{S}^3})^* : \Theta_9 \setminus bP_{10} \to \mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^3)$ follows from the above commutative diagram, as $(f_{\mathbb{C}P^3 \times \mathbb{S}^3})^* : \pi_9(G/O) \to [\mathbb{C}P^3 \times \mathbb{S}^3, G/O]$ is injective, observed in Lemma 6.2.3 (iii). □

Theorem 6.2.16. *The inertia group of $\mathbb{C}P^3 \times \mathbb{S}^3$ is zero.*

Proof. Since $\pi_3(\mathbb{C}P^3) = 0$, any self-homotopy equivalence of $\mathbb{C}P^3 \times \mathbb{S}^3$ can be diagonalized, by Proposition 2.6.21. Therefore, by Proposition 2.6.22 (b), to study the normal invariant of the element of $\mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^3)$, it suffices to consider self-homotopy equivalences induced from $\mathcal{E}(\mathbb{S}^3)$, $\mathcal{E}(\mathbb{C}P^3)$, $[\mathbb{C}P^3, E_1(\mathbb{S}^3)]$, and $\pi_3(E_1(\mathbb{C}P^3))$. Among these, those arising from $\mathcal{E}(\mathbb{S}^3)$ and $\mathcal{E}(\mathbb{C}P^3)$ are represented by diffeomorphisms, and thus have trivial normal invariant.

Let $[(\mathbb{C}P^3 \times \mathbb{S}^3 \# \Sigma_\mu, h_{\Sigma_\mu})] \in \mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^3)$, where $[\Sigma_\mu] \in \Theta_9$ corresponds to the generator $\mu \in \pi_9^s$ and $h_{\Sigma_\mu} : (\mathbb{C}P^3 \times \mathbb{S}^3)\#\Sigma_\mu \to \mathbb{C}P^3 \times \mathbb{S}^3$ is the canonical homeomorphism. Then by Proposition 5.3.6 and Lemma 6.2.3 (iii), we have $\eta^{Diff}(h_{\Sigma_\mu}) = (f_{\mathbb{C}P^3\times\mathbb{S}^3})^*([\mu])$.

Now, considering the decomposition of $\mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^3)$, and applying Fact 6.2.1 along with Lemma 6.2.10 (1), together with the observation that any self-homotopy equivalence of $\mathbb{C}P^3 \times \mathbb{S}^3$ arising from $\pi_3(U_4)$ is homotopic to a diffeomorphism, we conclude that

$$\eta^{Diff}(g \circ h_{\Sigma_\mu}) \neq 0, \text{ for all } g \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^3).$$

Therefore, $[\Sigma_\mu] \notin I(\mathbb{C}P^3 \times \mathbb{S}^3)$.

Let $[(\mathbb{C}P^3 \times \mathbb{S}^3 \# \Sigma_{\epsilon\eta}, h_{\Sigma_{\epsilon\eta}})]$ be an element of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^3)$, where $[\Sigma_{\epsilon\eta}]$ is the exotic 9-sphere associated to the generator $\epsilon\eta$ of π_9^s and $h_{\Sigma_{\epsilon\eta}} : \mathbb{C}P^3 \times \mathbb{S}^3 \# \Sigma_{\epsilon\eta} \to \mathbb{C}P^3 \times \mathbb{S}^3$ is the canonical homeomorphism. Then $\eta^{Diff}(h_{\Sigma_{\epsilon\eta}}) = (f_{\mathbb{C}P^3\times\mathbb{S}^3})^*([\epsilon\eta])$. Now applying Lemma 6.2.10 (1) in the composition formula for normal invariants, we get

$$\eta^{Diff}(f_{\eta^2} \circ h_{\Sigma_{\epsilon\eta}}) = 0.$$

This implies that $(\mathbb{C}P^3 \times \mathbb{S}^3 \# \Sigma_{\epsilon\eta}, f_{\eta^2} \circ h_{\Sigma_{\epsilon\eta}})$ is equivalent to $(\mathbb{C}P^3 \times \mathbb{S}^3 \# \Sigma, h_\Sigma)$ in $\mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^3)$ for some $[\Sigma] \in bP_{10}$. Hence

$$[\Sigma_{\epsilon\eta} \# \Sigma^{-1}] \in I(\mathbb{C}P^3 \times \mathbb{S}^3).$$

Now if $[\Sigma_{\epsilon\eta}] \in I(\mathbb{C}P^3 \times \mathbb{S}^3)$, then $[\Sigma]^{-1} \in I(\mathbb{C}P^3 \times \mathbb{S}^3)$, which contradicts Lemma 6.2.14. Thus $[\Sigma_{\epsilon\eta}] \notin I(\mathbb{C}P^3 \times \mathbb{S}^3)$.

Therefore, based on the above calculation with Lemma 6.2.14, we obtain $I(\mathbb{C}P^3 \times$

$\mathbb{S}^3) = 0$. ☐

Remark 6.2.17. *The exotic sphere corresponding to the Adams element $\mu \in \pi_9^s$ does not belong to the inertia group of $\mathbb{C}P^3 \times \mathbb{S}^3$ also follows directly from [62, Lemma 9.1].*

Theorem 6.2.16 allows us to determine all smooth structures on manifolds that are homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^3$.

Theorem 6.2.18. *Let N be a closed, oriented, smooth manifold homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^3$. Then N is (oriented) diffeomorphic to $(\mathbb{C}P^3 \times \mathbb{S}^3)\#\Sigma$ for some $[\Sigma] \in \Theta_9$ and two such manifolds are diffeomorphic if and only if the corresponding homotopy spheres represent the same element in Θ_9.*

Proof. Let $[(N, g)]$ be an element of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^3)$.

Suppose $\eta^{Diff}(g)$ is trivial. Then, by the smooth surgery exact sequence for $\mathbb{C}P^3 \times \mathbb{S}^3$, the pair (N, g) represents the same element in $\mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^3)$ as $(\mathbb{C}P^3 \times \mathbb{S}^3\#\Sigma, h_\Sigma)$ for some $[\Sigma] \in bP_{10}$.

Suppose $\eta^{Diff}(g)$ is non-trivial. Then, from Lemma 6.2.3 (iii), we note from the smooth surgery exact sequence (2.8) for$X = \mathbb{C}P^3 \times \mathbb{S}^3$ that the pairs (N, g) and $(\mathbb{C}P^3 \times \mathbb{S}^3\#\Sigma, h_\Sigma)$ are equivalent in $\mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^3)$, where $[\Sigma] \in \Theta_9 \setminus bP_{10}$. Theorem 6.2.16 then implies that N is not diffeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^3$.

Let $[(\mathbb{C}P^3 \times \mathbb{S}^3\#\Sigma_1, h_{\Sigma_1})]$ and $[(\mathbb{C}P^3 \times \mathbb{S}^3\#\Sigma_2, h_{\Sigma_2})]$ be two elements of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^3)$, where $[\Sigma_1], [\Sigma_2] \in \Theta_9 \subset \mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^3)$. If the pairs $(\mathbb{C}P^3 \times \mathbb{S}^3\#\Sigma_1, h_{\Sigma_1})$ and $(\mathbb{C}P^3 \times \mathbb{S}^3\#\Sigma_2, h_{\Sigma_2})$ represent the same element in $\mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^3)$, then $[\Sigma_1\#\Sigma_2]^{-1} \in I(\mathbb{C}P^3 \times \mathbb{S}^3)$. Since the inertia group of $\mathbb{C}P^3 \times \mathbb{S}^3$ is zero by Theorem 6.2.16, it follows that Σ_1 is oriented diffeomorphic to Σ_2. Thus, any two elements of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^3)$ arising from distinct elements of Θ_9 are pairwise non-diffeomorphic.

This completes the proof. ☐

We now proceed to classify smooth manifolds that are homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^4$. Before analyzing the action of self-homotopy equivalences of $\mathbb{C}P^3 \times \mathbb{S}^4$ on $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^4)$, we first show existence of a tangential self-homotopy equivalence of $\mathbb{C}P^3 \times \mathbb{S}^4$ with nontrivial normal invariant in $\pi_{10}(G/O)_{(3)} \subset [\mathbb{C}P^3 \times \mathbb{S}^4, G/O]$.

Lemma 6.2.19. *There exists a tangential homotopy equivalence* $f_{\alpha_1} : \mathbb{C}P^3 \times \mathbb{S}^4 \to \mathbb{C}P^3 \times \mathbb{S}^4$ *such that its normal invariant is given by*

$$\eta^{Diff}(f_{\alpha_1}) = (f_{\mathbb{C}P^3 \times \mathbb{S}^4})^*([\beta_1]) \neq 0,$$

where β_1 *is a generator of* $(\pi_{10}^s)_{(3)}$.

Proof. Fact 6.0.1 (d) and (e) give $\pi_4(F_{\mathbb{S}^1}(\mathbb{C}^4)) \cong \pi_4^s(\Sigma\mathbb{C}P^\infty) \oplus \pi_3^s$, and from Fact 6.0.1 (b), any element in this group induces a tangential self-homotopy equivalence of $\mathbb{C}P^3 \times \mathbb{S}^4$.

Let $f_{\alpha_1} : \mathbb{C}P^3 \times \mathbb{S}^4 \to \mathbb{C}P^3 \times \mathbb{S}^4$ be a tangential self-homotopy equivalence arising from the generator α_1 of $(\pi_3^s)_{(3)}$. Referring to Fact 6.0.1 (f), its normal invariant is given by the image under the map $j : G \to G/O$ of the composition

$$\Sigma^4\mathbb{C}P^3_+ \xrightarrow{\Sigma^3 t} \mathbb{S}^3 \xrightarrow{\alpha_1} \mathbb{S}^0$$

in $[\Sigma^4\mathbb{C}P^3, G/O] \subset [\mathbb{C}P^3 \times \mathbb{S}^4, G/O]$. We note that the restriction of $\alpha_1 \circ \Sigma^3 t$ to both $\Sigma^4\mathbb{C}P^1$ and $\Sigma^4\mathbb{C}P^2$ is trivial. Therefore, it suffices to compute the Toda bracket $\langle \alpha_1, \alpha_1, \alpha_1 \rangle$ to determine whether $\Sigma^3 t \circ \alpha_1 : \Sigma^4\mathbb{C}P^3 \to \mathbb{S}^0$ is trivial. By [98, Page 121], $\langle \alpha_1, \alpha_1, \alpha_1 \rangle = \beta_1$, which projects nontrivially to $[\beta_1] \in \pi_{10}(G/O)$.

Using $(f_{\mathbb{C}P^3 \times \mathbb{S}^4})^* = p^* \circ (\Sigma^4 f_{\mathbb{C}P^2})^*$, and noting that both $(\Sigma^4 f_{\mathbb{C}P^2})^* : \pi_{10}(G/O) \to [\Sigma^4\mathbb{C}P^3, G/O]$ and $p^* : [\Sigma^4\mathbb{C}P^3, G/O] \to [\mathbb{C}P^3 \times \mathbb{S}^4, G/O]$ are injective, we deduce that $\eta^{Diff}(f_{\alpha_1}) = (f_{\mathbb{C}P^3 \times \mathbb{S}^4})^*([\beta_1])$. □

Lemma 6.2.20. *There exists a self-homeomorphism of* $\mathbb{C}P^3 \times \mathbb{S}^4$ *whose normal invariant is* $(f_{\mathbb{C}P^3 \times \mathbb{S}^4})^*([\beta_1])$.

Proof. By the previous lemma, we have a self-homotopy equivalence $f_{\alpha_1} : \mathbb{C}P^3 \times \mathbb{S}^4 \to \mathbb{C}P^3 \times \mathbb{S}^4$ whose normal invariant is given by $(f_{\mathbb{C}P^3 \times \mathbb{S}^4})^*([\beta_1])$. By Lemma 6.2.3 (iv) and Proposition 5.3.6, there exists an exotic 10-sphere $[\Sigma_{\beta_1}]$ in Θ_{10} such that the normal invariant of the canonical homeomorphism $h_{\Sigma_{\beta_1}} : (\mathbb{C}P^3 \times \mathbb{S}^4)\#\Sigma_{\beta_1} \to \mathbb{C}P^3 \times \mathbb{S}^4$ is the same as that of f.

Hence, by the smooth surgery exact sequence (2.8) with $X = \mathbb{C}P^3 \times \mathbb{S}^4$, there exists a diffeomorphism $\mathcal{J} : (\mathbb{C}P^3 \times \mathbb{S}^4)\#\Sigma_{\beta_1} \to \mathbb{C}P^3 \times \mathbb{S}^4$ such that $f_{\alpha_1} \circ \mathcal{J} \simeq h_{\Sigma_{\beta_1}}$. This implies that $f_{\alpha_1} \simeq h_{\Sigma_{\beta_1}} \circ \mathcal{J}^{-1}$, showing that f_{α_1} is a self-homeomorphism of $\mathbb{C}P^3 \times \mathbb{S}^4$. This completes the proof. □

Lemma 6.2.21. *There exists a self-homeomorphism $f_{2\nu} : \mathbb{C}P^3 \times \mathbb{S}^4 \to \mathbb{C}P^3 \times \mathbb{S}^4$ such that $\eta^{Diff}(f_{2\nu})$ is nontrivial.*

Moreover, $\eta^{Diff}(f_{2\nu}) \in Im\left([\Sigma^4\mathbb{C}P^2, Top/O] \xrightarrow{\psi_*} [\Sigma^4\mathbb{C}P^2, G/O]\right) \subset [\mathbb{C}P^3 \times \mathbb{S}^4, G/O]$.

Proof. We aim to produce a tangential self-homotopy equivalence $f_{2\nu} : \mathbb{C}P^3 \times \mathbb{S}^4 \to \mathbb{C}P^3 \times \mathbb{S}^4$, whose normal invarinat is nontrivial and lies in the image of $\psi_* : [\Sigma^4\mathbb{C}P^2, Top/O] \to [\Sigma^4\mathbb{C}P^2, G/O]$. Once this is established, we apply same argument to that of Lemma 6.2.20 to show that $f_{2\nu} : \mathbb{C}P^3 \times \mathbb{S}^4 \to \mathbb{C}P^3 \times \mathbb{S}^4$ is a self-homeomorphism.

We now consider the tangential self-homotopy equivalence $f_{2\nu} : \mathbb{C}P^3 \times \mathbb{S}^4 \to \mathbb{C}P^3 \times \mathbb{S}^4$ induced by the generator $2\nu \in (\pi_3^s)_{(2)} \subset \pi_4(F_{\mathbb{S}^1})$. By Fact 6.0.1 (f), its normal invariant is the image of the composition $\Sigma^4\mathbb{C}P^3_+ \xrightarrow{\Sigma^3 t} \mathbb{S}^3 \xrightarrow{2\nu} \mathbb{S}^0$ under the canonical homomorphisms

$$[\Sigma^4\mathbb{C}P^3, G] \xrightarrow{j_*} [\Sigma^4\mathbb{C}P^3, G/O] \xrightarrow{p^*} [\mathbb{C}P^3 \times \mathbb{S}^4, G/O].$$

We observe that the restriction of $2\nu \circ \Sigma^3 t$ on $\Sigma^4\mathbb{C}P^1$ is $2\nu^2 = 0$ [125], while its restriction on $\Sigma^4\mathbb{C}P^2$ represents the Toda bracket $\langle \eta, \nu, 2\nu \rangle$. It is known that $\{\epsilon, \bar{\nu}\} \in \langle \eta, \nu, 2\nu \rangle$ [125, 103, 34] and is mapped nontrivially to $[\epsilon] \in \pi_8(G/O)$. Hence, $\eta^{Diff}(f_{2\nu})$ is nontrivial.

Moreover, there exists

$$\mathcal{T}' : \mathbb{S}^{10} \vee \mathbb{S}^8 \to \mathbb{S}^0$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{S}^7 \vee \mathbb{S}^9 \arrow[r, "(\eta,2\nu)"] & \mathbb{S}^6 \arrow[r, hook, "\Sigma^4\tilde{\tilde{i}}"] \arrow[dr, "\nu"'] & \Sigma^4\mathbb{C}P^3 \arrow[r, "\Sigma^4 q"] \arrow[d, "\Sigma^3 t"] & \mathbb{S}^8 \vee \mathbb{S}^{10} \arrow[r, "(\eta,2\nu)"] \arrow[ddl, dashed, "\mathcal{T}'"] & \mathbb{S}^7 \\
& & \mathbb{S}^3 \arrow[d, "2\nu"'] & & \\
& & \mathbb{S}^0, & &
\end{tikzcd}$$

where the restriction $\mathcal{T}' \mid_{\mathbb{S}^{10}} : \mathbb{S}^{10} \to \mathbb{S}^0$ is determined by the Toda bracket $\langle 2\nu, \nu, 2\nu \rangle$. Using the properties of Toda brackets and the fact that $\eta^2 \circ \mu \neq 0$, we conclude that $\langle 2\nu, \nu, 2\nu \rangle = 0$. Hence,

$$\eta^{Diff}(f_{2\nu}) = (\Sigma^4 q)^* \circ (\Sigma^4 Pr)^*([\epsilon]),$$

where $Pr : \mathbb{S}^4 \vee \mathbb{S}^6 \to \mathbb{S}^4$ is the projection map. □

Theorem 6.2.22. *The inertia group of* $\mathbb{C}P^3 \times \mathbb{S}^4$ *is* $\mathbb{Z}/3$.

Proof. Let $[\Sigma_{\beta_1}]$ be the exotic 10-sphere corresponding to the generator β_1 of $(\pi_{10}^s)_{(3)}$. Then $[(\Sigma_{\beta_1}, h_{\Sigma_{\beta_1}})]$ is an element of $\Theta_{10} \subset \mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^4)$.

Since $\pi_4(\mathbb{C}P^3)$ is trivial, it follows from Proposition 2.6.21 and Proposition 2.6.22 that we need to examine the action of the self-homotopy equivalence of $\mathbb{C}P^3 \times \mathbb{S}^4$ arising from each of the following: $\mathcal{E}(\mathbb{C}P^3), \mathcal{E}(\mathbb{S}^4), [\mathbb{C}P^3, E_1(\mathbb{S}^4)]$, and $\pi_4(E_1(\mathbb{C}P^3))$. Moreover, any self-homotopy equivalence of $\mathbb{C}P^3 \times \mathbb{S}^4$ arising from $\mathcal{E}(\mathbb{C}P^3)$ or $\mathcal{E}(\mathbb{S}^4)$ is homotopic to a diffeomorphism. By Fact 6.2.1, the normal invariant of a self-homotopy equivalence arising from $[\mathbb{C}P^3, E_1(\mathbb{S}^4)]$ lies in $[\mathbb{C}P^3, G/O]$, whereas $\eta^{Diff}(h_{\Sigma_{\beta_1}}) \in [\Sigma^4\mathbb{C}P^3, G/O]$. Thus, it remains to analyze the action of the self-homeomorphism f_{α_1} from Lemma 6.2.20 on $h_{\Sigma_{\beta_1}}$. Now the composition formula for normal invariants gives

$$\begin{aligned}
&\eta^{Diff}\left([(\mathbb{C}P^3 \times \mathbb{S}^4)\#\Sigma_{\beta_1}, f_{\alpha_1} \circ h_{\Sigma_{\beta_1}}]\right) \\
&= \eta^{Diff}\left([\mathbb{C}P^3 \times \mathbb{S}^4, f_{\alpha_1}]\right) + (f_{\alpha_1}^{-1})^* \eta^{Diff}\left([(\mathbb{C}P^3 \times \mathbb{S}^4)\#\Sigma_{\beta_1}, h_{\Sigma_{\beta_1}}]\right) \\
&= \eta^{Diff}\left([\mathbb{C}P^3 \times \mathbb{S}^4, f_{\alpha_1}]\right) + (f_{\alpha_1}^{-1})^* \circ (f_{\mathbb{C}P^3 \times \mathbb{S}^4})^* \left([\Sigma_{\beta_1}]\right) \\
&= (f_{\mathbb{C}P^3 \times \mathbb{S}^4})^*[\beta_1] \pm (f_{\mathbb{C}P^3 \times \mathbb{S}^4})^*[\beta_1].
\end{aligned}$$

If $\eta^{Diff}\left([(\mathbb{C}P^3 \times \mathbb{S}^4)\#\Sigma_{\beta_1}, f_{\alpha_1} \circ h_{\Sigma_{\beta_1}}]\right) = 0$, then by the smooth surgery exact sequence (2.8), we get $(\mathbb{C}P^3 \times \mathbb{S}^4\#\Sigma_{\beta_1}, f_{\alpha_1} \circ h_{\Sigma_{\beta_1}})$ is equivalent to $(\mathbb{C}P^3 \times \mathbb{S}^4, Id)$ in $\mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^4)$.

If $\eta^{Diff}\left([(\mathbb{C}P^3 \times \mathbb{S}^4)\#\Sigma_{\beta_1}, f_{\alpha_1} \circ h_{\Sigma_{\beta_1}}]\right) \neq 0$, then again by smooth surgery exact sequence for $\mathbb{C}P^3 \times \mathbb{S}^4$, the pairs $(\mathbb{C}P^3 \times \mathbb{S}^4\#\Sigma_{\beta_1}, f_{\alpha_1} \circ h_{\Sigma_{\beta_1}})$ and $(\mathbb{C}P^3 \times \mathbb{S}^4\#\Sigma_{2\beta_1}, h_{\Sigma_{2\beta_1}})$ represent the same element in $\mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^4)$. This implies that $[\Sigma_{2\beta_1}\#\Sigma_{\beta_1}^{-1}] \in I(\mathbb{C}P^3 \times \mathbb{S}^4)$. Since $[\Sigma_{2\beta_1}\#\Sigma_{\beta_1}^{-1}] = [\Sigma_{\beta_1}]$, it follows that

$$(\Theta_{10})_{(3)} \subseteq I(\mathbb{C}P^3 \times \mathbb{S}^4).$$

However, from [62, Lemma 9.1], $I(\mathbb{C}P^3 \times \mathbb{S}^4)$ contains only those homotopy spheres that bound spin manifolds. Since $\Theta_{10} \cong bspin_{10} \oplus \mathbb{Z}/2$ by [26], the 2-torsion component of Θ_{10} is not contained in $I(\mathbb{C}P^3 \times \mathbb{S}^4)$.

Therefore, $I(\mathbb{C}P^3 \times \mathbb{S}^4) \cong \mathbb{Z}/3$. □

Theorem 6.2.23. *Any closed, oriented, smooth manifold homeomorphic to* $\mathbb{C}P^3 \times \mathbb{S}^4$ *is (oriented) diffeomorphic to either* $\mathbb{C}P^3 \times \mathbb{S}^4$ *or* $(\mathbb{C}P^3 \times \mathbb{S}^4)\#\Sigma_{\eta\mu}$*, where* $[\Sigma_{\eta\mu}] \in \Theta_{10}$ *is the exotic sphere of order* 2.

Proof. Let $[(N, g)]$ be an element of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^4)$.

Suppose $\eta^{Diff}(g)$ is trivial. Then it follows from the smooth surgery exact sequence for $\mathbb{C}P^3 \times \mathbb{S}^4$ that (N, g) is equivalent to $(\mathbb{C}P^3 \times \mathbb{S}^4, Id)$ in $\mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^4)$.

Suppose $\eta^{Diff}(g)$ is nontrivial. Since $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^4) \cong [\Sigma^4\mathbb{C}P^3, Top/O]$, we have $\eta^{Diff}(g) \in [\Sigma^4\mathbb{C}P^3, G/O]$. Hence, by the discussion in Theorem 6.2.22, instead of considering the action of all self-homotopy equivalences of $\mathbb{C}P^3 \times \mathbb{S}^4$ on $[(N, g)]$, it suffices to consider the action of those arising from $\pi_4(F_{\mathbb{S}^1}(\mathbb{C}^4))$.

Let $[(N, g)]$ be an element of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^4)$ coming from $[\Sigma^4\mathbb{C}P^2, Top/O] \subset [\Sigma^4\mathbb{C}P^3, Top/O]$. Then the normal invariant of $[(N, g)]$ is determined by Lemma 6.2.3 (iv).

(a) Suppose the normal invariant of $[(N, g)]$ is $(\Sigma^4 q)^* \circ (\Sigma^4 Pr)^*[\epsilon]$. Now we study the action of the self-homeomorphism $f_{2\nu}$ from Lemma 6.2.21 on $[(N,g)]$ by computing the normal invariant as follows:

$$\eta^{Diff}(f_{2\nu}\circ g)=\eta(f_{2\nu})+(f_{2\nu}^{-1})^{*}\eta^{Diff}(g)$$
$$=(\Sigma^4 q)^{*}\circ(\Sigma^4 Pr)^{*}[\epsilon]\pm(\Sigma^4 q)^{*}\circ(\Sigma^4 Pr)^{*}[\epsilon]=0.$$

Therefore, by previous argument, N is (oriented) diffeomorphic to $\mathbb{C}P^3\times\mathbb{S}^4$.

(b) Let $[(N,g)]\in\mathcal{C}(\mathbb{C}P^3\times\mathbb{S}^4)$ such that its normal invariant is $(f_{\mathbb{C}P^3\times\mathbb{S}^4})^{*}[\eta\mu]+(\Sigma^4 q)^{*}\circ(\Sigma^4 Pr)^{*}[\epsilon]$. Then

$$\eta^{Diff}(f_{2\nu}\circ g)=\eta^{Diff}(f_{2\nu})+(f_{2\nu}^{-1})^{*}\eta^{Diff}(g)$$
$$=(\Sigma^4 q)^{*}\circ(\Sigma^4 Pr)^{*}[\epsilon]\pm\big((f_{\mathbb{C}P^3\times\mathbb{S}^4})^{*}[\eta\mu]+(\Sigma^4 q)^{*}\circ(\Sigma^4 Pr)^{*}[\epsilon]\big)$$
$$=\pm(f_{\mathbb{C}P^3\times\mathbb{S}^4})^{*}[\eta\mu].$$

Hence, by using Theorem 6.2.22, we conclude that N is (oriented) diffeomorphic to $(\mathbb{C}P^3\times\mathbb{S}^4)\#\Sigma_{\eta\mu}$.

(c) Let $[(N,g)]\in\mathcal{C}(\mathbb{C}P^3\times\mathbb{S}^4)$ such that $\eta^{Diff}(g)=(f_{\mathbb{C}P^3\times\mathbb{S}^4})^{*}[\beta_1]+(\Sigma^4 q)^{*}\circ(\Sigma^4 Pr)^{*}[\epsilon]$. The effect of the self-homeomorphism f_{α_1} of $\mathbb{C}P^3\times\mathbb{S}^4$, given in Lemma 6.2.20, on $[(N,g)]$ is described by the normal invariant:

$$\eta^{Diff}(f_{\alpha_1}\circ g)=\eta^{Diff}(f_{\alpha_1})+(f_{\alpha_1}^{-1})^{*}\eta^{Diff}(g)$$
$$=(f_{\mathbb{C}P^3\times\mathbb{S}^4})^{*}[\beta_1]-\big((f_{\mathbb{C}P^3\times\mathbb{S}^4})^{*}[\beta_1]+(\Sigma^4 q)^{*}\circ(\Sigma^4 Pr)^{*}[\epsilon]\big)$$
$$=(\Sigma^4 q)^{*}\circ(\Sigma^4 Pr)^{*}[\epsilon].$$

Hence, by Case (a), N is (oriented) diffeomorphic to $\mathbb{C}P^3\times\mathbb{S}^4$.

Combining all the above cases with Theorem 6.2.22 completes the proof. □

We begin the classification of smooth structures on $\mathbb{C}P^3\times\mathbb{S}^5$, fixing its homeomorphism type, by first computing the homotopy inertia group, which will subsequently be used to determine the inertia group.

Theorem 6.2.24. *The homotopy inertia group* $I_h(\mathbb{C}P^3 \times \mathbb{S}^5)$ *is* $\mathbb{Z}/62$.

Proof. The homotopy inertia group of $\mathbb{C}P^3 \times \mathbb{S}^5$ is given by the quotient $L_{12}(\mathbb{Z})/Im(\sigma_{12}^{Diff})$, where $\sigma_{12}^{Diff} : [\Sigma(\mathbb{C}P^3 \times \mathbb{S}^5), G/O] \to L_{12}(\mathbb{Z})$ is the surgery obstruction map. Thus, it suffices to compute the image of σ_{12}^{Diff}. Now, consider the following commutative diagram:

$$\begin{tikzcd}
{[\Sigma(\mathbb{C}P^3 \times \mathbb{S}^5), G/O]} \arrow[r, "\sigma_{12}^{Diff}"] \arrow[d, "p^*"'] & L_{12}(\mathbb{Z}) \arrow[d, "\cong"] \arrow[dr, "\omega^{Diff}"] & \\
{[\mathbb{C}P^3 \times \mathbb{S}^5 \times \mathbb{S}^1, G/O]} \arrow[r, "\sigma_{12}^{Diff}"'] & L_{12}(\mathbb{Z}[\mathbb{Z}]), & \mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^5)
\end{tikzcd}$$

where p^* is injective by (4.1); the rows are the surgery obstruction maps for $\mathbb{C}P^3 \times \mathbb{S}^5$ and $\mathbb{C}P^3 \times \mathbb{S}^5 \times \mathbb{S}^1$, respectively; and the isomorphism $L_{12}(\mathbb{Z}) \to L_{12}(\mathbb{Z}[\mathbb{Z}])$ is given in [113].

From the above commutative diagram, it follows that $\sigma_{12}^{Diff}(f) = \sigma_{12}^{Diff}(p^*(f))$ for any $f \in [\Sigma(\mathbb{C}P^3 \times \mathbb{S}^5), G/O]$. According to [28], $\sigma_{12}^{Diff}(f)$ is given by

$$\sigma_{12}^{Diff}(f) = \frac{1}{8}\langle L(\mathbb{C}P^3 \times \mathbb{S}^5 \times \mathbb{S}^1)(1 - L(\xi)), [\mathbb{C}P^3 \times \mathbb{S}^5 \times \mathbb{S}^1]\rangle,$$

where ξ denotes the image of f under the canonical map $i : G/O \to BSO$, and $L(\xi)$ is the L-genus of ξ.

We note that the image of the map $i_* : [\Sigma(\mathbb{C}P^3 \times \mathbb{S}^5), G/O] \to [\Sigma(\mathbb{C}P^3 \times \mathbb{S}^5), BSO]$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Let ξ_1 and ξ_2 denote generators of this image. Using [44, Theorem 2 (vi)], [122, Lemma 2.1 (2)], and [82, Corollary 15.5], we find that the total Pontrjagin classes of ξ_1 and ξ_2 are given by

$$p(\xi_1) = 1 + p_1(\xi_1) + p_2(\xi_1) + p_3(\xi_1) = 1 + 0 + 240 \cdot 12 \cdot yz + 504 \cdot 40 \cdot yz^3,$$

$$p(\xi_2) = 1 + p_1(\xi_2) + p_2(\xi_2) + p_3(\xi_2) = 1 + 0 + 0 + 504 \cdot 240 \cdot yz^3,$$

where $y \in H^6(\mathbb{S}^6;\mathbb{Z})$ and $z \in H^2(\mathbb{C}P^3;\mathbb{Z})$ are the generators. Thus, if $\xi = m\xi_1 + n\xi_2$, then

$$\begin{aligned}\sigma_{12}^{Diff}(f) &= -\frac{1}{8}\langle(1+\frac{4}{3}z^2)\left(\frac{7.240.12.m}{45}yz + \frac{62.504.40(m-6n)}{945}yz^3\right), [\mathbb{C}P^3 \times \mathbb{S}^5 \times \mathbb{S}^1]\rangle \\ &= -\frac{1}{8}\left[1920m - 7936n\right] = -240m + 992n = 2^4(-15m + 62n).\end{aligned}$$

Therefore, $I_h(\mathbb{C}P^3 \times \mathbb{S}^5) = \mathbb{Z}/62 \subset \Theta_{11}$. □

Lemma 6.2.25. *A self-homotopy equivalence $f \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^5)$ is homotopic to a diffeomorphism if and only if $\eta^{Diff}(f) = 0$.*

Proof. According to Proposition 2.6.21, every self-homotopy equivalence $f \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^5)$ is diagonalizable. Moreover, any element of $\mathcal{E}(\mathbb{C}P^3)$ or $\mathcal{E}(\mathbb{S}^5)$ corresponds to a diffeomorphism of $\mathbb{C}P^3 \times \mathbb{S}^5$. In view of Proposition 2.6.22, it is therefore sufficient to consider self-homotopy equivalences induced by elements in $\pi_5(E_1(\mathbb{C}P^3))$ and $[\mathbb{C}P^3, E_1(\mathbb{S}^5)] \cong [\mathbb{C}P^3, SG_6]$.

As $\pi_5(E_1(\mathbb{C}P^3)) \cong \pi_5(U_3)$, any element of $\mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^5)$ arising from $\pi_5(E_1(\mathbb{C}P^3))$ is homotopic to a diffeomorphism. Also, by [18, Proposition 10.2], a self-homotopy equivalence arising from $[\mathbb{C}P^3, SG_6]$ is homotopic to a diffeomorphism if its normal invariant is zero.

This establishes the result. □

Theorem 6.2.26. *The inertia group of $\mathbb{C}P^3 \times \mathbb{S}^5$ is $\mathbb{Z}/62$.*

Proof. We show that $I(\mathbb{C}P^3 \times \mathbb{S}^5) \subseteq I_h(\mathbb{C}P^3 \times \mathbb{S}^5)$. Suppose $[\Sigma] \in I(\mathbb{C}P^3 \times \mathbb{S}^5)$, and let $h_\Sigma : (\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma \to \mathbb{C}P^3 \times \mathbb{S}^5$ denote the canonical homeomorphism. Then, by Lemma 6.2.3 (v), $\eta^{Diff}(h_\Sigma) = 0$, and there exists an (oriented) diffeomorphism $g : (\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma \to \mathbb{C}P^3 \times \mathbb{S}^5$. It follows that $g \circ h_\Sigma^{-1} \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^5)$. Applying the composition formula for the normal invariant, we obtain

$$\eta^{Diff}(g \circ h_\Sigma^{-1}) = \eta^{Diff}(g) + (g^{-1})^* \eta^{Diff}(h_\Sigma^{-1}) = 0.$$

Therefore, by Lemma 6.2.25, the map $g \circ h_\Sigma^{-1}$ is homotopic to a diffeomorphism, and hence $[\Sigma] \in I_h(\mathbb{C}P^3 \times \mathbb{S}^5)$. Consequently, Theorem 6.2.24 implies that $I(\mathbb{C}P^3 \times \mathbb{S}^5) = \mathbb{Z}/62$. □

Theorem 6.2.27. *If N is a closed, smooth, oriented manifold homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^5$, then it is (oriented) diffeomorphic to $(\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma$, for some homotopy sphere $[\Sigma] \in \mathbb{Z}/2^4 \subset \Theta_{11}$.*

Moreover, two such manifolds are (oriented) diffeomorphic if and only if the corresponding homotopy spheres represent the same class in $\mathbb{Z}/2^4 \subset \Theta_{11}$.

Proof. Let $[(N, g)] \in \mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^5)$. Then, by Lemma 6.2.3 (v), the normal invariant $\eta^{Diff}(g)$ is trivial. It now follows from the smooth surgery exact sequence (2.8) with $X = \mathbb{C}P^3 \times \mathbb{S}^5$ that N is (oriented) diffeomorphic to $(\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma^{11}$, where $[\Sigma^{11}] \in bP_{12}$. By Lemma 6.2.26, we conclude that N is (oriented) diffeomorphic to $(\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma$ for some $[\Sigma] \in \mathbb{Z}/2^4 \subset bP_{12} \cong \Theta_{11}$.

Let $[\Sigma_1], [\Sigma_2] \in \mathbb{Z}/2^4 \subset \Theta_{11}$ be distinct classes such that $(\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma_1$ is (oriented) diffeomorphic to $(\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma_2$. Then $[\Sigma_1 \# \Sigma_2^{-1}] \in I(\mathbb{C}P^3 \times \mathbb{S}^5)$, and hence its order divides 62. Therefore, the only possibility is that $[\Sigma_1 \# \Sigma_2^{-1}]$ is trivial as $[\Sigma_1], [\Sigma_2] \in \mathbb{Z}/2^4$. This implies that $[\Sigma_1] = [\Sigma_2]$, which contradicts the assumption that they are distinct in Θ_{11}. Thus, $(\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma_1$ cannot be (oriented) diffeomorphic to $(\mathbb{C}P^3 \times \mathbb{S}^5)\#\Sigma_2$. □

Next, we determine the diffeomorphism classification of all closed, oriented, smooth manifolds homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^6$.

Lemma 6.2.28. *Let $f : \mathbb{C}P^3 \times \mathbb{S}^6 \to \mathbb{C}P^3 \times \mathbb{S}^6$ be a self-homotopy equivalence induced by an element of $\pi_6(F_{\mathbb{S}^1}(\mathbb{C}^4))$. If the normal invariant $\eta^{Diff}(f)$ is nontrivial, then it lies in the image of $\pi_6(G/O) \subset [\mathbb{C}P^3 \times \mathbb{S}^6, G/O]$.*

Proof. Let $f \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^6)$ arising from an element $t \in \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^4))$. Consider the following commutative diagram:

$$\begin{tikzcd}
& \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^4)) \arrow[dl, "\Psi"'] \arrow[dr, "\eta_{\text{rel}}^{Diff} \circ \Psi"] & \\
\mathcal{S}_6^{Diff}(\mathbb{C}P^3) \arrow[rr, "\eta_{\text{rel}}^{Diff}"] \arrow[d, "\Gamma"'] & & {[\Sigma^6\mathbb{C}P^3_+, G/O] \cong \mathbb{Z}/3 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \pi_6(G/O)} \arrow[d, "p^*"] \\
\mathcal{S}_h^{diff}(\mathbb{C}P^3 \times \mathbb{S}^6) \arrow[rr, "\eta^{Diff}"'] & & {[\mathbb{C}P^3 \times \mathbb{S}^6, G/O],}
\end{tikzcd}$$

where $\Gamma : \mathcal{S}_6^{Diff}(\mathbb{C}P^3) \to \mathcal{S}_h^{Diff}(\mathbb{C}P^3 \times \mathbb{S}^6)$ is the canonical mapping obtained by extension via diffeomorphism, $p^* : [\Sigma^6\mathbb{C}P^3_+, G/O] \to [\mathbb{C}P^3 \times \mathbb{S}^6, G/O]$ is induced by the quotient map $p : \mathbb{C}P^3 \times \mathbb{S}^6 \to \Sigma^6\mathbb{C}P^3_+$ and is injective by (4.1), while $[\Sigma^6\mathbb{C}P^3, G/O] \cong \mathbb{Z}/3 \oplus \mathbb{Z} \oplus \mathbb{Z}$, follows from the long exact sequence induced from (5.9) along G/O.

Given $\eta^{Diff}(f)$ is nontrivial, the above commutative diagram implies that $\eta_{\text{rel}}^{Diff}(f)$ is also nontrivial. Since $\pi_6(F_{\mathbb{S}^1}(\mathbb{C}^4)) \cong \pi_6^s(\Sigma\mathbb{C}P^\infty) \cong \mathbb{Z}/2$ by [16, 92] and both $\Psi : \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^4)) \to \mathcal{S}_6^{Diff}(\mathbb{C}P^3)$, $\eta_{\text{rel}}^{Diff} : \mathcal{S}_6^{Diff}(\mathbb{C}P^3) \to [\Sigma^6\mathbb{C}P^3_+, G/O]$ are group homomorphisms, it follows that $\eta_{\text{rel}}^{Diff} \circ \Psi(t) \in \pi_6(G/O) \subset [\Sigma^6\mathbb{C}P^3_+, G/O]$, and hence $\eta^{Diff}(f) \in \pi_6(G/O) \subset [\mathbb{C}P^3 \times \mathbb{S}^6, G/O]$. This completes the proof. □

Before stating the smooth classification result for $\mathbb{C}P^3 \times \mathbb{S}^6$, we consider a closed, oriented, smooth 12-dimensional manifold $\widetilde{N}$, together with a homeomorphism $h : \widetilde{N} \to \mathbb{C}P^3 \times \mathbb{S}^6$, such that the normal invariant $\eta^{Diff}(h)$ is nontrivial and lies in the subgroup $\pi_{10}(G/O)_{(3)} \subset [\mathbb{C}P^3 \times \mathbb{S}^6, G/O]$.

Theorem 6.2.29. *Let N be a closed, oriented, smooth manifold homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^6$. Then N is (oriented) diffeomorphic to exactly one of the manifolds $\mathbb{C}P^3 \times \mathbb{S}^6$ or $\widetilde{N}$.*

Proof. Let $[(N, g)]$ be an element of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^6)$. Then from Proposition 5.3.6 (ii) and Lemma 6.2.3 (vi), $\eta^{Diff}(g) \in (\pi_{10}(G/O))_{(3)} \subset [\mathbb{C}P^3 \times \mathbb{S}^6, G/O]$.

Suppose $\eta^{Diff}(g)$ is trivial, then the surgery exact sequence for $\mathbb{C}P^3 \times \mathbb{S}^6$ implies that (N, g) and $(\mathbb{C}P^3 \times \mathbb{S}^6, Id)$ represent the same element in $\mathcal{S}^{Diff}_h(\mathbb{C}P^3 \times \mathbb{S}^6)$.

Suppose $\eta^{Diff}(g)$ be nontrivial. In order to understand the implications of this assumption, we analyze the action of the elements of $\mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^6)$ on $[(N, g)]$. Since $\pi_6(\mathbb{C}P^3) = 0$, it follows from Proposition 2.6.21 and Proposition 2.6.22 that any self-homotopy equivalence of $\mathbb{C}P^3 \times \mathbb{S}^6$ arises from $\mathcal{E}(\mathbb{S}^6)$, $\mathcal{E}(\mathbb{C}P^3)$, $[\mathbb{C}P^3, E_1(\mathbb{S}^6)]$, or $\pi_6(E_1(\mathbb{C}P^3))$; those coming from $\mathcal{E}(\mathbb{S}^6)$ or $\mathcal{E}(\mathbb{C}P^3)$ are homotopic to diffeomorphisms, so for any such f, $\eta^{Diff}(f \circ g)$ is nontrivial if and only if $\eta^{Diff}(g)$ is nontrivial.

Let $f \in \mathcal{E}(\mathbb{C}P^3 \times \mathbb{S}^6)$ be a self-homotopy equivalence coming from an element of $\pi_6(E_1(\mathbb{C}P^3)) \cong \pi_6(F_{\mathbb{S}^1}(\mathbb{C}^4))$ (respectively, from $[\mathbb{C}P^3, E_1(\mathbb{S}^6)] \cong [\mathbb{C}P^3, SG_7]$). Then $\eta^{Diff}(f \circ g)$ is nontrivial, because $\eta^{Diff}(f) \in \pi_6(G/O) \subset [\mathbb{C}P^3 \times \mathbb{S}^6, G/O]$ by Lemma 6.2.28 (respectively, $\eta^{Diff}(f) \in [\mathbb{C}P^3, G/O] \subset [\mathbb{C}P^3 \times \mathbb{S}^6, G/O]$ by Fact 6.2.1), while $\eta^{Diff}(g) \in [\Sigma^6\mathbb{C}P^3, G/O] \subset [\mathbb{C}P^3 \times \mathbb{S}^6, G/O]$ as per Lemma 6.2.3 (vi). This implies that N is not diffeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^6$, by the smooth surgery exact sequence for $\mathbb{C}P^3 \times \mathbb{S}^6$.

Let $[(N_1, g_1)]$ and $[(N_2, g_2)]$ be two elements of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^6)$ such that $\eta^{Diff}(g_2) = -\eta^{Diff}(g_1) \neq 0$. Let $f = Id_{\mathbb{C}P^3} \times (-Id_{\mathbb{S}^6}) : \mathbb{C}P^3 \times \mathbb{S}^6 \to \mathbb{C}P^3 \times \mathbb{S}^6$ be the self-homotopy equivalence induced from $-Id_{\mathbb{S}^6} \in \mathcal{E}(\mathbb{S}^6)$. We claim that $(f^{-1})^*\eta^{Diff}(g_i) = -\eta^{Diff}(g_i)$ for $i = 1$ and 2.

Note that the map $f : \mathbb{C}P^3 \times \mathbb{S}^6 \to \mathbb{C}P^3 \times \mathbb{S}^6$ induces the map $(-Id_{\mathbb{S}^6}) \wedge Id_{\mathbb{C}P^3} : \mathbb{S}^6 \wedge \mathbb{C}P^3 \to \mathbb{S}^6 \wedge \mathbb{C}P^3$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}P^3 \times \mathbb{S}^6 \arrow[r, "p"] \arrow[d, "f = Id_{\mathbb{C}P^3} \times (-Id_{\mathbb{S}^6})"'] & \mathbb{S}^6 \wedge \mathbb{C}P^3 \arrow[d, "(-Id_{\mathbb{S}^6}) \wedge Id_{\mathbb{C}P^3}"] \\
\mathbb{C}P^3 \times \mathbb{S}^6 \arrow[r, "p"'] & \mathbb{S}^6 \wedge \mathbb{C}P^3.
\end{tikzcd}$$

Since $\eta^{Diff}(g_i) \in [\Sigma^6\mathbb{C}P^3, G/O]$, it follows from the above diagram that

$$(f^{-1})^*\eta^{Diff}(g_i) = \left((-Id_{\mathbb{S}^6} \wedge Id_{\mathbb{C}P^3})^{-1}\right)^* \eta^{Diff}(g_i).$$

Moreover, as $\eta^{Diff}(g_i) \in \mathbb{Z}/3 \subset [\Sigma^6\mathbb{C}P^3, G/O]$ and $\psi_* : [\Sigma^6\mathbb{C}P^3, Top/O] \cong \mathbb{Z}/3 \to$

$[\Sigma^6\mathbb{C}P^3, G/O]$ is injective, it suffices to show that

$$((-Id_{\mathbb{S}^6} \wedge Id_{\mathbb{C}P^3})^{-1})^* : [\Sigma^6\mathbb{C}P^3, Top/O] \to [\Sigma^6\mathbb{C}P^3, Top/O]$$

sends the generator $\bar{1} \in [\Sigma^6\mathbb{C}P^3, Top/O]$ to its inverse $-\bar{1}$.

To show this, we observe that $(-Id_{\mathbb{S}^6}) \wedge Id_{\mathbb{C}P^3} : \mathbb{S}^6 \wedge \mathbb{C}P^3 \to \mathbb{S}^6 \wedge \mathbb{C}P^3$ fits into the following commutative diagram:

$$\begin{tikzcd}
\mathbb{S}^6 \wedge \mathbb{C}P^3 \arrow[r, "Id_{\mathbb{S}^6}\wedge q"] \arrow[d, "(-Id_{\mathbb{S}^6})\wedge Id_{\mathbb{C}P^3}"'] & \mathbb{S}^6 \wedge (\mathbb{S}^4 \vee \mathbb{S}^6) \arrow[r, "Id_{\mathbb{S}^6}\wedge Pr"] \arrow[d, "(-Id_{\mathbb{S}^6})\wedge(Id_{\mathbb{S}^4\vee\mathbb{S}^6})"] & \mathbb{S}^6 \wedge \mathbb{S}^4 \arrow[d, "(-Id_{\mathbb{S}^6})\wedge Id_{\mathbb{S}^4}\simeq -Id_{\mathbb{S}^{10}}"] \\
\mathbb{S}^6 \wedge \mathbb{C}P^3 \arrow[r, "Id_{\mathbb{S}^6}\wedge q"'] & \mathbb{S}^6 \wedge (\mathbb{S}^4 \vee \mathbb{S}^6) \arrow[r, "Id_{\mathbb{S}^6}\wedge Pr"'] & \mathbb{S}^6 \wedge \mathbb{S}^4,
\end{tikzcd}$$

where the map $(-Id_{\mathbb{S}^6}) \wedge Id_{\mathbb{S}^4} : \mathbb{S}^6 \wedge \mathbb{S}^4 \to \mathbb{S}^6 \wedge \mathbb{S}^4$ is homotopic to the map $-Id_{\mathbb{S}^{10}} : \mathbb{S}^{10} \to \mathbb{S}^{10}$ for degree reason. This diagram gives rise to the subsequent commutative diagram along Top/O:

$$\begin{tikzcd}
(\pi_{10}(Top/O))_{(3)} \arrow[r, "(Id_{\mathbb{S}^6}\wedge q)^*_{(3)}\circ(Id_{\mathbb{S}^6}\wedge Pr)^*_{(3)}"] \arrow[d, "(-Id_{\mathbb{S}^{10}})^*_{(3)}"'] & {[\mathbb{S}^6 \wedge \mathbb{C}P^3, Top/O]_{(3)}} \arrow[d, "(-Id_{\mathbb{S}^6}\wedge Id_{\mathbb{C}P^3})^*_{(3)}"] \\
(\pi_{10}(Top/O))_{(3)} \arrow[r, "(Id_{\mathbb{S}^6}\wedge q)^*_{(3)}\circ(Id_{\mathbb{S}^6}\wedge Pr)^*_{(3)}"'] & {[\mathbb{S}^6 \wedge \mathbb{C}P^3, Top/O]_{(3)}}.
\end{tikzcd}$$

The map $(-Id_{\mathbb{S}^{10}})^*_{(3)} = (-Id_{\mathbb{S}^6} \wedge Id_{\mathbb{S}^4})^*_{(3)} : \pi_{10}(Top/O)_{(3)} \to \pi_{10}(Top/O)_{(3)}$ sends the generator $\bar{1}$ of $\pi_{10}(Top/O)_{(3)}$ to the generator $-\bar{1}$ in $\pi_{10}(Top/O)_{(3)}$. Consequently, from the above diagram, it is evident that $(-Id_{\mathbb{S}^6} \wedge Id_{\mathbb{C}P^3})^*_{(3)} : [\Sigma^6\mathbb{C}P^3, Top/O]_{(3)} \to [\Sigma^6\mathbb{C}P^3, Top/O]_{(3)}$ takes the generator $\bar{1}$ of $[\Sigma^6\mathbb{C}P^3, Top/O]_{(3)}$ to the generator $-\bar{1}$ in $[\Sigma^6\mathbb{C}P^3, Top/O]_{(3)}$. Therefore

$$\eta^{Diff}(f \circ g_1) = (f^{-1})^*\eta^{Diff}(g_1) = -\eta^{Diff}(g_1) = \eta^{Diff}(g_2),$$

and

$$\eta^{Diff}(f \circ g_2) = (f^{-1})^*\eta^{Diff}(g_2) = -\eta^{Diff}(g_2) = \eta^{Diff}(g_1).$$

Thus, N_1 and N_2 are diffeomorphic. We may therefore take $\widetilde{N}$ to be either N_1 or N_2.

This concludes the proof. □

We begin the classification of smooth manifolds homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^7$ by establishing the existence of a self-homotopy equivalence of $\mathbb{C}P^3 \times \mathbb{S}^7$ with nontrivial normal invariant.

Lemma 6.2.30. *There exists a self-homotopy equivalence $f_{\nu^2} : \mathbb{C}P^3 \times \mathbb{S}^7 \to \mathbb{C}P^3 \times \mathbb{S}^7$ with nontrivial normal invariant. In particular, the normal invariant of f_{ν^2} equals $[\nu^3]$, where $[\nu^3]$ represents the image of $\nu^3 \in \pi_9^s$ in $\pi_9(G/O)$.*

Proof. Let $f_{\nu^2} : \mathbb{C}P^3 \times \mathbb{S}^7 \to \mathbb{C}P^3 \times \mathbb{S}^7$ be a self-homotopy equivalence induced by the nontrivial element $\nu^2 \in \pi_6^s \subset \pi_7(F_{\mathbb{S}^1})$. Again by Fact 6.0.1 (f), the normal invariant $\eta^{Diff}(f_{\nu^2})$ is the image of the composition

$$\Sigma^7\mathbb{C}P^3_+ \xrightarrow{\Sigma^6 t} \mathbb{S}^6 \xrightarrow{\nu^2} \mathbb{S}^0$$

under the map $[\Sigma^7\mathbb{C}P^3_+, \mathbb{S}^0] \to [\Sigma^7\mathbb{C}P^3_+, G/O] \subset [\mathbb{C}P^3 \times \mathbb{S}^7, G/O]$.

Since the restriction $(\nu^2 \circ \Sigma^6 t)|_{\mathbb{S}^9}$ equals ν^3, the normal invariant $\eta^{Diff}(f_{\nu^2})|_{\Sigma^7\mathbb{C}P^1}$ is the image of $\nu^3 \in \pi_9^s$ in $\pi_9(G/O)$. Consider the commutative diagram:

$$\begin{tikzcd}
\pi_{10}^s \arrow[r, "\eta^*"] & \pi_{11}^s \arrow[r, "(\Sigma^7 f_{\mathbb{C}P^2})^*"] \arrow[d, "j_*"'] & {[\Sigma^7\mathbb{C}P^2, G]} \arrow[r, "(\Sigma^7 i)^*"] \arrow[d, "j_*"] & \pi_9^s \arrow[r, "\eta^*"] \arrow[d, "j_*"] & \pi_{10}^s \arrow[d, "\cong"', "j_*"] \\
 & 0 \arrow[r] & {[\Sigma^7\mathbb{C}P^2, G/O] \cong \mathbb{Z}/2\{[\nu^3]\}} \arrow[r, "(\Sigma^7 i)^*"'] & \pi_9(G/O) \arrow[r, "\eta^*"'] & \pi_{10}(G/O).
\end{tikzcd}$$

From this diagram, we see that the image of $j_* : [\Sigma^7\mathbb{C}P^2, G] \to [\Sigma^7\mathbb{C}P^2, G/O]$ is $\mathbb{Z}/2\{[\nu^3]\}$. Therefore, $\eta^{Diff}(f_{\nu^2})$ restricted to $\Sigma^7\mathbb{C}P^2$ equals to $[\nu^3]$. Hence, $\eta^{Diff}(f_{\nu^2})$ is nontrivial. Since $[\Sigma^7\mathbb{C}P^3, G] = [\Sigma^7\mathbb{C}P^2, G]$, it follows that $\eta^{Diff}(f_{\nu^2})$ is generated by $[\nu^3]$. □

Proposition 6.2.31. *If the normal invariant of $[(N, g)] \in \mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^7)$ is trivial, then N is (oriented) diffeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^7$.*

Proof. Since the normal invariant $\eta^{Diff}(g)$ is trivial, the smooth surgery exact sequence for $\mathbb{C}P^3 \times \mathbb{S}^7$ implies that N is oriented diffeomorphic to $(\mathbb{C}P^3 \times \mathbb{S}^7)\#\Sigma$, for some $[\Sigma] \in bP_{14}$. By Theorem 2.1.10, $bP_{14} = 0$, so the manifold N is (oriented) diffeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^7$. □

Theorem 6.2.32. *If a closed, oriented, smooth manifold N is homeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^7$, then N is (oriented) diffeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^7$.*

Proof. Let $[(N, g)]$ be an element of $\mathcal{C}(\mathbb{C}P^3 \times \mathbb{S}^7)$.

If the normal invariant of g is trivial, then N is (oriented) diffeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^7$, by Proposition 6.2.31.

If the normal invariant $\eta^{Diff}(g)$ is nontrivial, then Lemma 6.2.3 (vii) and Proposition 5.3.6 (ii) show that $\eta^{Diff}(g) = [\nu^3]$. By Lemma 6.2.30 and the composition formula for normal invariants, we obtain

$$\eta^{Diff}(f_{\nu^2} \circ g) = \eta(f_{\nu^2}) + (f_{\nu^2}^{-1})^* \eta^{Diff}(g) = [\nu^3] \pm [\nu^3] = 0.$$

Hence, Proposition 6.2.31 implies that N is (oriented) diffeomorphic to $\mathbb{C}P^3 \times \mathbb{S}^7$. □

6.3 Smooth Classification of the product of Smooth Projective Plane-Like 8-Manifold with $\mathbb{S}^1$

We note that every closed $(n-1)$-connected manifold of dimension $2n \geq 4$ with rank 1 is a projective plane-like manifold. Recall that a *projective plane-like manifold* is a simply connected, closed topological manifold M such that $H_*(M;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Examples of such manifolds include the projective planes $\mathbb{C}P^2$, $\mathbb{H}P^2$, and $\mathbb{O}P^2$, which are defined over the complex numbers, quaternions, and octonions, respectively. Eells and Kuiper [40] established several fundamental results in the study of projective plane-like manifolds. For instance, they showed that the integral cohomology ring of such a projective plane-like manifold M is isomorphic to that of a projective plane, i.e.,

$$H^*(M;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^3).$$

This result, combined with the solution of the Hopf invariant one problem, implies that the dimension of M must be $4, 8$, or 16 (cf. [40, §5]). Moreover, they showed that there are six homotopy types of projective plane-like manifolds of dimension 8 and sixty of dimension 16 [40, §6].

Furthermore, Eells and Kuiper classified smooth projective plane-like manifolds of dimensions 8 and 16 up to connected sum with homotopy spheres. This classification was later completed by Kramer and Stolz [68], who showed that the diffeomorphism class of a smooth projective plane-like manifold M of dimension $2n \geq 8$ is determined by its Pontryagin number

$$p^2_{\frac{n}{4}}(M)[M] \in \mathbb{Z}.$$

The results of Eells and Kuiper, combined with those of Wall [126], established that an integer k equals the Pontryagin number $p^2_{\frac{n}{4}}(M)[M]$ of such a manifold if and only if k takes the form:

$$k = 2^2(1+2t)^2 \quad \text{with } t \equiv 0, 7, 48, 55 \pmod{56} \quad (\text{for } n = 4)$$

$$k = 6^2(1+2t)^2 \quad \text{with } t \equiv 0, 127, 16128, 16255 \pmod{16256} \quad (\text{for } n = 8)$$

(see, for example, [68, Theorem 1.3]). These results provide an infinite family of smooth projective plane-like manifolds in dimensions 4 and 8 each of which possesses a unique differentiable structure, in the sense that any manifold homeomorphic to M is, in fact, diffeomorphic to M. Throughout this section, $\widetilde{\mathbb{H}P}^2$ denotes a projective plane-like smooth manifold of dimension 8.

Let $\tilde{\pi}_0(Diff(M))$ denote the group of isotopy classes of self-diffeomorphisms of M preserving orientation. We know that there is a natural map $\gamma : \Theta_{n+1} \to \tilde{\pi}_0(Diff(M))$ defined as follows: given $\Sigma \in \Theta_{n+1}$, it corresponds to a diffeomorphism $f : \mathbb{D}^n \to \mathbb{D}^n$ with $f|_{\mathbb{S}^{n-1}} = Id$, then $\gamma(\Sigma) \in Diff(M)$ with $\gamma(\Sigma)|_{M-\mathbb{D}^n} = Id$ and $\gamma(\Sigma)|_{\mathbb{D}^n} = f$. We now recall a lemma describing the inertia group of the product manifold $M \times \mathbb{S}^1$.

Lemma 6.3.1. *[71, Proposition 1] Let M be a closed, oriented, smooth manifold of* *dimension $n \geq 5$. Then the inertia group of $M \times \mathbb{S}^1$ is given by*

$$I(M \times \mathbb{S}^1) = Ker\left(\Theta_{n+1} \xrightarrow{\gamma} \tilde{\pi}_0(Diff(M))\right).$$

Moreover, for $[\Sigma_1]$, $[\Sigma_2] \in \Theta_{n+1}$, if $\gamma([\Sigma_1])$ is isotopic to $\gamma([\Sigma_2])$, then $(M \times \mathbb{S}^1)\#\Sigma_1$ is oriented diffeomorphic to $(M \times \mathbb{S}^1)\#\Sigma_2$.

Recall from [6] that an exotic m-sphere $[\Sigma^m]$ does not bound spin manifolds if and only if $m \equiv 1, 2 \pmod 8$, and $m > 8$, which is equivalent to the condition that $\alpha([\Sigma^m]) = 0$, where $\alpha : \Theta_m \to KO^{-m}$ is the α-invariant. In particular, $\Theta_9 = bspin_{10} \oplus \mathbb{Z}/2\{[\Sigma_\mu]\}$, where $bspin_{10}$ denotes the group of homotopy 9-spheres bounding spin manifolds and $[\Sigma_\mu]$ is the exotic 9-sphere corresponding to the generator $[\mu] \in \pi_9^s/Im(J)$.

Lemma 6.3.2. *$I(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1) = bSpin_{10} = bP_{10} \oplus \mathbb{Z}/2\{[\Sigma_{\nu^3}]\}$, where $[\Sigma_{\nu^3}]$ denotes the exotic 9-sphere corresponding to the generator $[\nu^3] \in \pi_9^s/Im(J)$.*

Proof. It is known that $\Theta_9 \cong bP_{10} \oplus \mathbb{Z}/2\{[\Sigma_\mu]\} \oplus \mathbb{Z}/2\{[\Sigma_{\nu^3}]\}$, where $[\Sigma_\mu]$ and $[\Sigma_{\nu^3}]$ are exotic 9-spheres corresponding to the elements $[\mu], [\nu^3] \in \pi_9^s/Im(J)$, respectively. The map $\gamma : \Theta_9 \to \tilde{\pi}_0(Diff(\widetilde{\mathbb{H}P}^2))$ is surjective, as shown in [120, Page 13], and $\tilde{\pi}_0(Diff(\widetilde{\mathbb{H}P}^2)) \cong \mathbb{Z}/2$, generated by $\gamma([\Sigma_\mu])$, according to [120, Theorem 1.1]. It follows that

$$Ker\left(\Theta_9 \xrightarrow{\gamma} \tilde{\pi}_0(Diff(\widetilde{\mathbb{H}P}^2))\right) \cong bP_{10} \oplus \mathbb{Z}/2\{[\Sigma_{\nu^3}]\},$$

which, by Lemma 6.3.1, is the inertia group $I(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1)$. □

Theorem 6.3.3. *Any closed, oriented, smooth manifold N homeomorphic to $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$ is oriented diffeomorphic to exactly one of the manifolds $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$ or $(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1)\#\Sigma_\mu$, where $[\Sigma_\mu]$ is the exotic 9-sphere associated with $[\mu] \in \pi_9^s/Im(J)$.*

Proof. Let $[(N, g)]$ be an element of $\mathcal{C}(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1)$.

If the normal invariant $\eta^{Diff}(g)$ is trivial, then the smooth surgery exact sequence for $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$ implies that N is (oriented) diffeomorphic to $(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1)\#\Sigma$, for some

$[\Sigma] \in bP_{10}$. Since $bP_{10} \cap I(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1) \neq \emptyset$, it follows that N is (oriented) diffeomorphic to $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$.

Suppose $\eta^{Diff}(g)$ is nontrivial. By Theorem 5.4.2 (i), we have $\mathcal{C}(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1) \cong \Theta_9 \oplus \Theta_8$. If $[(N, g)] \in \Theta_9$, then Lemma 6.3.2 implies that N is (oriented) diffeomorphic to either $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$ or $(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1)\#\Sigma_\mu$. On the other hand, if $[(N, g)] \in \Theta_8 \subset \mathcal{C}(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1)$, then $\eta^{Diff}(g) = [\epsilon] \in \pi_8(G/O)$, where $[\epsilon]$ is the generators of $\pi_8^s/Im(J)$ [103, Theorem 1.1.14]. Since $[\widetilde{\mathbb{H}P}^2, Top/O] = \Theta_8$ and $I(\widetilde{\mathbb{H}P}^2) = \mathbb{Z}/2$ [57, 34], there exists a self-homotopy equivalence $f : \widetilde{\mathbb{H}P}^2 \to \widetilde{\mathbb{H}P}^2$ whose normal invariant is $\eta^{Diff}(f) = [\epsilon] \in \pi_8(G/O)$. As $\widetilde{\mathbb{H}P}^2$ is simply connected, Proposition 2.6.21 implies that every self-homotopy equivalence of $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$ is diagonalizable. Hence, by Proposition 2.6.22, $h = f \times Id : \widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1 \to \widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$ is a self-homotopy equivalence of $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$, and $\eta^{Diff}(h) = \eta^{Diff}(f)$. Now, the composition formula for normal invariant gives

$$\begin{aligned} \eta^{Diff}(h \circ g) &= \eta^{Diff}(h) + (h^{-1})^* \eta^{Diff}(g) \\ &= [\epsilon] \pm [\epsilon] = 0. \end{aligned}$$

Thus, it follows from the previous case that N is (oriented) diffeomorphic to $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^1$. This completes the proof. □

Lemma 6.3.4. *Let M be a smooth projective plane-like manifold of dimension 16. Then $I(M \times \mathbb{S}^1) = bP_{18} \oplus \mathbb{Z}/2\{[\Sigma_{\eta\eta^*}]\}$, where $[\Sigma_{\eta\eta^*}]$ is the exotic 17-sphere associated with the generator $[\eta\eta^*] \in \pi_{17}^s/Im(J)$.*

Proof. According to [120, Pages 13–14], the quotient $\Theta_{17}/Ker(\gamma)$ is isomorphic to $\mathbb{Z}/2\{[\Sigma_{\bar{\mu}}]\} \oplus \mathbb{Z}/2\{[\Sigma_{\nu\kappa}]\}$, where $[\Sigma_{\bar{\mu}}]$ and $[\Sigma_{\nu\kappa}]$ are exotic 17-spheres corresponding to the elements $[\bar{\mu}]$ and $[\nu\kappa]$ in $\pi_{17}^s/Im(J)$.

Since Θ_{17} fits into the following split short exact sequence

$$0 \longrightarrow bP_{18} \longrightarrow \Theta_{17} \longrightarrow \pi_{17}^s/Im(J) \longrightarrow 0,$$

we conclude that $Ker(\gamma) \cong bP_{18} \oplus \mathbb{Z}/2\{[\Sigma_{\eta\eta^*}]\}$, where $[\Sigma_{\eta\eta^*}]$ denotes the exotic 17-sphere corresponding to $[\eta\eta^*] \in \pi_{17}^s/Im(J)$.

Thus, by Lemma 6.3.1, the inertia group $I(M \times \mathbb{S}^1)$ is given by $bP_{18} \oplus \mathbb{Z}/2\{[\Sigma_{\eta\eta^*}]\}$.

□

Theorem 6.3.5. *$bP_{18} \cap I_h(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9) = \emptyset$.*

Proof. To prove the theorem, it is enough to show that the surgery obstruction map $\sigma_{18}^{Diff} : [\Sigma(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9), G/O] \to L_{18}(\mathbb{Z})$ is trivial, as follows from the smooth surgery exact sequence for $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9$. The surgery obstruction map $\sigma_{18}^{Diff} : [\Sigma(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9), G/O] \to L_{18}(\mathbb{Z})$ fits into the following commutative diagram:

$$\begin{tikzcd}
{[\Sigma(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9), G/O]} \arrow[r, "\sigma_{18}^{Diff}"] \arrow[d, "p^*"'] & L_{18}(\mathbb{Z}) \arrow[r, "\omega^{Diff}"] \arrow[d, "\cong"] & \mathcal{S}^{Diff}(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9) \\
{[\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9 \times \mathbb{S}^1, G/O]} \arrow[r, "\sigma_{18}^{Diff}"'] & L_{18}(\mathbb{Z}[\mathbb{Z}]), &
\end{tikzcd}$$

where $p^* : [\Sigma(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9), G/O] \to [\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9 \times \mathbb{S}^1, G/O]$ is injective by (4.1), and $L_{18}(\mathbb{Z}) \to L_{18}(\mathbb{Z}[\mathbb{Z}])$ is an isomorphism from [113].

Let $f : \Sigma(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9) \to G/O$. Then from the above commutative diagram and [28], the surgery obstruction of f is given by:

$$\sigma_{18}^{Diff}(f) = \sigma_{18}^{Diff}(p^*(f)) = \langle v_4^2(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9 \times \mathbb{S}^1)(p^*(f))^*(\mathcal{K}_2), [\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9 \times \mathbb{S}^1]\rangle = 0,$$

where $\mathcal{K}_2 \in H^2(G/O; \mathbb{Z}/2) = \mathbb{Z}/2$ is the generator, the 4-th Wu class $v_4(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9 \times \mathbb{S}^1)$ of $\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9 \times \mathbb{S}^1$ is zero from [67, Lemma 3.5]and [82]. □

Combining [62, Lemma 9.1] with Theorems 4.4.1 (iii) and 6.3.5, we obtain the following result.

Proposition 6.3.6. *The homotopy inertia group $I_h(\widetilde{\mathbb{H}P}^2 \times \mathbb{S}^9)$ is either $\mathbb{Z}/2\{[\Sigma_{\nu\kappa}]\}$ or $\mathbb{Z}/2\{[\Sigma_{\nu\kappa}]\} \oplus \mathbb{Z}/2\{[\Sigma_{\eta\eta^*}]\}$, where $[\Sigma_{\nu\kappa}]$ and $[\Sigma_{\eta\eta^*}]$ are exotic 17-spheres corresponding to the elements $[\nu\kappa]$ and $[\eta\eta^*]$ in $\pi_{17}^s/Im(J)$, respectively.*

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