Problems on Hilbert schemes and quiver bundles

By

Saurav Holme Choudhury

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The Institute of Mathematical Sciences, Chennai

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Dedicated to *Maa, Baba and Jethamoni*

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Contents

Summary

This thesis is divided into two distinct projects.

(I) **Stratified bundles on Hilbert scheme of points on a surface** Let k be an algebraically closed field of characteristic $p > 3$ and S be a smooth projective surface over k with k-rational point x. For $n \geq 2$, let $S^{[n]}$ denote the Hilbert scheme of n points on S. We compute the fundamental group scheme $\pi^{\mathrm{alg}}(S^{[n]}, \tilde{nx})$ defined by the Tannakian category of stratified bundles on $S^{[n]}$.

(II) **Criteria for rationality of moduli of chains** Let X be a compact Riemann surface of genus ≥ 2. We study the birational geometry of the moduli of holomorphic chains of type $\underline{t}$ on X, which are stable with respect to a fixed parameter θ. For suitable $\underline{t}$ and θ, we establish a criteria for the rationality of these moduli spaces.

Notation

$\mathcal{D}_X$	The sheaf of differential operators on the scheme X
$G_{\mathbf{ab}}$	Abelianization of group (scheme) G
$\pi^{\textbf{ét}}(X, x)$	The étale fundamental group of $\mathcal{S}(X)$ wrt basepoint x
$\pi^{\mathbf{alg}}(X, x)$	Tannakian group scheme of $\mathcal{S}(X)$ wrt basepoint x
$\mathcal{S}(X)$	Category of stratified bundles on X
$\mathbf{Rep}_k(G)$	Category of k-representations of G
Vec_k	Category of k vector spaces
$X^{(n)}$	The n-th symmetric product of X
$X^{[n]}$	The Hilbert scheme of n points of X
$\mathcal{O}_X$	The structure sheaf of the scheme X
$M^s_\theta(\underline{n}, \underline{d})$	The coarse moduli space of θ-stable chains of type $(\underline{n}, \underline{d})$.
$\mathcal{M}^s_\theta(\underline{n}, \underline{d})$	The moduli stack of θ-stable chains of type $(\underline{n}, \underline{d})$.
$M^s_C(r, d)$	The coarse moduli space of stable vector bundles of rank r and degree d.
$\mathcal{H}ecke^l_{\mathcal{E}/\mathcal{T}}$	The stack of length l Hecke correspondences of $\mathcal{E}$
$\mathbf{Quot}^{(r,P)}_{T\times X/T}(\mathcal{E})$	The Quot scheme of rank r quotients of $\mathcal{E}$ with Hilbert polynomial P
$\mathcal{B}un_{n,d}$	The moduli stack of vector bundles of rank n and degree d.
$Coh(n, d)$	The moduli stack of coherent sheaves of rank n and degree d.

$Ch\,(\underline{r},\underline{d})$	The moduli stack of chains of type $(\underline{n},\underline{d})$.
$Ch^{\mathbf{inj}}(\underline{r},\underline{d})$	The moduli stack of injective chains of type $(\underline{n},\underline{d})$.
$Ch^{\underline{\theta}-\mathbf{ss}}(\underline{r},\underline{d})$	The moduli stack of $\underline{\theta}$-semistable chains of type $(\underline{n},\underline{d})$.

Chapter 1

Introduction

This thesis is divided into two parts. The first part is on stratified bundles on the Hilbert scheme of n points on a smooth projective surface. The second part is on the moduli of chains on a smooth projective curve and establishes a criterion for rationality of these moduli spaces.

Stratified bundles on Hilbert Scheme of points on a surface

Hilbert schemes of closed subschemes of a fixed projective scheme X are one of the fundamental examples of moduli spaces. Since their construction by Grothendieck, Hilbert schemes have been extensively studied and have played a foundational role in the construction and study of other moduli spaces. In particular, the Hilbert scheme of zero-dimensional subschemes of X have appeared in diverse areas such as representation theory, enumerative geometry, moduli of sheaves etc.

The Hilbert scheme of n points on X, denoted $X^{[n]}$, gives us a canonical compactification of the configuration space of n points on X. When X is a smooth projective curve, $X^{[n]}$ is isomorphic to the symmetric n-product of X. For X a smooth projective surface, Fogarty showed that $X^{[n]}$ is a smooth projective variety of dimension $2n$. Thus for surfaces, $X^{[n]}$ is a resolution of singularities for $Sym^n(X)$, which is never smooth for $\dim X \geq 2$.

In this project we study stratified bundles on $X^{[n]}$, when X is a smooth projective surface over an algebraically closed field k of characteristic $p > 3$.

Stratified bundles are the positive characteristic analogues of flat connections. The Riemann Hilbert correspondence identifies the category of representations of the fundamental group of a smooth projective variety X over $\mathbb{C}$ with the category of flat connections (E, ∇) on X. Over $\mathbb{C}$, this category is equivalent to the category of D_X-modules E over X, where D_X is the sheaf of differential operators on X.

However, in positive characteristic, the category of flat connections on X is not well-behaved (local freeness fails) and the two categories are not equivalent (as D_X is not the universal enveloping algebra of $\mathrm{Der}_k(X)$). So, in positive characteristic, one directly works with the category of D_X-modules E over X.

Gieseker [Gie75] gave an alternate description of the category of D_X-modules in positive characteristic in terms of *stratified bundles*, which is technically easier to work with. Stratified bundles on X are vector bundles which satisfies infinite Frobenius descent. More precisely, one has

Definition 1.0.1. *Let X be a smooth projective variety over k, where $\mathrm{char}\, k > 0$. Then X is equipped with the Frobenius morphism $F : X \to X$. A stratified bundle on X is a sequence of vector bundles $(E_0, E_1, \dots)$ equipped with isomorphisms $\alpha_i : F^* E_{i+1} \to E_i$*

The category of stratified bundles on X is denoted as $\mathcal{S}(X)$. We use the formalism of Tannakian categories [DM82] to study $\mathcal{S}(X)$.

One shows that $\mathcal{S}(X)$ is a rigid tensor category. For any $x \in X$, the fiber functor $\mathcal{S}(X) \to Vect_k$ sending $(E_0, E_1, \dots)$ to $E_0|_x$ makes $\mathcal{S}(X)$ into a neutral Tannakian category. The corresponding Tannakian group scheme is the algebraic fundamental group $\pi^{\mathrm{alg}}(X, x)$. This proalgebraic group was studied in [dS07].

Let S be a smooth projective surface over k. Let $S^{(n)}$ be the symmetric n-product of S and $S^{[n]}$ be the Hilbert scheme of n points on S. Fogarty showed that $S^{[n]}$ is a smooth projective variety.

Our main theorem is the following:

Theorem 1.0.2. *Let char $k \geq 3$ and $n \geq 2$, then there is a natural isomorphism of affine group schemes over k*

$$\pi^{alg}(S, x)_{ab} \to \pi^{alg}(S^{[n]}, \tilde{nx})$$

Remark 1.0.3. *This project was motivated by earlier calculations of the Nori fundamental group scheme and the S-fundamental group scheme in [PS20].*

Criterion of rationality for moduli of chains

Two irreducible varieties X and Y over $\mathbb{C}$ are said to be birational if their function fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are isomorphic over $\mathbb{C}$. A variety X is said to be rational if it is birational to $\mathbb{P}^n$ for some n. Determining whether a variety is rational is a longstanding problem in algebraic geometry. The rationality of a moduli space, in particular, implies that the objects it represents can be systematically described using a simpler space (an open subset of affine space), with each object appearing uniquely. In other words, this mean we can write down polynomial equations defining almost all the objects of the moduli space, such that the coefficients of the polynomials depend on n independent parameters and the generic object occurs exactly once in this description.

The study of the rationality of moduli spaces has a long history. The rationality (or lack thereof) for moduli of curves has been investigated by several mathematicians, including Severi, Harris, Mumford, and others. More directly related to our research is the work of Newstead, King, and Schofield on the rationality of moduli of vector bundles on a smooth projective curve.

In our work (jointly with S Manikandan), we investigate the rationality of moduli of chains on a smooth projective curve. Chains of length n consist of $n + 1$ vector bundles $E_0, \ldots, E_n$ alongwith n vector bundle morphisms $\phi_i : E_{i+1} \to E_i$. There is a notion of θ-stability for chains for $\theta \in \mathbb{R}^{n+1}$, which we recall in the second chapter. Let $(\underline{r}, \underline{L}) = (r_0, \ldots, r_n, L_0, \ldots L_n)$ encode the ranks and determinant of the vector bundles E_i. We investigate the rationality properties of the moduli of θ-stable chains with fixed invariants $(\underline{r}, \underline{L})$.

We first investigate the related moduli space of holomorphic chains of rank $\underline{n}$ and degree $\underline{d}$ whose components are stable vector bundles and morphisms are non-zero. We denote this moduli space by $N^s(\underline{n}, \underline{d})$. Under certain assumptions on $(\underline{n}, \underline{d})$, one can show that $N^s(\underline{n}, \underline{d})$ can be explicitly described as an iterated projective bundle over the product of moduli spaces of stable vector bundles $\prod_{i=0}^{n} M^s(r_i, d_i)$.

Let $N_\theta^s(\underline{t})$ denote the open subset of $M_\theta^s(\underline{t})$ consisting of θ-stable chains whose underlying vector bundles E_i are stable. We then establish the following result.

Theorem 1.0.4. *Let $\underline{t} = (\bar{r}, \bar{d}) \in \mathbb{N}^{n+1} \times \mathbb{Z}^{n+1}$ and $\theta \in \mathbb{R}^n_{>\theta_{Higgs}}$. Assume $(\bar{r}, \bar{d})$ satisfy $r_i d_{i-1} - r_{i-1} d_i > r_{i-1} r_i (2g-2)$ for all $1 \leq i \leq n$.. Let $\theta \in Stability_{\bar{r}}^{\bar{d}}$ and satisfy condition C_0. If $N_\theta^s(\underline{t})$ is non-empty, then $M_\theta^s(\underline{t})$ is birational to an iterated projective bundle on $\prod_{i=0}^{n} M^s(r_i, d_i)$.*

Remark 1.0.5. *For the definitions of $Stability_{\bar{r}}^{\bar{d}}$ and condition C_0, see 3.2.2.*

This allows us to establish a criterion for rationality of $M_\theta^s(\underline{n}, \underline{L})$, by establishing the non-emptiness of $N_\theta^s(\underline{t})$ under the following assumptions.

Corollary 1.0.6. *Let $\underline{t} = (\underline{r}, \underline{d})$ and $\underline{\theta}$ satisfy*

- $\underline{r} = (n, n, \ldots, n)$.
- $d_{i-1} - d_i > n(2g-2)$ *for all* i
- $\underline{\theta}$ *be any stability parameter satisfying* $\theta_i - \theta_{i-1} > d_{i-1} - d_i$.

Then for generic choice of $\underline{L}$, irreducibility of $M^{\theta-ss}(\underline{n}, \underline{L})$ implies rationality of $M^{\theta-ss}(\underline{n}, \underline{L})$.

1.1 Arrangement of thesis

The thesis is divided into three chapters.

We give a brief introduction of the objects of study and the main theorems in the first chapter. In the second chapter, we begin by revisiting the concept of Tannakian categories and review key properties about Tannakian formalism that will be useful later. The subsequent part of the chapter focuses on stratified bundles. We reiterate the relevant definitions and demonstrate how the category of stratified bundles can be structured as a neutral Tannakian category (with respect to a fixed base point). The next section reviews the geometric properties of the Hilbert scheme of points and the Hilbert-Chow morphism. The following section defines a natural functor between Tannakian categories, employing a descent lemma and purity for stratified bundles. We conclude the proof by showing that the natural functor defined in the previous section induces an isomorphism between the associated fundamental group schemes. In the concluding section, we recover the computation of the étale fundamental group from our theorem.

In the third chapter, after introducing the problem of interest, we recall definitions and properties of moduli of chains and other preliminaries. Moduli of chains with stable components and their birational geometry are studied in the third section of the chapter. We show that under suitable conditions these moduli spaces can be realized as iterated projective bundles over a product of moduli of vector bundles. The final section establishes the non-emptiness of $N_{\theta}^{s}(\underline{t})$ under the conditions as in 1.0.6. This allows us to formulate our rationality criterion for $M_{\theta}^{s}(\underline{r}, \underline{L})$.

Chapter 2

Stratified bundles on the Hilbert scheme of n points on a surface

For a variety X over $\mathbb{C}$, one has the classical notion of the fundamental group $\pi_1(X^{\mathrm{an}}, x)$ defined using the analytic topology on X. Over arbitrary base fields k, one has several analogues of the fundamental group defined in terms of algebro-geometric information.

In [GR02], Grothendieck introduced the notion of étale fundamental group $\pi^{\text{ét}}(X, x)$, where X is a scheme and x is a geometric point of X, in terms of the finite etale covers of X. In [Nor76], Nori defined the Nori fundamental group scheme $\pi^N(X, x)$, where X is a connected, reduced and complete scheme over a perfect field k and x is a k-rational point, via Tannakian reconstruction using the category of essentially finite vector bundles on X. The definition of $\pi^N(X, x)$ was extended to the case of connected and reduced k-schemes in [Nor82]. Another analogue, the S-fundamental group scheme $\pi^S(X, x)$ was introduced and studied by Langer in [Lan11] and [Lan12] for smooth projective varieties X over an algebraically closed field k. It is defined via Tannakian reconstruction using the category of numerically flat vector bundles on X. The S-fundamental group scheme for a smooth projective curve C over an algebraically closed field k was already introduced and studied in [BPS06].

The variant of the fundamental group scheme which is of prime importance in this note is the algebraic fundamental group $\pi^{\text{alg}}(X, x)$. In [Gie75], Gieseker defined $\pi^{\text{alg}}(X, x)$ as the fundamental group scheme corresponding to the Tannakian category of $\mathcal{D}_X$-modules, where $\mathcal{D}_X$ is the sheaf of differential operators on X. For X smooth over a field of positive characteristic, Gieseker introduced the notion of *stratified bundles* and showed that the category of $\mathcal{D}_X$-modules is tensor equivalent to the category of stratified bundles on X. Stratifed bundles were further studied in [dS07] and [BHdS21]. Precise definitions and statements are given in the following sections.

2.1 Tannakian formalism

In this section we recall the definition of Tannakian categories and the details of the Tannakian correspondence which we use in the sequel. We begin by recalling the definition of a *tensor category*.

Let C be a category equipped with a functor

$$\otimes : C \times C \to C$$

$$(X, Y) \mapsto X \otimes Y$$

In addition, we ask for functorial isomorphisms

$$\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

and

$$\psi_{X,Y} : X \otimes Y \to Y \otimes X$$

which satisfy some natural compatibility relations - the pentagon axiom and the hexagon axiom respectively (see [DM82], diagrams 1.0.1 and 1.0.2).

A pair (U, u) consisting of an object U of C and an isomorphism $u : U \to U \otimes U$ is an identity object of $(C, \otimes)$ if $X \mapsto X \otimes U : C \to C$ is an equivalence of categories.

Definition 2.1.1. *A collection* $(C, \otimes, \phi, \psi)$ *as above is a* ***tensor category*** *if there exists an identity object* $\mathbb{1}$.

Next we recall the definition of *internal Hom* in a tensor category. Let $(C, \otimes)$ be a tensor category. Consider the functor

$$C^{\text{opp}} \to \text{Set}$$

$$T \mapsto \text{Hom}(T \otimes X, Y)$$

Assume this functor is representable. Then we denote the representing object by $\underline{\text{Hom}}(X, Y)$ and

$$ev_{X,Y} : \underline{\text{Hom}}(X, Y) \otimes X \to Y$$

denotes the morphism corresponding to $\text{id}_{\underline{\text{Hom}}(X,Y)}$. In other words, we have a natural isomorphism, for any object T in C

$$\text{Hom}(T, \underline{\text{Hom}}(X, Y)) \simeq \text{Hom}(T \otimes X, Y)$$

.

Assume that $\underline{\text{Hom}}(X, Y)$ exists for each pair of objects (X, Y) of C. Then there is a composition map

$$\underline{\mathrm{Hom}}(X, Y) \otimes \underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z)$$

and an isomorphism

$$\underline{\mathrm{Hom}}(Z, \underline{\mathrm{Hom}}(X, Y)) \to \underline{\mathrm{Hom}}(Z \otimes X, Y)$$

We define the *dual* of an object X, denoted $X^\vee$, to be $\underline{\mathrm{Hom}}(X, \mathbb{1})$. This comes equipped with a map $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$ inducing a functorial isomorphism

$$\mathrm{Hom}(T, X^\vee) \to \mathrm{Hom}(T \otimes X, \mathbb{1})$$

Using above notations, let $i_X : X \to X^{\vee\vee}$ be the morphism corresponding the composition $\mathrm{ev}_X \circ \psi : X \otimes X^\vee \to \mathbb{1}$. If i_X is an isomorphism, then X is said to be *reflexive*.

The above notions allow us to define the notion of a *rigid* tensor category.

Definition 2.1.2. *A tensor category* $(C, \otimes)$ *is said to be* ***rigid*** *if*

1. $\underline{Hom}(X, Y)$ *exists for all objects X and Y.*

2. *The natural morphisms*

$$\underline{Hom}(X_1, Y_1) \otimes \underline{Hom}(X_2, Y_2) \to \underline{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

   are isomorphisms for all X_1, X_2, Y_1, Y_2

3. *All objects are reflexive.*

We now recall the definition of *tensor functors* between tensor categories and morphisms of tensor functors. Let $(C, \otimes)$ and $(C', \otimes')$ be tensor categories.

Definition 2.1.3. *A* ***tensor functor*** $(C, \otimes) \to (C', \otimes')$ *is a tuple* (F, c) *consisting of a functor* $F : C \to C'$ *and a functorial isomorphism* $c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ *with the following properties*

1. *for all objects* X, Y, Z *of* C*, we have a commutative diagram*

$$\begin{array}{ccccc} FX \otimes (FY \otimes FZ) & \xrightarrow{id \otimes c} & FX \otimes F(Y \otimes Z) & \xrightarrow{c} & F(X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle \phi'} & & & & \downarrow{\scriptstyle F(\phi)} \\ (FX \otimes FY) \otimes FZ & \xrightarrow[c \otimes id]{} & F(X \otimes Y) \otimes FZ & \xrightarrow[c]{} & F((X \otimes Y) \otimes Z) \end{array}$$

2. *for all objects* X, Y *in* C*, we have a commuting square*

$$\begin{array}{ccc} FX \otimes FY & \xrightarrow{c} & F(X \otimes Y) \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle F(\psi)} \\ FY \otimes FX & \xrightarrow[c]{} & F(Y \otimes X) \end{array}$$

3. *If* (U, u) *is an identity objecto of* C*, then* $(F(U), F(u))$ *is an identity object of* C'*.*

A tensor functor $(F, c) : (C, \otimes) \to (C', \otimes')$ *is a* ***tensor equivalence*** *if* $F : C \to C'$ *is an equivalence of categories.*

2.1.1 Neutral Tannakian categories: recovering an affine group scheme from its representations

Let G be an affine group scheme over k and ω be the forgetful functor $\mathrm{Rep}_k(G) \to Vec_k$. For R a k-algebra, $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ consists of tuples (ϕ_X) for $X \in \mathrm{ob}(\mathrm{Rep}_k(G))$, where ϕ_X is a R-linear automorphism of $X \otimes R$ such that

- $\phi_{X_1 \otimes X_2} = \phi_{X_1} \otimes \phi_{X_2}$

- $\phi_{\mathbb{1}} = \mathrm{id}_R$

- For all G-equivariant maps $\alpha : X \to Y$

$$\phi_Y \circ (\alpha \otimes 1) = (\alpha \otimes 1) \circ \phi_X : X \otimes R \to Y \otimes R$$

As any element $g \in G(R)$ defines an element of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$, we get a natural morphism

$$\Psi : G \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$$

The Tannakian reconstruction theory begins with the two following results which allow us to recover the underlying group scheme (and morphisms) from the representation category and compatible functors between them.

Proposition ([DM82], Proposition 2.8)**.** *The natural map Ψ is an isomorphism of functors of k-algebras.*

A morphism of group schemes $f : G \to G'$ defines a tensor functor $\omega^f : \mathrm{Rep}_k(G') \to \mathrm{Rep}_k(G)$ such that $\omega^G \circ \omega^f = \omega^{G'}$.

Proposition. *Let G and G′ be affine group schemes over k and let $F : Rep_k(G') \to Rep_k(G)$ be a tensor functor such that $\omega^G \circ F = \omega^{G'}$. Then there exists a unique homomorphism $f : G \to G'$ such that $F = \omega^f$.*

This leads to the main theorem of Tannakian reconstruction of an affine group scheme

Theorem 2.1.4. *Let $(C, \otimes)$ be a rigid abelian tensor category such that $End(\mathbb{1}) = k$. Let $\omega : C \to Vec_k$ be an exact faithful k-linear tensor functor. Then*

1. *the functor $Aut^{\otimes}(\omega)$ of k-algebras is represented by an affine group scheme G;*

2. *the functor $C \to Rep_k(G)$ defined by ω is an equivalence of tensor categories.*

For a proof of this theorem, see theorem 2.11 in [DM82].

The main theorem justifies the following definition.

Definition 2.1.5. *A rigid abelian tensor category C with $End(\mathbb{1}) = k$ is a **neutral Tannakian category** over k if it admits an exact faithful k-linear tensor functor $\omega : C \to Vec_k$. Any such functor ω is called a **fiber functor** for C.*

The main theorem then says that any neutral Tannakian category is equivalent to the category of finite-dimensional representations of an affine group scheme.

We recall another important result which allows us to read off the properties of a morphism of affine group schemes $f : G \to G'$ from the associated functor $\omega^f : \mathrm{Rep}_k(G') \to \mathrm{Rep}_k(G)$

Proposition 2.1.6. *For f and ω^f as above, we have the following:*

1. *f is faithfully flat if and only if ω^f is fully faithful and every subobject of $\omega^f(X')$, for $X' \in ob(Rep_k(G'))$, is isomorphic to the image of a subobject of X'.*

2. *f is a closed immersion if and only if every object of $Rep_k(G)$ is isomorphic to a subquotient of an object of the form $\omega^f(X')$, for $X' \in ob(Rep_k(G'))$.*

We end this section by recalling another result which is useful in determining the representation of G_{ab}. We recall from [Wat12] , section 10.1 the notion of the derived subgroup $\mathcal{D}G$ which is a closed normal subgroup. There exists a quotient morphism $\alpha : G \to G_{\mathrm{ab}}$, whose kernel is precisely $\mathcal{D}G$. By the definition of $\mathcal{D}G$, G_{ab} is an abelian affine group scheme.

For an integer $n \geq 2$, we denote by G^n the group scheme $G \times \cdots \times G$ (the n-fold product of G with itself). Then the symmetric group $\mathfrak{S}_n$ acts on G^n by permuting the factors. Let f_0 be the following composite group homomorphism

$$G^n \xrightarrow{\alpha^n} G^n_{\mathrm{ab}} \xrightarrow{m} G_{\mathrm{ab}}$$

where m denotes the multiplication homomorphism.

Proposition 2.1.7. *A homomorphism of k-group schemes $f : G^n \to H$ is $\mathfrak{S}_n$-invariant if and only if there is a homomorphism $\overline{f} : G_{ab} \to H$ such that $\overline{f} \circ f_0 = f$.*

For a proof, see lemma 4.1.1 of [PS20].

2.2 Stratified bundles

In the rest of the chapter, k will always be an algebraically closed field of characteristic p and X a noetherian scheme over k. We begin by recalling the definition of the Frobenius endomorphism of X, which plays an important role in positive characteristic geometry.

Given a ring R of characteristic p, we know the morphism $r \mapsto r^p$ defines a ring endomorphism of R called the Frobenius of R, denoted by F_R. Given a ring homomorphism of characteristic p rings $\phi : R \to S$, we have that

$$\phi \circ F_R = F_S \circ \phi$$

It is easy to see that topologically F_R induces the identity map on Spec R. Thus given a scheme X defined over a characteristic p field k, the Frobenius on affine open subsets glue together to give a Frobenius endomorphisms of the scheme X, denoted as $F_X : X \to X$. This is called the absolute Frobenius of X.

Let $X^{(1)}$ denote the pullback of X under the Frobenius F_k on Spec (k). The geometric or relative Frobenius is the induced k-morphism $F : X \to X^{(1)}$. In what follows, we will overlook this and work only with the absolute Frobenius, leaving the reader with the task of checking k-linearity if needed.

Stratified bundles on X are sequences of coherent sheaves on X satisfying infinite Frobenius descent. More precisely, the category of stratified bundles on X, denoted $\mathcal{S}(X)$, consists of

- **Objects** $(\mathcal{E}_i, \alpha_i)$ are sequences of coherent $\mathcal{O}_X$-modules $\mathcal{E}_i$, $i \in \mathbb{N}$ along with isomorphisms

$$\alpha_i : F^*\mathcal{E}_{i+1} \to \mathcal{E}_i$$

for all $i \in \mathbb{N}$, where F is the absolute Frobenius on X.

- **Morphisms** $\phi : (\mathcal{E}_i, \alpha_i) \to (\mathcal{F}_i, \beta_i)$ consists of a sequence of $\mathcal{O}_X$-module morphisms $\phi_i : \mathcal{E}_i \to \mathcal{F}_i$ such that $\phi_i \circ \alpha_i = \beta_i \circ F^*(\phi_{i+1})$

Let $f : Y \to X$ be a morphism and $(\mathcal{E}_i, \alpha_i)$ be a stratified bundle on X. Then we can define the pullback along f, denoted $f^*(\mathcal{E}_i, \alpha_i)$, as consisting of the sequence of $\mathcal{O}_Y$ coherent sheaves $f^*\mathcal{E}_i$ and isomorphisms are given by the composite maps

$$F^* f^* \mathcal{E}_{i+1} \xrightarrow{\gamma_{\mathcal{E}_{i+1}}} f^* F^* \mathcal{E}_{i+1} \xrightarrow{f^*(\alpha_i)} f^* \mathcal{E}_i$$

where $\gamma : F^* f^* \to f^* F^*$ is the natural isomorphism of functors.

Thus $\mathcal{S}(X)$ is contravariant functor in X. One also has a tensor product on $\mathcal{S}(X)$ defined by taking term by term tensor product. We can also define an unit element for this tensor product.

An important result about stratified bundles is the following.

Proposition. *If $(\mathcal{E}_i, \alpha_i)$ is a stratified bundle on X, then $\mathcal{E}_i$ is a locally free $\mathcal{O}_X$-module for all $i \in \mathbb{N}$.*

For a proof, see lemma 6 of [dS07].

This allows us to show that $\mathcal{S}(X)$ has the structure of an abelian category as follows:

Let $(\phi_i) : (\mathcal{E}_i, \alpha_i) \to (\mathcal{F}_i, \beta_i)$ be a morphism in $\mathcal{S}(X)$. Then we can define

$$\mathcal{K}_i := \text{ker}\,(\alpha_i),\ \mathcal{C}_i := \text{coker}\,(\alpha_i),\ \mathcal{I}_i := \text{im}\,(\alpha_i)$$

It is easy to see the $C_\bullet$ is a stratified bundle i.e there exists unique isomorphisms

$$\tau_n : F_X^* C_{i+1} \to C_i$$

which makes the following diagram commute

$$\begin{array}{ccccccc}
F_X^*\mathcal{E}_{i+1} & \xrightarrow{\phi_{i+1}} & F_X^*\mathcal{F}_{i+1} & \longrightarrow & F_X^*C_{i+1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \beta_i} & & \downarrow & & \\
\mathcal{E}_i & \xrightarrow[\phi_i]{} & \mathcal{F}_i & \longrightarrow & C_i & \longrightarrow & 0
\end{array}$$

This implies that C_i is locally free for all $i \in \mathbb{N}$. This together with the fact that each $\mathcal{F}_i$ is locally free implies that $\mathcal{I}_i$ is locally free and hence $\mathcal{K}_i$ is locally free. Hence the following pullback sequence is exact.

$$0 \to F_X^*\mathcal{K}_i \to F_X^*\mathcal{E}_i \to F_X^*\mathcal{F}_i$$

Hence each $\alpha_i : F_X^*\mathcal{E}_{i+1} \to \mathcal{E}_i$ induces an isomorphism $F_X^*\mathcal{K}_{i+1} \to \mathcal{K}_i$.

We can also define duals of stratified bundles, making $\mathcal{S}(X)$ into an *abelian rigid tensor category*.

The *rank* of a stratified bundle $(\mathcal{E}_i, \alpha_i)$ is defined to be the rank of $\mathcal{E}_0$. The *trivial stratified bundles* on X are of the form $\oplus(O_X, ...; F^*, ...)$.

Let $\mathcal{D}_X$ be the sheaf of differential operators on X. The category of $\mathcal{D}_X$ modules consists of

- **Objects** coherent O_X modules $\mathcal{E}$ equipped with a $\mathcal{D}_X$ action i.e a morphism of O_X-algebras

$$\mathcal{D}_X \to \mathcal{E}\mathrm{nd}_k(\mathcal{E})$$

- **Morphisms** $\mathcal{O}_X$-linear maps $\mathcal{E} \to \mathcal{F}$ compatible with the $\mathcal{D}_X$ action

A theorem of Katz [[Gie75], Theorem 1.3] shows that for X smooth over k, then the category of stratified bundles on X and the category of $\mathcal{D}_X$ modules are tensor equivalent to each other.

2.2.1 The group scheme $\pi^{\mathbf{alg}}(X, x)$

Classically, over $\mathbb{C}$, the Riemann-Hilbert correspondence identifies the category of vector bundles equipped with *integrable* or *flat* connections on a smooth connected projective variety $X/\mathbb{C}$ with the category of representations of the topological fundamental group $\pi^{\mathrm{top}}(X, x)$ for some chosen base point x. Via GAGA, this gives a purely algebraic description of the category of representations of the topological fundamental group $\pi(X, x)$. This category (equipped with the fiber functor $(E, \nabla) \to E_x$) is a neutral Tannakian category and can be identified, via the Tannakian formalism, with the representation category of the proalgebraic completion of the topological fundamental group, denoted as $\pi^{\mathrm{top}}(X, x)^{\mathrm{alg}}$.

Over a field k of characteristic 0, the category of flat connections on a smooth variety X is tensor equivalent to the category of $\mathcal{D}_X$-modules. However over a field of characteristic p, the category of flat connections on X is not as well behaved as the category of $\mathcal{D}_X$-modules and one defines a fundamental group scheme for X by Tannakian formalism using the category of $\mathcal{D}_X$-modules. By Katz's theorem mentioned before, the fundamental group scheme coincides with the one defined using $\mathcal{S}(X)$ below.

Let $x \in X(k)$ be a k-rational point. Then the abelian rigid tensor category $\mathcal{S}(X)$ is neutralized by the fiber functor

$$T_x : \mathcal{S}(X) \to Vec_k$$

$$(\mathcal{E}_i, \alpha_i) \mapsto \mathcal{E}_o|_x$$

The fundamental group scheme defined by the neutral Tannakian category $(\mathcal{S}(X), \otimes, T_x, (O_X, F^*))$ is called the algebraic fundamental group of X based at x and is denoted by $\pi^{\mathrm{alg}}(X, x)$.

The following basic properties of π^{alg} are well known and are used in the sequel.

Theorem. *(Independence of basepoint) Let X be a geometrically connected, smooth projective k-scheme. Then for all $x_1, x_2 \in X(k)$, one has*

$$\pi^{alg}(X, x_1) \simeq \pi^{alg}(X, x_2)$$

Theorem. *(Product rule) [[Bis09], Theorem 3.4] For X_1, X_2 geometrically connected and smooth over k and $x_i \in X_i(k)$, there is an isomorphism*

$$\pi^{alg}(X_1, x_1) \times \pi^{alg}(X_2, x_2) \to \pi^{alg}(X_1 \times X_2, (x_1, x_2))$$

Theorem. *(Purity) [[Gie75], Theorem 3.14] For X smooth and an open immersion $U \xrightarrow{i} X$ such that the complement of U in X has codimension ≥ 2 and $x \in U(k)$, then the homomorphism*

$$\pi^{alg}(U, x) \to \pi^{alg}(X, x)$$

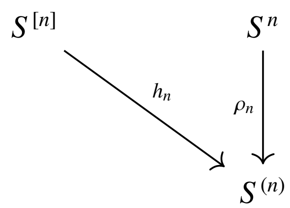
associated to the restriction functor $i^* : \mathcal{S}(X) \to \mathcal{S}(U)$ is an isomorphism.

2.3 The Hilbert scheme of n points on a surface S

Let S be a smooth projective surface over k. We fix notation as follows

- S^n denotes the n-fold cartesian product of S with itself.
- $S^{(n)}$ denotes the nth symmetric product of S defined as the quotient $S^n/\mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the symmetric group on n letters.
- $S^{[n]}$ denotes the Hilbert scheme of n points on S.

Let $\rho : S^n \to S^{(n)}$ be the quotient map and $h : S^{[n]} \to S^{(n)}$ be the Hilbert-Chow morphism. We write $S^{(n)}_\circ$ for the open subset of $S^{(n)}$ consisting of distinct points with $S^{[n]}_\circ := h^{-1}(S^{(n)}_\circ)$ and $S^n_\circ := \rho^{-1}(S^{(n)}_\circ)$. The map $h_{n,\circ} : S^{[n]}_\circ \to S^{(n)}_\circ$ is an isomorphism. We have the diagram:



In general, Hilbert schemes of points on a projective variety display a lot of pathological features - formalized by Murphy's law as in [Vak06]. But in the case of smooth projective surface S, we have the following theorem of Fogarty:

Theorem ([Fog68], Theorem 2.4)**.** *Let S be a nonsingular surface over the field k. Then $S^{[n]}$ is a nonsingular scheme of dimension $2n$*

Thus, in this case, the Hilbert-Chow morphism $h : S^{[n]} \to S^{(n)}$ is a resolution of singularities.

One can consider $S^{(n)}$ as the set of effective 0-cycles of degree n on S. We recall the notion of type of an effective 0-cycle. When y is an effective 0-cycle of degree n, the type of y is a tuple $(n_1, \ldots, n_r)$ where y can be written as

$$y = \Sigma_{j=1}^{r} n_j x_j$$

where x_j are distinct points of S with multiplicities $n_1 \geq n_2 \geq \cdots \geq n_r$, where n_j are positive integers, such that $n = \sum_{j=1}^{n} n_j$. This allows us to stratify $S^{(n)}$ by *type*.

Let $C(n_1, \ldots, n_r)$ denote the subset of $S^{(n)}$ consisting of points of the type $(n_1, \ldots, n_r)$. Let $S_*^{(n)} = C(1, \ldots, 1) \bigcup C(2, 1, \ldots, 1)$ denote the open subset of $S^{(n)}$ consisting of points of type $(1, \ldots, 1)$ and $(2, 1, \ldots, 1)$. Let $S_*^{[n]}$ and S_*^{n} denote the preimage of $S_*^{(n)}$ under h and ρ respectively.

We recall some basic properties below which we will need later (we refer to [Fog68], [PS20] for details).

- The subsets $C(n_1, \ldots, n_r)$ are nonsingular of dimension $2r$.
- The closed subset $S^{(n)} \setminus S_*^{(n)}$ is of codimension ≥ 2 in $S^{(n)}$.
- The closed subset $S^{[n]} \setminus S_*^{[n]}$ is of codimension 2 in $S^{[n]}$.
- The closed subset $S^{n} \setminus S_*^{n}$ is of codimension ≥ 4 in S^{n}.
- The closed subset $S_*^{(n)} \setminus S_\circ^{(n)}$ is of codimension 2 in $S_*^{(n)}$.
- When characteristic of $k \neq 2$, for $y \in C(2, 1, \ldots, 1)$, the scheme theoretic fiber $h^{-1}(y)$ is isomorphic to $\mathbb{P}_k^1$. In fact, $S_*^{[n]}$ is the blowup of $S_*^{(n)}$ along $C(2, 1, \ldots, 1)$.

We end this section by recalling a result of Fogarty ([Fog77], Proposition 3.6).

Proposition. *If L is a $\mathfrak{S}_n$-invariant line bundle on S^{n}, there exists a line bundle L' on $S^{(n)}$ such that $\rho_n^* L' \simeq L$.*

It follows that L' in the proposition is isomorphic to $(\rho_n)_*(L)^{\mathfrak{S}_n}$

2.4 The functor between Tannakian categories

Let S be a smooth projective surface over k and $(\mathcal{E}_i, \alpha_i)$ be a stratified bundle on $S^{[n]}$. Restricting to $S^{[n]}_*$ gives us a functor

$$i^* : \mathcal{S}(S^{[n]}) \to \mathcal{S}(S^{[n]}_*)$$

which is a equivalence of categories as $S^{[n]}_*$ is the complement of a codimension 2 closed subset of $S^{[n]}$.

Next we show that a stratified bundle on $S^{[n]}_*$ can be pushed forward under h to get a stratified bundle on $S^{(n)}_*$. First we begin by a result on descent of vector bundles along the morphism $h : S^{[n]}_* \to S^{(n)}_*$. Similar results have been established by authors in [Ish83] and [PS20].

Proposition 2.4.1. *Assume char $k \neq 2$. Let $\mathcal{E}$ be a vector bundle on $S^{[n]}_*$ which restricts to trivial vector bundles on the fibers of h over $S^{(n)}_*$. Then $h_*\mathcal{E}$ is a locally free $\mathcal{O}_{S^{(n)}_*}$-module. Moreover the natural map*

$$h^*h_*(\mathcal{E}) \to \mathcal{E}$$

is an isomorphism.

Proof. Let $x \in S^{(n)}_*$ be a point of type $(2, 1, \ldots, 1)$. Then by assumption on characteristic and the basic properties from previous section, the fiber of h over x is isomorphic to $\mathbb{P}^1_k$. Let $\mathcal{J}$ be the ideal sheaf of the closed subscheme $h^{-1}(x)$ and $\mathcal{I}_x$ be the ideal sheaf of the closed point x. We have

$$\mathcal{J} = \mathcal{I}_x \mathcal{O}_{S^{[n]}_*}$$

For all $m \geq 1$, let Y_m denote the closed subscheme of $S_*^{[n]}$ corresponding to the ideal sheaf $\mathcal{J}^m$. Consider the following short exact sequence of sheaves on $S_*^{[n]}$

$$0 \to \mathcal{J} \otimes \mathcal{E} \to \mathcal{E} \to \mathcal{E}|_{Y_1} \to 0$$

Pushing forward by h, we get the following exact sequence of sheaves on $S_*^{(n)}$

$$h_*\mathcal{E} \to H^0(Y_1, \mathcal{E}|_{Y_1}) \to R^1h_*(\mathcal{J} \otimes \mathcal{E})$$

We claim that the completion of $R^1h_*(\mathcal{J} \otimes \mathcal{E})$ at the maximal ideal m_x in $O_{S_*^{(n)},x}$ is 0. The proof uses the theorem of formal functions (see Theoreme 4.1.5 in [Gro61]) which says that for h proper, we have,

$$(R^1h_*(\mathcal{J} \otimes \mathcal{E}))^\wedge \simeq \varprojlim H^1(Y_n, \mathcal{J} \otimes \mathcal{E} \otimes O_{S_*^{[n]}}/\mathcal{J}^n)$$

We prove by induction that $H^1(Y_m, \mathcal{J} \otimes \mathcal{E} \otimes O_{S_*^{[n]}}/\mathcal{J}^m) = 0$. As $Y_1 \simeq \mathbb{P}_k^1$, the sheaves $\mathcal{J}^m/\mathcal{J}^{m+1}$ are locally free. These sheaves are also globally generated over Y_1 as we have the surjection

$$\mathfrak{m}_x^m/\mathfrak{m}_x^{m+1} \otimes_{O_{S_*^{(n)},x}} O_{S_*^{[n]}} \simeq \mathcal{I}_x^m/\mathcal{I}_x^{m+1} \otimes_{O_{S_*^{(n)}}} O_{S_*^{[n]}} \twoheadrightarrow \mathcal{J}^m/\mathcal{J}^{m+1}$$

As $\mathcal{J}^m/\mathcal{J}^{m+1}$ is locally free on $Y_1 \simeq \mathbb{P}_k^1$ and globally generated, it is a direct sum of line bundles each of which has degree ≥ 0. Thus one gets the base case of induction from degree considerations, as

$$H^1(Y_1, \mathcal{J} \otimes \mathcal{E} \otimes O_{S_*^{[n]}}/\mathcal{J}) = H^1(Y_1, \mathcal{J}/\mathcal{J}^2 \otimes \mathcal{E}_{Y_1}) = 0$$

Assume that the claim is true for m. Then the proof for $m+1$ follows from the long exact sequence in cohomology attached to the short exact sequence of sheaves on Y_{m+1}

$$0 \to \mathcal{J}^{m+1}/\mathcal{J}^{m+2} \otimes \mathcal{E} \to \mathcal{J}/\mathcal{J}^{m+2} \otimes \mathcal{E} \to \mathcal{J}/\mathcal{J}^{m+1} \otimes \mathcal{E} \to 0$$

which gives us the exact sequence

$$H^1(Y_{m+1}, \mathcal{J}^{m+1}/\mathcal{J}^{m+2} \otimes \mathcal{E}) \to H^1(Y_{m+1}, \mathcal{J}/\mathcal{J}^{m+2} \otimes \mathcal{E}) \to H^1(Y_{m+1}, \mathcal{J}/\mathcal{J}^{m+1} \otimes \mathcal{E})$$

We know $H^1(Y_m, \mathcal{J}^{m+1}/\mathcal{J}^{m+2} \otimes \mathcal{E}) = H^1(Y_1, \mathcal{J}^{m+1}/\mathcal{J}^{m+2} \otimes \mathcal{E}) = 0$ (by degree consideration) and $H^1(Y_{m+1}, \mathcal{J}/\mathcal{J}^{m+1} \otimes \mathcal{E}) = H^1(Y_m, \mathcal{J}/\mathcal{J}^{m+1} \otimes \mathcal{E}) = 0$ (by induction hypothesis), thus we get

$$H^1(Y_{m+1}, \mathcal{J}/\mathcal{J}^{m+2} \otimes \mathcal{E}) = 0$$

Thus the stalk of $R^1 h_*(\mathcal{J} \otimes \mathcal{E})$ at x is 0.

This shows that the natural map $h_*\mathcal{E} \to H^0(Y_1, \mathcal{E}|_{Y_1})$ is surjective in a neighbourhood of x. Let $f_1, ..., f_r$ be a basis of $H^0(Y_1, \mathcal{E}|_{Y_1})$. Let Spec$(R)$ be an affine neighbourhood of x where the natural map is surjective and let $\tilde{f}_i \in \Gamma(\text{Spec}(R), h_*\mathcal{E}) = \Gamma(h^{-1}(\text{Spec}(R)), \mathcal{E})$ be lifts of f_i. Using $\tilde{f}_i$ one defines a homomorphism

$$\mathcal{O}^{\oplus r}_{S^{[n]}_*}|_{h^{-1}(\text{Spec}(R))} \to \mathcal{E}$$

on h^{-1}(Spec(R) which is a surjection (and hence an isomorphism) on Y_1. As h is proper, there exists a smaller affine neighbourhood U of x over which there is an isomorphism

$$\mathcal{O}^{\oplus r}_V \simeq \mathcal{E}$$

where $V = h^{-1}(U)$. Applying h_*, we get

$$(h_*\mathcal{O}_V)^{\oplus r} \simeq h_*\mathcal{E}$$

As $S_*^{(n)}$ is normal and $h : S_*^{[n]} \to S_*^{(n)}$ is birational with connected fibers, by a form of Zariski's main theorem [cf [Har13], Corollary 11.3 and 11.4], we have that $h_*\mathcal{O}_V \simeq \mathcal{O}_U$ and thus $h_*\mathcal{E}$ is locally free. The natural morphism

$$h^*h_*(\mathcal{E}) \to \mathcal{E}$$

is clearly an isomorphism.

□

Let $\mathbb{VB}_{S_*^{(n)}}$ be the category of locally free sheaves on $S_*^{(n)}$ and $\mathbb{VB}^h_{S_*^{[n]}}$ be the category of locally free sheaves on $S_*^{[n]}$ which restrict to trivial vector bundles on the fibers of h. Proposition 1 above gives us an equivalence of categories.

Proposition 2.4.2. *Assume char $k \neq 2$. The pushforward functor*

$$h_* : \mathbb{VB}^h_{S_*^{[n]}} \to \mathbb{VB}_{S_*^{(n)}}$$

is an equivalence of categories with the inverse given by

$$h^* : \mathbb{VB}_{S_*^{(n)}} \to \mathbb{VB}^h_{S_*^{[n]}}$$

Proof. We observe that if $\mathcal{E}' \simeq h^*(\mathcal{E})$, then $\mathcal{E} \simeq h_*\mathcal{E}'$. This shows that h_* is essentially surjective. The natural map

$$\mathrm{Hom}_{S_*^{(n)}}(h_*\mathcal{E}, h_*\mathcal{F}) \to \mathrm{Hom}_{S_*^{[n]}}(\mathcal{E}, \mathcal{F})$$

is bijective. Thus h_* is an equivalence of categories. □

Corollary. *For all $\mathcal{E} \in \mathbb{VB}^h_{S_*^{[n]}}$, the natural map*

$$F^* h_*(\mathcal{E}) \to h_* F^*(\mathcal{E})$$

is an isomorphism over $S_*^{(n)}$.

Proof. As $F^*\mathcal{E}$ is also an object of $\mathbb{VB}^h_{S_*^{[n]}}$, both sheaves are locally free of the same rank. Thus it suffices to show that the natural map

$$F^* h_*(\mathcal{E}) \to h_* F^*(\mathcal{E})$$

is surjective. As F is faithfully flat on the smooth locus of $S_*^{(n)}$ [1], the claim holds on the smooth locus. Let $x \in S_*^{(n)}$ be of type $(2, 1, \ldots, 1)$. Then the restriction of $F^* h_*(\mathcal{E})$ to x is naturally isomorphic to $H^0(Y_1, \mathcal{E}|_{Y_1})$ and the restriction of $h_* F^*(\mathcal{E})$ to x is $H^0(Y_1, F^*(\mathcal{E}|_{Y_1})$. The restriction of the natural map to x is the map

$$F^* : H^0(Y_1, \mathcal{E}_1) \to H^0(Y_1, F^* \mathcal{E}_1)$$

which is surjective. □

By Theorem 2.2 of [Gie75], we have that every stratified bundle on $\mathbb{P}^1_k$ is trivial. Thus the above results give us

Proposition 2.4.3. *Assume char $k \neq 2$. Let $(\mathcal{E}_i, \alpha_i)$ be a stratified bundle on $S_*^{[n]}$. Then $h_*(\mathcal{E}_i)$ is locally free $\mathcal{O}_{S_*^{(n)}}$-module for all $i \in \mathbb{N}$. Moreover the natural map*

$$h^* h_*(\mathcal{E}_i) \to \mathcal{E}_i$$

[1] By a result of Kunz, a ring R of characteristic $p > 0$ is regular if and only if the Frobenius endomorphism is flat.

is an isomorphism. Furthermore the natural map

$$F^*h_*(\mathcal{E}_i) \to h_*F^*(\mathcal{E}_i)$$

is an isomorphism over $S_*^{(n)}$.

This allows us to define the pushforward of a stratified bundle $(\mathcal{E}_i, \alpha_i)$ on $S_*^{[n]}$. The pushforward denoted $h_*(\mathcal{E}_i, \alpha_i)$ is given by the sequence of vector bundles $h_*\mathcal{E}_i$ for all $i \in \mathbb{N}$ and the isomorphisms are given by the composite

$$F^*h_*(\mathcal{E}_{i+1}) \xrightarrow{\eta_{\mathcal{E}_{i+1}}} h_*F^*(\mathcal{E}_{i+1}) \xrightarrow{h_*(\alpha_i)} h_*(\mathcal{E}_i)$$

where $\eta : F^*h_* \to h_*F^*$ is the natural transformation.

Thus we get a functor

$$h_* : \mathcal{S}(S_*^{[n]}) \to \mathcal{S}(S_*^{(n)})$$

On the smooth locus $S_\circ^{(n)}$ we have the isomorphisms

$$h_*((\mathcal{E}_i, \alpha_i) \oplus (\mathcal{F}_i, \beta_i))|_{S_\circ^{(n)}} \simeq h_*(\mathcal{E}_i, \alpha_i)|_{S_\circ^{(n)}} \oplus h_*(\mathcal{F}_i, \beta_i))|_{S_\circ^{(n)}}$$

$$h_*((\mathcal{E}_i, \alpha_i) \otimes (\mathcal{F}_i, \beta_i))|_{S_\circ^{(n)}} \simeq h_*(\mathcal{E}_i, \alpha_i)|_{S_\circ^{(n)}} \otimes h_*(\mathcal{F}_i, \beta_i))|_{S_\circ^{(n)}}$$

which extend to $S_*^{(n)}$ due to codimension reasons (as the complement of $S_\circ^{(n)}$ in $S_*^{(n)}$ has codimension ≥ 2. Thus h_* is additive tensor functor.

The following commutative diagram shows that $h^*h_*(\mathcal{E}_i, \alpha_i)$ is isomorphic to $(\mathcal{E}_i, \alpha_i)$ as stratified bundles with the isomorphism given by the natural morphisms $h^*h_*\mathcal{E}_i \to \mathcal{E}_i$.

$$
\begin{CD}
F^*h^*h_*\mathcal{E}_{i+1} @>{\gamma_{h_*\mathcal{E}_{i+1}}}>> h^*F^*h_*\mathcal{E}_{i+1} @>{h^*\eta_{\mathcal{E}_{i+1}}}>> h^*h_*F^*\mathcal{E}_{i+1} @>{h_*h^*\alpha_i}>{\simeq}> h^*h_*\mathcal{E}_i \\
@V{\simeq}VV @. @V{\simeq}VV @V{\simeq}VV \\
F^*\mathcal{E}_{i+1} @>{=}>> @>>> F^*\mathcal{E}_{i+1} @>{\alpha_i}>{\simeq}> \mathcal{E}_i
\end{CD}
$$

Consider the pullback functor

$$\rho^* : \mathcal{S}(S_*^{(n)}) \to \mathcal{S}(S_*^n)$$

which takes values in the category of $\mathfrak{S}_n$-equivariant stratified bundles on S_*^n. Also we have the extension functor

$$j_* : \mathcal{S}(S_*^n) \to \mathcal{S}(S^n)$$

which is an equivalence of categories, due to codimension reasons. Composing these functors together, we get a functor

$$T : \mathcal{S}(S^{[n]}) \to \mathcal{S}(S^n)$$

given by

$$T = j_* \circ \rho^* \circ h_* \circ i^*$$

Clearly T is an additive tensor functor. Note that h_* is fully faithful, $\rho^* : \mathcal{S}(S_*^{(n)}) \to \mathcal{S}(S_*^n)$ is fully faithful (as $\rho : S_\circ^n \to S_\circ^{(n)}$ is finite étale) and $j_* : \mathcal{S}(S_*^n) \to \mathcal{S}(S^n)$ is an equivalence of categories (due to codimension reasons). Thus T is fully faithful.

2.4.1 The homomorphism

Fix n distinct k-valued points $x_1, \ldots, x_k \in S(k)$. Let $\tilde{x} \in S^{[n]}$ such that $h(\tilde{x}) = \rho_n(x_1, \ldots, x_n) = z \in S^{(n)}_\circ$. Then the categories $\mathcal{S}(S^{[n]})$ and $\mathcal{S}(S^n)$ are neutralized by the respective fiber functors

$$\tau_{\tilde{x}} : \mathcal{S}(S^{[n]}) \to \mathrm{Vec}_k$$

$$(\mathcal{E}_i, \alpha_i) \mapsto (\mathcal{E}_0)_{\tilde{x}}$$

$$\tau_{(x_1,\ldots,x_n)} : \mathcal{S}(S^n) \to \mathrm{Vec}_k$$

$$(\mathcal{F}_i, \beta_i) \mapsto (\mathcal{F}_0)_{(x_1,\ldots,x_n)}$$

If $T((\mathcal{E}_i, \alpha_i)) = (\mathcal{F}_i, \beta_i)$ that we have natural isomorphisms $(\mathcal{E}_0)_{\tilde{x}} \simeq h_*(\mathcal{E}_0)_z \simeq (\mathcal{F}_0)_{(x_1,\ldots,x_n)}$.

Thus we have a functor of Tannakian categories

$$T : (\mathcal{S}(S^{[n]}), \otimes, \tau_{\tilde{x}}, (O_{S^{[n]}}, d)) \to (\mathcal{S}(S^n), \otimes, \tau_{(x_1,\ldots,x_n)}, (O_{S^n}, d))$$

which by the independence of basepoint property of $\mathcal{S}$ induces a functor of Tannakian categories

$$T : (\mathcal{S}(S^{[n]}), \otimes, \tau_{\tilde{nx}}, (O_{S^{[n]}}, d)) \to (\mathcal{S}(S^n), \otimes, \tau_{(x,\ldots,x)}, (O_{S^n}, d))$$

and hence a morphisms of the associated fundamental group schemes

$$\tilde{f} : \pi^{\mathrm{alg}}(S^n, (x, \ldots, x)) \to \pi^{\mathrm{alg}}(S^{[n]}, \tilde{nx})$$

Note that, by the product rule as mentioned in section 2.2.1, we have

$$\pi^{\mathrm{alg}}(S^n, (x, \ldots, x)) \simeq \pi^{\mathrm{alg}}(S, x)^n$$

.

As

$$T : (\mathcal{S}(S^{[n]}), \otimes, T_{\tilde{nx}}, (\mathcal{O}_{S^{[n]}}, d)) \to (\mathcal{S}(S^n), \otimes, T_{(x,\ldots,x)}, (\mathcal{O}_{S^n}, d))$$

takes stratified bundles on $S^{[n]}$ to $\mathfrak{S}_n$-equivariant stratified bundles on S^n and a $\mathfrak{S}_n$-equivariant stratified bundles on S^n corresponds to a $\mathfrak{S}_n$-invariant representation of $\pi^{\mathrm{alg}}(S, x)^n$, by 2.1.7, $\tilde{f}$ factors uniquely through

$$f : \pi^{\mathrm{alg}}(S, x)_{\mathrm{ab}} \to \pi^{\mathrm{alg}}(S^{[n]}, \tilde{nx})$$

2.5 Isomorphism of fundamental group schemes

In this section, we show that f is an isomorphism of affine group schemes. We begin by proving a result about $\mathfrak{S}_n$-equivariant stratified line bundles on S^n.

Proposition 2.5.1. *Let (L_i, α_i) be a $\mathfrak{S}_n$-equivariant stratified line bundles on S^n. Then there exists a stratified line bundle $(\mathcal{L}_i, \beta_i)$ such that $\rho^*(\mathcal{L}_i, \beta_i) \simeq (L_i, \alpha_i)$*

Proof. By Fogarty's result mentioned above, for any $\mathfrak{S}_n$-equivariant line bundle L_i there exists line bundle $\mathcal{L}_i \simeq \rho_* L_i^{\mathfrak{S}_n}$ such that $\rho^* \mathcal{L}_i \simeq L_i$. Pushing forward α_i and taking $\mathfrak{S}_n$ invariants we get the isomorphism

$$\rho_*(F^* L_{i+1})^{\mathfrak{S}_n} \xrightarrow{\rho_*(\alpha_i)^{\mathfrak{S}_n}} \rho_*(L_i)^{\mathfrak{S}_n}$$

We show that the natural homomorphism

$$F^*(\rho_*(L_i)^{\mathfrak{S}_n}) \to (F^*\rho_*(L_i))^{\mathfrak{S}_n} \to (\rho_* F^*(L_i))^{\mathfrak{S}_n}$$

is an isomorphism. Pulling back the morphism under ρ, we get the commutative diagram

$$\begin{array}{ccc} \rho^* F^*((\rho_* L_i)^{\mathfrak{S}_n}) & \longrightarrow & \rho^*((\rho_* F^* L_i)^{\mathfrak{S}_n}) \\ \downarrow & & \downarrow \\ F^* L_i & \xrightarrow{=} & F^* L_i \end{array}$$

where the vertical morphisms are the natural morphisms, which are isomorphisms by Fogarty's theorem. By pushing forward the top isomorphism under ρ and taking $\mathfrak{S}_n$ invariants we get that

$$F^*(\rho_*(L_i)^{\mathfrak{S}_n}) \to (\rho_* F^*(L_i)^{\mathfrak{S}_n})$$

is an isomorphism. We define β_i to be the composite isomorphism

$$F^*(\rho_*(L_i)^{\mathfrak{S}_n}) \to (\rho_* F^*(L_i)^{\mathfrak{S}_n}) \xrightarrow{\rho_*(\alpha_i)^{\mathfrak{S}_n}} \rho_*(L_i)^{\mathfrak{S}_n}$$

The commutative diagram also gives us that $\rho^*(\mathcal{L}_i, \beta_i) \simeq (L_i, \alpha_i)$ □

2.5.1 Faithfully flat

Next we show that the morphism f is faithfully flat

Proposition 2.5.2. *The homomorphism*

$$f : \pi^{alg}(S, x)_{ab} \to \pi^{alg}(S^{[n]}, \tilde{nx})$$

is faithfully flat.

Proof. By [[DM82] Theorem 2.21], this is equivalent to showing that the functor

$$T : \mathcal{S}(S^{[n]}) \to \mathcal{S}(S^n)$$

is fully faithful and the essential image of T is closed under taking subobjects. We already know that T is fully faithful. Let $\mathcal{E}_\bullet = (\mathcal{E}_i, \alpha_i)$ be a stratified bundle on $S^{[n]}$ and $\mathcal{F}_\bullet := T(\mathcal{E}_\bullet)$ be the corresponding $\mathfrak{S}_n$-equivariant stratified bundle on S^n. If $\mathcal{F}'_\bullet \subset \mathcal{F}_\bullet$ is a $\mathfrak{S}_n$-equivariant stratified subbundle, then we need to show there exists $\mathcal{E}'_\bullet \subset \mathcal{E}_\bullet$ such that $T(\mathcal{E}'_\bullet) = \mathcal{F}'_\bullet$.

The proof proceeds by induction on the rank of $\mathcal{E}_\bullet$. If rank $\mathcal{E}_\bullet = 1$, the proof is immediate. Let rank $\mathcal{E}_\bullet \geq 2$

Then the stratified bundles $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ correspond to the representations

$$\pi^{\text{alg}}(S^n, (x, \ldots, x) \to \pi^{\text{alg}}(S, x)_{\text{ab}} \to GL(V)$$

and

$$\pi^{\text{alg}}(S^n, (x, \ldots, x) \to \pi^{\text{alg}}(S, x)_{\text{ab}} \to GL(V')$$

respectively.

As $\pi^{\text{alg}}(S, x)_{\text{ab}}$ is an abelian affine group scheme over k, all its irreducible representations are one dimensional. Thus one gets that the $\pi^{\text{alg}}(S, x)_{\text{ab}}$-module V/V' has a one dimensional quotient W. Thus there is a $\pi^{\text{alg}}(S, x)_{\text{ab}}$-module surjection $V \to W$ such that the kernel contains V'. Let $\mathcal{L}_\bullet$ be the $\mathfrak{S}_n$-equivariant stratified bundle corresponding to W. Thus we have a short exact sequence of $\mathfrak{S}_n$-equivariant stratified bundles

$$0 \to \mathcal{K}_\bullet \to \mathcal{F}_\bullet \to \mathcal{L}_\bullet \to 0$$

where $\mathcal{F}'_\bullet \subset \mathcal{K}_\bullet$.

By proposition 1 above, we know that $L_i := \rho_* \mathcal{L}_i^{\mathfrak{S}_n}$ is a line bundle on $S^{(n)}$ and $\rho^* L_i = \mathcal{L}_i$

We claim that the following complex of sheaves on $S_*^{(n)}$ is exact for all $i \in \mathbb{N}$

$$0 \to (\rho_* \mathcal{K}_i)^{\mathfrak{S}_n}|_{S_*^{(n)}} \to (\rho_* \mathcal{F}_i)^{\mathfrak{S}_n}|_{S_*^{(n)}} \to (\rho_* \mathcal{L}_i)^{\mathfrak{S}_n}|_{S_*^{(n)}} \to 0 \tag{2.1}$$

It is enough to show that $(\rho_* \mathcal{F}_i)^{\mathfrak{S}_n}|_{S_*^{(n)}} \to (\rho_* \mathcal{L}_i)^{\mathfrak{S}_n}|_{S_*^{(n)}}$ is surjective. We note that $(\rho_* \mathcal{F}_i)^{\mathfrak{S}_n}|_{S_*^{(n)}} = h_*(\mathcal{E}_i|_{S_*^{[n]}})$. Let C be the cokernel

$$h_*(\mathcal{E}_i|_{S_*^{[n]}}) \to (\rho_* \mathcal{L}_i)^{\mathfrak{S}_n}|_{S_*^{(n)}} \to C \to 0$$

Pulling back under ρ, we get the following commutative diagram on S_*^n

$$\begin{array}{ccccccc}
\rho^* h_*(\mathcal{E}_i|_{S_*^{[n]}}) & \longrightarrow & \rho^*((\rho_* \mathcal{L}_i)^{\mathfrak{S}_n}|_{S_*^{(n)}}) & \longrightarrow & \rho^* C & \longrightarrow & 0 \\
\downarrow = & & \downarrow = & & & & \\
\mathcal{F}_i & \longrightarrow & \mathcal{L}_i|_{S_*^n} & \longrightarrow & 0 & &
\end{array}$$

The rows are exact and hence $\rho^* C = 0$. As ρ is surjective, this implies $C = 0$. Thus $K_i := (\rho_* \mathcal{K}_i)^{\mathfrak{S}_n}|_{S_*^{(n)}}$ is locally free on $S_*^{(n)}$

Pulling back the exact sequence (2.1) under h, we get a short exact sequence of locally free sheaves on $S_*^{[n]}$

$$0 \to h^* K_i|_{S_*^{[n]}} \to \mathcal{E}_i|_{S_*^{[n]}} \to \tilde{L}_i|_{S_*^{[n]}} \to 0$$

where $\tilde{L}_i := h^*L_i$.

As the complement of $S^{[n]}_*$ in $S^{[n]}$ is of codimension ≥ 2 and $\mathcal{E}_i, L$ are locally free, the surjective morphism

$$\mathcal{E}_i|_{S^{[n]}_*} \to \tilde{L}_i|_{S^{[n]}_*}$$

extends to a unique morphism $\tau_i : \mathcal{E}_i \to \tilde{L}_i$. This is surjective as L is of rank 1 and $\tau := (\tau_i)$ give a nonzero morphism of stratified bundles

$$\mathcal{E}_\bullet \to \tilde{L}_\bullet$$

where $\tilde{L}_\bullet := h^*(\rho_*(\mathcal{L}_\bullet)^{\mathfrak{S}_n})$. Let $\kappa_\bullet$ be the kernel of the morphism $\mathcal{E}_\bullet \to \tilde{L}_\bullet$. Then $T(\kappa_\bullet) = \mathcal{K}_\bullet$. Thus, by the induction hypothesis on rank, there exists a stratified subbundle $\mathcal{E}'_\bullet \subset \kappa_\bullet \subset \mathcal{E}_\bullet$ such that $T(\mathcal{E}'_\bullet) = \mathcal{F}'_\bullet$.

□

2.5.2 Closed immersion

We begin by recalling a result from [PS20].

Let $p \in S^{(n)}$ be a point of type $(n_1, n_2, \ldots, n_r)$. Let p'_i, for $i = 1, 2, \ldots m$ be the points in the fiber $h^{-1}(p)$. Let A be the local ring $\mathcal{O}_{S^{(n)},p}$ and B be the semilocal ring $\mathcal{O}_{S^n} \otimes_{\mathcal{O}_{S^{(n)}}} A$. Then B is a finite A module and $B^{\mathfrak{S}_n} = A$.

Lemma 2.5.3. *When char $k > n_1$, any $\mathfrak{S}_n$-equivariant surjective B-module homomorphism $f : M \to N$ of finitely generated B modules descends to surjective A-module homomorphism of the $\mathfrak{S}_n$-invariants $M^{\mathfrak{S}_n} \to N^{\mathfrak{S}_n}$*

This allows us to prove the following analogue of Proposition 5.3.6 in [PS20].

Proposition. *Let $\mathcal{E}_\bullet = (\mathcal{E}_i, \alpha_i)$ be a $\mathfrak{S}_n$-equivariant stratified bundle on S^n*

1. *Let $p \in S^{(n)}$ be a point of type $(n_1, n_2, \dots, n_r)$. If char $k > n_1$, then the sheaf $\rho_* \mathcal{E}_i^{\mathfrak{S}_n}$ is locally free in a neighbourhood of p for all i.*
2. *Let U denote the largest open subset where $\rho_* \mathcal{E}_i^{\mathfrak{S}_n}$ is locally free, then on $\rho^{-1}(U)$, the natural morphism*

$$\rho^* \rho_* \mathcal{E}_i^{\mathfrak{S}_n} \to \mathcal{E}_i$$

is an isomorphism for all $i \in \mathbb{N}$

Proof. The first assertion is proved by induction on the rank of $\mathcal{E}_\bullet$. If $\mathcal{E}_\bullet$ is a $\mathfrak{S}_n$-equivariant stratified bundle of rank 1, then by proposition 1, $\rho_* \mathcal{E}_i^{\mathfrak{S}_n}$ is locally free on $S^{(n)}$ for all i. In general, as $\mathcal{E}_\bullet$ corresponds to a representation of the abelian group scheme $\pi^{\text{alg}}(S, x)_{\text{ab}}$, there exists a $\mathfrak{S}_n$-equivariant short exact sequence of locally free sheaves on S^n

$$0 \to \mathcal{K}_\bullet \to \mathcal{E}_\bullet \to \mathcal{L}_\bullet \to 0$$

Pushing forward by ρ and taking $\mathfrak{S}_n$-invariants we get the exact sequence for all i

$$0 \to \rho(\mathcal{K}_i)^{\mathfrak{S}_n} \to \rho(\mathcal{E}_i)^{\mathfrak{S}_n} \to \rho(\mathcal{L}_i)^{\mathfrak{S}_n}$$

We claim that the homomorphism on the right is surjective in the neighbourhood of a point p of type $(n_1, n_2, \dots, n_r)$. Surjectivity can be checked after passing to a formal neighbourhood of p and thus reduces to lemma 2.5.3. By induction hypothesis on rank, both $\rho(\mathcal{K}_i)^{\mathfrak{S}_n}$ and $\rho(\mathcal{L}_i)^{\mathfrak{S}_n}$ are locally free on a neighbourhood of p and hence so is $\rho(\mathcal{E}_i)^{\mathfrak{S}_n}$.

The second assertion follows from the observation that the natural homomorphism

$$\rho^*\rho_*\mathcal{E}_i^{\mathfrak{S}_n} \to \mathcal{E}_i$$

is an isomorphism on $\rho^{-1}(S_\circ^{(n)})$ as as $\rho : S_\circ^n \to S_\circ^{(n)}$ is finite étale. As the complement of $S_\circ^n$ in $\rho^{-1}(U)$ is of codimension ≥ 2 and both sheaves are locally free on $\rho^{-1}(U)$, thus the natural morphism is an isomorphism. □

Proposition 2.5.4. *Let char $k > 3$. The homomorphism*

$$f : \pi^{alg}(S, x)_{ab} \to \pi^{alg}(S^{[n]}, \tilde{nx})$$

is faithfully flat.

Proof. By [[DM82], Theorem 2.21], it is enough to show that the functor

$$T : \mathcal{S}(S^{[n]}) \to \mathcal{S}(S^n)$$

is essentially surjective. Thus we want to show that for any $\mathfrak{S}_n$-equivariant stratified bundle $\mathcal{E}_\bullet$ on S^n, there exists a stratified bundle $\mathcal{F}_\bullet$ on $S^{[n]}$ such that $T(\mathcal{F}_\bullet) = \mathcal{E}_\bullet$.

Let U be the open subset of $S^{(n)}$ consisting of points of type $(1, 1, \ldots, 1), (2, 1, \ldots, 1), (3, 1, \ldots, 1)$ and $(2, 1, 1, \ldots, 1)$. By assumption on characteristic of k and the previous proposition, we get that $\rho_*\mathcal{E}_i^{\mathfrak{S}_n}$ is locally free on U. Also we have on $\rho^{-1}(U)$, the natural morphism

$$\rho^*\rho_*\mathcal{E}_i^{\mathfrak{S}_n} \to \mathcal{E}_i$$

is an isomorphism. Imitating proposition 1 above, this allows us to define a stratified bundle $(\rho_*\mathcal{E}_i^{\mathfrak{S}_n}, \beta_i)$ on U such that $\rho^*(\rho_*\mathcal{E}_i^{\mathfrak{S}_n}, \beta_i) \simeq \mathcal{E}_\bullet$. Pulling back under h to $h^{-1}(U)$ (whose complement in $S^{[n]}$ has codimension ≥ 3) and extending to $S^{[n]}$, we get a stratified bundle $\mathcal{F}_\bullet$ such that $T(\mathcal{F}_\bullet) = \mathcal{E}_\bullet$ □

As f is both faithfully flat and a closed immersion, we get the following theorem

Theorem 2.5.5. *Let char $k > 3$. The homomorphism*

$$f : \pi^{alg}(S, x)_{ab} \to \pi^{alg}(S^{[n]}, \tilde{nx})$$

is an isomorphism.

Remark. *This theorem is analogous to the results in [Bea83], [BH15] and [PS20] where the authors compute the topological fundamental group, the etale fundamental group, the Nori fundamental group scheme and the S-fundamental group scheme of the Hilbert scheme of n points on S.*

Remark. *If we considered the case of integrable connections on $S^{[n]}$ when $k = \mathbb{C}$, then the Beauville's result computing the fundamental group of $S^{[n]}$, together with the Riemann-Hilbert correspondence, shows that the category of integrable connections on $S^{[n]}$ is tensor equivalent to the representation category of $\pi(S, x)_{ab}$, where $\pi(S, s)$ is the topological fundamental group of S.*

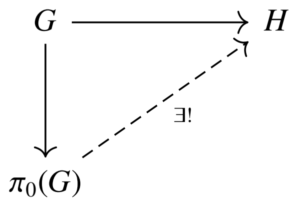
2.6 Relation with the etale fundamental group of $S^{[n]}$

The main theorem above allows us to compute the etale fundamental group of $S^{[n]}$. We begin by recalling the notion of connected components of an affine group scheme.

Let G be an affine group scheme and $A = k[G]$ be its ring of co-ordinate functions. Let B be the largest separable subalgebra of A. We denote Spec (B) by $\pi_0(G)$. From [Wat12], section 6.7, one notes that the Hopf algebra structure on A restricts to a Hopf algebra structure on B. Thus, in this case, $\pi_0(G)$ is also an affine group scheme and the canonical morphism

$$G \to \pi_0(G)$$

is a morphism of affine group schemes. This satisfies an universal property that any morphism from G to an étale group scheme H has a unique factorization as follows -



Recall that G_{ab} denotes the abelianization of the group scheme G. The following lemma follows directly from the universal properties of $\pi_0(G)$ and G_{ab}.

Lemma 2.6.1. *There is a canonical isomorphism*

$$\pi_0(G_{ab}) \xrightarrow{\sim} \pi_0(G)_{ab}$$

Next we recall a result of dos Santos.

Proposition ([dS07], Proposition 13)**.** *There is a natural quotient homomorphism of group schemes*

$$\mu : \pi^{\text{alg}}(X, x) \to \pi^{\text{et}}(X, x)$$

which identifies $\pi_0(\pi^{\text{alg}}(X, x))$ *with* $\pi^{\text{et}}(X, x)$.

Here $\pi^{\text{et}}(X, x)$ is considered as a constant group scheme.

This allows us to recover the computation of the étale fundamental group of $S^{[n]}$ as in [BH15].

Proposition 2.6.2. *Let S be a smooth projective surface over an algebraically closed field of characteristic $p > 3$ and $x \in S(k)$. Then we have a canonical isomorphism*

$$\pi^{\acute{e}t}(S, x)_{ab} \to \pi^{\acute{e}t}((S^{[n]}, \tilde{nx})$$

Chapter 3

Criterion for rationality of moduli of chains

3.1 Introduction

Let X be a compact Riemann surface of genus ≥ 2. A *chain* on X is a tuple $(E_0, \ldots, E_n; \phi_1, \ldots, \phi_n)$ where E_i are vector bundles on X and $\phi_i : E_i \to E_{i-1}$ are morphisms between vector bundles. In [ÁCGP01], a concept of stability dependent on n real parameters was introduced (for the definition see section 3.2.2). For $\theta \in \mathbb{R}^{n+1}$ and type $\underline{t}$, Schmitt constructed the moduli space of θ-stable chains of type $\underline{t}$ denoted $M^s_\theta(\underline{t})$, as a smooth quasi-projective variety via GIT in [Sch03]. In [ACGPS06] the extensions and deformations of chains and the variation of the moduli spaces with respect to the stability parameters were studied.

The study of chains began with the study of solutions to gauge-theoretic equations on X called vortex equations which are obtained by dimensional reduction of the Hermitian-Einstein equations in 4 dimensions (see, for instance, [ÁCGP01] and [ACGP03]). Moduli of chains also arise in the study of moduli spaces of representations of the fundamental group of X. It was shown that these moduli spaces are diffeomorphic to moduli spaces of semistable Higgs bundles on X with fixed invariants n and d denoted $H^{ss}_{n,d}$. There is a natural $\mathbb{G}_m$-action on the moduli space $H^{ss}_{n,d}$. The fixed loci for this action on $H^{ss}_{n,d}$ are moduli spaces of chains that are semistable with respect to certain stability parameters. This observation has been used to study the topology of the moduli space by several authors (for instance, in [BGPG03]). Recently, this relation between $H^{ss}_{n,d}$ and the moduli spaces of θ-semistable chains on X has allowed the computation of the motive of $H^{ss}_{n,d}$ in [GPHS14] and [HL21].

In this chapter, we establish a sufficient criteria for the rationality of the moduli spaces of chains $M^s_\theta(\underline{r}, \underline{L})$, which are stable with respect to a parameter θ. We begin by recalling some preliminaries on projective bundles. Then we revisit the definition and properties of the moduli space of θ-stable chains of a fixed type $\underline{t}$ - we will consider both the fixed degree and the fixed determinant variants of these spaces. Of particular importance is the result in [BGPGH18] which establishes necessary and sufficient conditions for the non-emptiness and irreducibility of the moduli stack $\mathcal{M}^s_\theta(\underline{t})$ and the coarse moduli space $M^s_\theta(\underline{t})$. These conditions allow us to establish our criterion in the sequel.

Then we study the related moduli space of chains of type $\underline{t}$ whose components are stable vector bundles and morphisms are non-zero. We denote this moduli space by $N^s(\underline{t})$. Under certain assumptions on $\underline{t}$, one can show that $N^s(\underline{t})$ can be explicitly described as an iterated projective bundle over the product of moduli spaces of stable vector bundles $\prod_{i=0}^n M^s(r_i, d_i)$. This is done in section 3.

In the final section, we derive the above-mentioned criterion for the rationality of $M^s_\theta(\underline{r}, \underline{L})$ in the case when $\underline{n} = (n, n, \ldots, n)$, $d_{i-1} - d_i \geq 0$ for all i and any stability parameter $\underline{\theta}$ satisfying $\theta_i - \theta_{i-1} > d_{i-1} - d_i$ for all i.

3.2 Preliminaries

3.2.1 Preliminaries on projective bundles and rationality results

In this section we recall the definition of $\mathbb{P}^n$-bundles on a variety X.

Definition 3.2.1. *A morphism $f : X \to Y$ is said to be a $\mathbb{P}^n$-bundle if f is flat, proper and the scheme theoretic fiber $f^{-1}(y) \simeq \mathbb{P}^n$ for all $y \in Y$*

One way of constructing $\mathbb{P}^n$ bundles over X is to take projectivizations $\mathbb{P}(\mathcal{E})$ of locally free sheaves $\mathcal{E}$ of rank $n+1$.

It is well-known that given a $\mathbb{P}^n$ bundles $f : X \to Y$ and $y \in Y$, there exists an etale neighbourhood $\rho : V \to Y$ of y such that $V \times_Y X \simeq V \times \mathbb{P}^n$. In other words, any $\mathbb{P}^n$-bundle is étale locally trivial.

In particular, there exist examples of $\mathbb{P}^n$-bundles which are not Zariski-locally trivial. It is easy to show that the Zariski-locally trivial $\mathbb{P}^n$-bundles are exactly the projectivized vector bundles $\mathbb{P}(\mathcal{E})$.

Of particular interest to us is the following well known result.

Proposition. *On a smooth rational variety X, any $\mathbb{P}^n$-bundle is Zariski locally trivial.*

Remark 3.2.2. *An easy but useful observation is that for any $\mathbb{P}^n$-bundle $X \to Y$ with Y irreducible and rational, X is also rational. This follows from the Zariski-local triviality of $X \to Y$. Say U is a trivializing open subset of Y, then $X_U \simeq U \times \mathbb{P}^n$. By shrinking U if necessary, we can identify U with an open subset of $\mathbb{P}^k$, which gives rationality of X.*

3.2.2 Preliminaries on chains

For the convenience of the reader, we recall the definition of chains, the notion of their stability, and state some known results of their moduli spaces.

Chains: Let X be a compact Riemann surface of genus ≥ 2. A $(n+1)-$chain over X is a following diagram

$$(E_\bullet, \phi_\bullet): \quad E_n \xrightarrow{\phi_n} E_{n-1} \xrightarrow{\phi_{n-1}} ...E_2 \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} E_0,$$

where E_i are vector bundle on X and ϕ_j are morphism of vector bundles, for each $i = 0, ..., n$ and $j = 1, ..., n$. If $(E_\bullet, \phi_\bullet)$ and $(E'_\bullet, \phi'_\bullet)$ are two holomorhic chains over X then a *morphism from* $(E_\bullet, \phi_\bullet)$ *to* $(E'_\bullet, \phi'_\bullet)$ is a family $f = (f_i)_{i=0,...,n}$ of vector bundle morphisms $f_i : E_i \to E'_i$ such that $f_{i-1} \circ \phi_i = \phi'_i \circ f_i$, for every $i = 1, ..., n$.

Further, if each $f_i : E_i \to E'_i$ is an isomorphism then we say that the family $f = (f_i)_{i=o,...,n}$ is an isomorphism $(E_\bullet, \phi_\bullet)$ to $(E'_\bullet, \phi'_\bullet)$. A *subchain* of $(E_\bullet, \phi_\bullet)$ is another chain $(E'_\bullet, \phi'_\bullet)$ such that E'_i is a *subbundle* of E_i, $\phi(E'_j) \subset E'_{j-1}$ and ϕ'_j is the restriction of ϕ_j, for each $i = 0, ..., n$ and for all $j = 1, ..., n$. If E is a vector bundle on X, we write rk E for its rank and $\deg(E)$ for its degree.

Moduli space of chains: Now, fix a weight $\theta = (\theta_i)_{i=0,...,n}$ in $\mathbb{R}^{n+1}$. For a non trivial chain $(E_\bullet, \phi_\bullet)$, we define the θ-degree as

$$\deg_\theta(E_\bullet, \phi_\bullet) = \sum_{i=0}^{n} (\deg(E_i) + \theta_i \text{ rk } E_i),$$

and rk $E_\bullet, \phi_\bullet = \sum_{i=0}^{n} \text{rk } E_i$.

The real number

$$\mu_\theta(E_\bullet, \phi_\bullet) = \frac{\deg_\theta(E_\bullet, \phi_\bullet)}{\text{rk } E_\bullet, \phi_\bullet},$$

is called the θ-slope of $(E_\bullet, \phi_\bullet)$.

A chain $(E_\bullet, \phi_\bullet)$ is called θ-semistable (respectively, θ-stable) if

$$\mu_\theta(F_\bullet, \psi_\bullet) \leq \mu_\theta(E_\bullet, \phi_\bullet) \quad \text{(respectively, } \mu_\theta(F_\bullet, \psi_\bullet) < \mu_\theta(E_\bullet, \phi_\bullet)\text{)}$$

for every non-trivial proper *subchain* $(F_\bullet, \psi_\bullet)$ of $(E_\bullet, \phi_\bullet)$. We say that a chain is θ-polystable if it is θ-semistable, and is isomorphic to the direct sum of a finite family of θ-stable chains.

For a chain $(E_\bullet, \phi_\bullet)$, its type $\underline{t}((E_\bullet, \phi_\bullet))$ is the pair $((rk(E_i)), (deg(E_i)))$. For a fixed weight $\theta \in \mathbb{R}^{n+1}$ and a type $\underline{t} = (\underline{r}, \underline{d}) \in \mathbb{N}^{n+1} \times \mathbb{Z}^{n+1}$, we denote $M^s_\theta(\underline{t})$ the moduli space of θ-stable chains of type $\underline{t}$. The dimension of the moduli space $M^s_\theta(\underline{t})$ at a smooth point $(E_\bullet, \phi_\bullet)$ is given by $(g-1)(\sum_{i=0}^n r_i^2 - \sum_{i=1}^n r_i r_{i-1}) + \sum_{i=1}^n (r_i d_{i-1} - r_{i-1} d_i) + 1$ (see [ACGPS06], Theorem 3.8).

Conditions for smoothness and irreducibility of $M^s_\theta(\underline{t})$ We recall from [ACGPS06] that a sufficient condition for smoothness of the moduli space.

Theorem 3.2.3 ([ACGPS06], Theorem 3.8)**.** *If $\theta_i - \theta_{i-1} \geq 2g-2$, for all $i = 1, ..., n$, then $M^s_\theta(\underline{t})$ is smooth.*

Now, we recall the necessary and sufficient conditions for irreducibility and non-emptiness of the moduli space from [BGPGH18]. For a chain $(E_\bullet, \phi_\bullet)$, we define the following natural subchains which we call the standard subchains of $(E_\bullet, \phi_\bullet)$.

(1)

$$E'^{,\geq k}_\bullet : \quad 0 \to ... \to 0 \to E_k \to ... \to E_0,$$

for $0 \leq k < n$.

(2)

$$E'^{,[k,j]}_\bullet : \quad E_n \to ... \to E_{j+1} \to E_j = ... = E_j \to E_{k-1} \to E_0,$$

for $0 \leq k < j \leq n$ such that $r_j < min\{r_k, ..., r_{j-1}\}$.

Dually the following quotient chains which we call the standard quotient chains.

(3)

$$E''^{,\leq k}_{\bullet}: \quad E_n \to ... \to E_k \to 0 \to ... \to 0,$$

for $0 \leq k < n$.

(4)

$$E''^{,[k,j]}_{\bullet}: \quad E_n \to ... \to E_{j+1} \to E_k = ... = E_k \to E_{k-1} \to E_0,$$

for $0 \leq k < j \leq n$ such that $r_k < min\{r_{k+1}, ..., r_j\}$.

Let us recall the necessary conditions for the existence of θ-semistable chains of type $\underline{t} = (\underline{r}, \underline{d})$.

C_0 : For all i such that $r_i = r_{i-1}$, we have

$$d_i \leq d_{i-1}.$$

C_1 : For all $0 \leq k < n$, we have

$$\mu_\theta(E'^{,\geq k}_{\bullet}) \leq \mu_\theta((E_\bullet, \phi_\bullet)).$$

C_2 : For all $0 \leq k < j \leq n$ such that $n_j < min\{n_k, ..., n_{j-1}\}$, we have

$$\mu_\theta(E'^{,[k,j]}_{\bullet}) \leq \mu_\theta((E_\bullet, \phi_\bullet)).$$

C_3 : For all $0 \leq k < j \leq n$ such that $n_k < min\{n_{k+1}, ..., n_j\}$, we have

$$\mu_\theta(E''^{,[k,j]}_{\bullet}) \geq \mu_\theta((E_\bullet, \phi_\bullet)).$$

For $(\underline{r}, \underline{d})$ satisfying condition C_0, we define

$$\text{Stability}_{\underline{r}}^{\underline{d}} := \{\theta \in \mathbb{R}^n : \theta > \theta_{Higgs} \textit{ and } \theta \textit{ satisfies } C_1 - C_3\}.$$

Remark 3.2.4. *We are using abuse of notion in calling chains in* (2) – (4) *the natural subchains as the canonical morphism from those chains to* $(E_\bullet, \phi_\bullet)$ *need not be injective. In [BGPGH18], it was shown that semistable chains of type* $(\underline{r}, \underline{d})$ *can exist only if the slopes of the standard subchains bounded above by that slope of* $(E_\bullet, \phi_\bullet)$*. Thus the conditions* $C_0 - C_3$ *depend only on the types and the weights of the chains.*

Theorem 3.2.5 ([BGPGH18], Theorem 3.2)**.** *Assume that* $g \geq 1$*. Let* $\underline{t} = (\underline{r}, \underline{d}) \in \mathbb{N}^{n+1} \times \mathbb{Z}^{n+1}$ *and* $\theta \in \mathbb{R}^n_{>\theta_{Higgs}}$*. Then the moduli stack* $\mathcal{M}^s_\theta(\underline{t})$ *is irreducible and non-empty if and only if* $(\underline{r}, \underline{d})$ *satisfy condition* C_0 *and* $\theta \in \textit{Stability}_{\underline{r}}^{\underline{d}}$*.*

Remark 3.2.6. *For the case of constant rank i.e when* $\underline{r} = (r, r, \ldots, r)$*, the conditions* C_2 *and* C_3 *are vacuous.*

We conclude this section by recalling the definition of a non-critical stability parameter θ. θ is said to be non-critical for type $\underline{t}$ if there exists no properly θ-semistable chains of type $\underline{t}$, or equivalently, all θ-semistable chains of type $\underline{t}$ are θ-stable. In the rest of the paper, we assume $\underline{\theta}$ is always non-critical.

3.3 Moduli spaces of chains with stable components

Let X be a compact Riemann surface of genus ≥ 2. We fix $\theta = (\theta_0, ..., \theta_n) \in \mathbb{R}^{n+1}$ and $\underline{t} = (r_0, ..., r_n; d_0, ..., d_n)$ such that $r_i \geq 2$.

For clarity we recall our notation

- $\mathcal{M}^{\theta-ss}(\underline{r}, \underline{d})$ denoted the moduli stack of θ-semistable chains of type $\underline{t} = (\underline{r}, \underline{d})$.
- $M^{\theta-ss}(\underline{r}, \underline{d})$ denoted the coarse moduli space of θ-semistable chains of type $\underline{t} = (\underline{r}, \underline{d})$.

In what follows we will also consider the fixed determinant analogues of these spaces. Let $\underline{L}$ be a $r+1$-tuple of line bundles on X. Then we define

- $\mathcal{M}^{\theta-ss}(\underline{r}, \underline{L})$ denoted the moduli stack of θ-semistable chains of type $\underline{t} = (\underline{r}, \underline{L})$.
- $M^{\theta-ss}(\underline{r}, \underline{L})$ denoted the coarse moduli space of θ-semistable chains of type $\underline{t} = (\underline{r}, \underline{L})$.

Here by saying that a chain $(E_0, \ldots, E_n; \phi_1, \ldots, \phi_n)$ has type $\underline{t} = (\underline{r}, \underline{L})$ we mean the component vector bundles E_i has rank r_i and determinant L_i.

We begin with the following useful observation about chains whose component vector bundles are stable.

Lemma 3.3.1. *Let $C = (E_0, ..., E_n; \phi_1, ..., \phi_n)$ and $C' = (E_0, ..., E_n; \psi_1, ..., \psi_n)$ be two chains of type $\underline{t}$. Assume that each E_i is stable. Then, C is isomorphic to C' if and only if there exists $\lambda_i \in \mathbb{C}^*$ such that $\psi_i = \lambda_i \phi_i$ for all $i = 1, ..., n$.*

Proof. This follows from the observation that the automorphism group $\mathrm{Aut}(E_i)$ is isomorphic to $\mathbb{C}^*$, for each $i = 0, ..., n$ as E_i are stable. □

We also recall the following well-known fact about the non-vanishing of the vector bundle morphisms in a θ-stable chain.

Lemma 3.3.2. *If $C = (E_0, ..., E_n; \phi_1, ..., \phi_n)$ is a θ-stable chain of type $\underline{t}$ then $\phi_i \neq 0$ for all $i = 1, ..., n$.*

Proof. Suppose that C is a θ-stable chain of type $\underline{t}$. If there exists j such that $\phi_j = 0$, then the quotient $C/C_j = (0, ..., 0, E_j, ..., E_n; 0, ..., \phi_{j+1}, ..., \phi_n)$ is also a subchain of C, and $\mu_\theta(C/C_j) < \mu_\theta(C)$. Since C is θ-stable, we have $\mu_\theta(C) < \mu_\theta(C/C_j)$. This is absurd. Hence, the result follows. □

Now we recall the definition of the projective Picard bundle $\mathcal{PW}$ (see for example [BBPN09]) which is a $\mathbb{P}^N$-bundle defined over the open subset of the fixed determinant moduli space

$M^s(r, L)$ consisting of vector bundles E such that $H^1(X, E) = 0$. The fiber of $\mathcal{PW}$ over any point E is $\mathbb{P}(H^0(X, E))$.

Let us fix $n+1$ line bundles L_i such that $d_i = \deg L_i$. If we assume $r_i d_{i-1} - r_{i-1} d_i > r_{i-1} r_i (2g-2)$, then the projective Picard bundle $\mathcal{PW}_i$ is defined over the whole of $M^s(r_{i-1} r_i, L_{i-1}^{\otimes r_i} \otimes L_i^{\otimes - r_{i-1}})$ (refer to [BBPN09] for details).

Consider the morphisms (for $1 \leq i \leq n$)

$$\rho_i : M^s(r_0, L_0) \times ... \times M^s(r_n, L_n) \to M^s(r_{i-1} r_i, L_{i-1}^{\otimes r_i} \otimes L_i^{\otimes - r_{i-1}})$$

$$(E_0, ..., E_n) \mapsto E_{i-1} \otimes E_i^*$$

.

By construction, the fiber of the fiber product $\mathcal{P} := \prod_{i=1}^n \mathcal{P}_i$ over a point $(E_0, ..., E_n)$ in $M^s(r_0, L_0) \times ... \times M^s(r_n, L_n)$ is the product $\prod_{i=1}^n \mathbb{P}(\mathrm{Hom}(E_i, E_{i-1}))$. By lemma 3.3.1 the iterated projective bundle $\mathcal{P}$ parametrizes chains of type t with each component E_i stable.

Let $N^s(\underline{r}, \underline{L})$ be the moduli space of chains $(E_0, ..., E_n; \phi_1, ..., \phi_n)$ of type $(\underline{r}, \underline{L})$ such that each underlying vector bundle E_i is stable and each $\phi_i \neq 0$. When $r_i d_{i-1} - r_{i-1} d_i > r_{i-1} r_i (2g-2)$ for all i, the forgetful morphism

$$N^s(\underline{r}, \underline{L}) \to M^s(r_0, L_0) \times ... \times M^s(r_n, L_n)$$

allows us to identify $N^s(\underline{r}, \underline{L})$ with an iterated projective bundle over $M^s(r_0, L_0) \times ... \times M^s(r_n, L_n)$.

The above observations can be summarized in the following proposition.

Proposition 3.3.3. *Let* $\underline{t} = (r_0, ..., r_n; L_0, ..., L_n)$ *and* $d_i = deg(L_i)$ *Assume* $r_i d_{i-1} - r_{i-1} d_i >$

$r_{i-1}r_i(2g-2)$ for all $1 \le i \le n$. Then we have an isomorphism over $M^s(r_0, L_0) \times ... \times M^s(r_n, L_n)$

$$N^s(\underline{r}, \underline{L}) \simeq \mathcal{P}$$

Remark 3.3.4. *The analogue of this proposition also holds for the fixed degree variant $N^s(\underline{r}, \underline{d})$ of the moduli space. This will be used in the next section. It is also easy to see that the dimension of $N^s(\underline{r}, \underline{d})$ agrees with the dimension of $M^s_\theta(\underline{r}, \underline{d})$.*

Remark 3.3.5. *By a theorem of Hirschowitz and Russo-Teixidor (see [Hof10], Theorem A.10), one can show that under the weaker assumption that $r_i d_{i-1} - r_{i-1} d_i > r_{i-1} r_i (g-1)$ for all i, $N^s(\underline{r}, \underline{L})$ is birational to a $\mathbb{P}^n$-bundle defined over an open subset of $M^s(r_0, L_0) \times ... \times M^s(r_n, L_n)$. However for our purpose, we will work with the stronger assumptions.*

Remark 3.3.6. *If $(r_i, d_i) = 1$ for all i, then $M^s(r_i, L_i)$ are rational smooth projective varieties, by [KS99]. Thus, in this case, $\mathcal{P}_i$ is Zariski locally trivial on $M^s(r_0, L_0) \times ... \times M^s(r_n, L_n)$ which shows that $\mathcal{P}_i$ is rational. Iterating we get that, under the coprimality assumptions, $\mathcal{P}$ is rational.*

Let $N^s_\theta(\underline{r}, \underline{d}) \subset M^s_\theta(\underline{r}, \underline{d})$ be the open subset of the moduli space of θ-stable chains $(E_0, ..., E_n; \phi_1, ..., \phi_n)$ of type $(\underline{r}, \underline{d})$ such that each underlying vector bundle E_i is also stable. Lemma 3.3.2 allows us to identify $N^s_\theta(\underline{r}, \underline{d})$ with an open subset of $N^s(\underline{r}, \underline{d})$. Thus we have open immersions

$$N^s(\underline{r}, \underline{d}) \hookleftarrow N^s_\theta(\underline{r}, \underline{d}) \hookrightarrow M^s_\theta(\underline{r}, \underline{d})$$

As $N^s(\underline{r}, \underline{d})$ and $M^s_\theta(\underline{r}, \underline{d})$ are irreducible under our assumptions, we have the following proposition.

Proposition 3.3.7. *Let $\underline{t} = (\underline{r}, \underline{d}) \in \mathbb{N}^{n+1} \times \mathbb{Z}^{n+1}$ and $\theta \in \mathbb{R}^n_{>\theta_{Higgs}}$. Assume $(\underline{r}, \underline{d})$ satisfy $r_i d_{i-1} - r_{i-1} d_i > r_{i-1} r_i (2g-2)$ for all $1 \le i \le n$ and condition C_0. Let $\theta \in Stability^{\underline{d}}_{\underline{r}}$ and $(r_i, d_i) = 1$ for all i. If $N^s_\theta(\underline{t})$ is non-empty, then $M^s_\theta(\underline{t})$ is birational to $N^s(\underline{t})$.*

Remark 3.3.8. *We use the above propostion to derive a criteria for the rationality of the fixed determinant moduli of chains i.e for $M^s_\theta(\underline{r}, \underline{L})$*

3.4 Non-emptiness of $N^s_\theta(\underline{r}, \underline{d})$

In this section we establish the non-emptiness of $N^s_\theta(\underline{r}, \underline{d})$ when $\underline{r} = (n, n, \ldots, n)$, $d_{i-1} - d_i \geq 0$ for all i and any stability parameter $\underline{\theta}$ satisfying $\theta_i - \theta_{i-1} > d_{i-1} - d_i$ for all i

We work with moduli stacks instead of coarse moduli spaces in this section. We begin by recalling the definition and properties of the stack of Hecke correspondences and the moduli stack of injective chains from [GPHS14] and [HL21].

3.4.1 The stack of Hecke correspondences

Let $l \in \mathbb{N}$ and $\mathcal{E}$ be a family of rank n and degree d vector bundles over X parametrized by an algebraic stack $\mathcal{T}$, we define a category fibered in groupoids $\mathcal{H}ecke^l_{\mathcal{E}/\mathcal{T}}$ over Sch/k as follows. For every $S \in$ Sch/k, the groupoid is defined as

$$\mathcal{H}ecke^l_{\mathcal{E}/\mathcal{T}}(S) = \left\{ g \in \mathcal{T}(S), \phi : \mathcal{F} \hookrightarrow (g \times id_X)^*\mathcal{E} : \begin{array}{c} \mathcal{F} \to S \times X \text{ vector bundle} \\ rk(\mathcal{F}) = n, \deg(\mathcal{F}) = d - l, rk(\phi) = n \end{array} \right\}$$

Given a morphism $f : S' \to S$ in $/k$, a morphism over f from $(g', \phi') \in \mathcal{H}ecke^l_{\mathcal{E}/\mathcal{T}}(S')$ to $(g, \phi) \in \mathcal{H}ecke^l_{\mathcal{E}/\mathcal{T}}(S)$ is a morphism $\alpha : g' \to g \circ f$ in $\mathcal{T}(S')$ and an isomorphism $(f \times id_X)^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}'$ which fits into the diagram

$$\begin{array}{ccc} (f \times id_X)^*\mathcal{F} & \hookrightarrow & (f \times id_X)^*(g \times id_X)^*\mathcal{E} \\ \downarrow \sim & & \alpha^* \downarrow \sim \\ \mathcal{F}' & \hookrightarrow & (g' \times id_X)^*\mathcal{E}. \end{array}$$

This stack $\mathcal{H}ecke^l_{\mathcal{E}/\mathcal{T}}$ is referred to as the stack of length l Hecke correspondences of $\mathcal{E}$. There is a natural morphism $\mathcal{H}ecke^l_{\mathcal{E}/\mathcal{T}} \to \mathcal{T}$ which basechanges functorially in $\mathcal{E}/\mathcal{T}$. This shows that the natural morphism is relatively representable, smooth and projective as when T is a scheme,

$\mathcal{H}ecke^l_{\mathcal{E}/T}$ is representable by the Quot scheme

$$\mathrm{Quot}^{(0,l)}_{T\times X/T}(\mathcal{E})$$

which parametrizes families of quotients of $\mathcal{E}$ of rank 0 and degree l, which is smooth and projective over T. Thus $\mathcal{H}ecke^l_{\mathcal{E}/\mathcal{T}}$ is an algebraic stack.

For $\mathcal{T} = \mathcal{B}un_{n,d}$ and $\mathcal{E} = \mathcal{U}$ the universal bundle of rank n and degree d on $\mathcal{B}un_{n,d} \times X$, there is a natural morphism

$$gr : \mathcal{H}ecke^l_{\mathcal{U}/\mathcal{B}un_{n,d}} \to \mathcal{B}un(n, d-l) \times Coh(0.l)$$

which sends $(\mathcal{F} \hookrightarrow (g \times id_X)^*\mathcal{U} \twoheadrightarrow Q)$ to $(\mathcal{F}, Q)$. Here $Coh(n, d)$ denotes the stack classifying coherent sheaves of rank n and degree d on X.

This allows us to identify $\mathcal{H}ecke^l_{\mathcal{U}/\mathcal{B}un_{n,d}}$ with an open substack of

$$\underline{\mathrm{Ext}}((n, d-l), (0, l))$$

the stack classifying extensions $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ of coherent sheaves with $(rk(\mathcal{F}'), deg(\mathcal{F}')) = (n, d-l)$ and $(rk(\mathcal{F}''), deg(\mathcal{F}'')) = (n, d)$. We get the following commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^l_{\mathcal{U}/\mathcal{B}un(n,d)} & \hookrightarrow & \underline{\mathrm{Ext}}((n, d-l), (0, l)) \\
\downarrow{\scriptstyle gr} & & \downarrow{\scriptstyle gr} \\
\mathcal{B}un(n, d-l) \times Coh(0, l) & \hookrightarrow & Coh(n, d-l) \times Coh(0, l)
\end{array}$$

It is known that $gr : \underline{\mathrm{Ext}}((n, d-l), (0, l)) \to Coh(n, d-l) \times Coh(0, l)$ is a vector bundle stack (see

Corollary 3.2 in [GPHS14]). This allows us to conclude that $gr : \mathcal{H}ecke^{l}_{\mathcal{U}/\mathcal{B}un_{n,d}} \to \mathcal{B}un(n, d - l) \times Coh(0.l)$ is smooth.

As X is a curve, $Coh(n, d)$ is a smooth algebraic stack over k. Thus the forgetful morphism $\mathcal{H}ecke^{l}_{\mathcal{U}/\mathcal{B}un_{n,d}} \to \mathcal{B}un(n, d - l)$ is a smooth morphism.

3.4.2 The stack of injective chains

For $\underline{r} = (r_0, r_1, \ldots, r_s) \in \mathbb{N}^{s+1}$ and $\underline{d} \in \mathbb{Z}^{s+1}$, let $Ch^{\text{inj}}(\underline{r}, \underline{d})$ be the stack classifying chains $(E_s \to E_{s-1} \to \cdots \to E_0)$ of rank $\underline{r}$ and degree $\underline{d}$ such that $E_i \to E_{i-1}$ is injective for all i. This is an algebraic stack as it can be identified with an open substack of the algebraic stack $Ch(\underline{r}, \underline{d})$ classifying all chains of rank $\underline{r}$ and degree $\underline{d}$.

In the case when $\underline{r} = (n, n, \ldots, n)$ for a fixed $n \in \mathbb{N}$ and $d_{i-1} - d_i \geq 0$ for all i, $Ch^{\text{inj}}(\underline{r}, \underline{d})$ is an iterated Hecke correspondence stack over $\mathcal{B}un(n_0, d_0)$. This is well known (see for example Proposition 5.4 in [HL21]). We recall the details for completeness.

Let $\underline{r}_{\leq i} := (n_0, \ldots, n_i)$ and $\underline{d}_{\leq i} := (d_0, \ldots, d_i)$. Let $Ch^{\text{inj}}_{\leq i} := Ch^{\text{inj}}(\underline{r}_{\leq i}, \underline{d}_{\leq i})$ and

$$\mathcal{U}^{i}_{\leq i} \to \cdots \to \mathcal{U}^{0}_{\leq i}$$

be the universal chain over $Ch^{\text{inj}}_{\leq i}$. There are forgetful morphisms

$$Ch^{\text{inj}}_{\leq i} \to Ch^{\text{inj}}_{\leq i-1}$$

which identifies $Ch^{\text{inj}}_{\leq i}$ with $\mathcal{H}ecke^{l_i}(\mathcal{U}^{i-1}_{\leq i-1}/Ch^{\text{inj}}_{\leq i-1})$. This shows that the forgetful morphisms (for $0 \leq i \leq s$)

$$Ch^{\text{inj}}(\underline{r}, \underline{d}) \to \mathcal{B}un(n_i, d_i)$$

are smooth as the same holds for the forgetful morphisms on the stack of Hecke correspondences. Thus $Ch^{\text{inj}}_{\leq i}$ contains a dense open substack of chains $(E_s \to E_{s-1} \to \cdots \to E_0)$ where each E_i is a stable vector bundle.

In [GPHS14], proposition 6.9, the authors showed that for $\underline{r} = (n, n, \ldots, n)$, $d_{i-1} - d_i \geq 0$ for all i and any stability parameter $\underline{\theta}$ satisfying $\theta_i - \theta_{i-1} > d_{i-1} - d_i$, we have that

$$Ch^{\underline{\theta}\text{-ss}}(\underline{r}, \underline{d}) \subset Ch^{\text{inj}}(\underline{r}, \underline{d})$$

As the forgetful morphisms,

$$Ch^{\text{inj}}(\underline{r}, \underline{d}) \to \mathcal{B}un(n_i, d_i)$$

are smooth, we see that, in this case, $Ch^{\underline{\theta}\text{-ss}}(\underline{r}, \underline{d})$ contains a dense open substack of $\underline{\theta}$-semistable chains such that each vector bundle is stable.

Remark 3.4.1. *This observation is used by the authors in [BGPGH18] (Theorem 3.2) to show non-emptiness of $Ch^{\underline{\theta}\text{-}ss}(\underline{r}, \underline{d})$ in above case.*

Thus, under these assumptions on $\underline{t}$ and $\underline{\theta}$, $N^s_{\underline{\theta}}(\underline{r}, \underline{d})$ is non-empty, which gives us the following.

Theorem 3.4.2. *Let $\underline{t} = (\underline{r}, \underline{d})$ satisfy $\underline{r} = (n, n, \ldots, n)$ and $d_{i-1} - d_i > n(2g-2)$ for all i and $\underline{\theta}$ be any stability parameter satisfying $\theta_i - \theta_{i-1} > d_{i-1} - d_i$. Then $M^s_{\underline{\theta}}(\underline{t})$ is birational to $N^s(\underline{t})$, which is an iterated projective bundle over $M^s(n, d_0) \times \ldots \times M^s(n, d_n)$.*

Remark 3.4.3. *We note that for $n > 0$, the assumptions in above proposition already imply condition C_0 and $\underline{\theta} > \underline{\theta}_H$. Furthermore, for $\underline{r} = (n, n, \ldots, n)$, C_1 is the only nontrivial condition in theorem 3.2.5. Thus above assumptions already give us non-emptiness and irreducibility of $M^s_{\underline{\theta}}(\underline{t})$.*

The fixed determinant case

In what follows we assume that $(n, d_i) = 1$ for all i.

To specialize to the fixed determinant case, we note that under the assumptions $(n, d_i) = 1$, the determinant morphism $Bun(n, d_i) \to Pic^{d_i}(X)$ is smooth. In particular the morphism

$$Ch^{\underline{\theta}\text{-ss}}(\underline{r}, \underline{d}) \to Bun(n, d_i) \to Pic^{d_i}(X)$$

is open. Thus for generic choice of L_i (for all i), we get that the coarse moduli space $N^s_\theta(\underline{n}, \underline{L})$ is nonempty. Thus we have

$$N^s(\underline{r}, \underline{L}) \hookleftarrow N^s_\theta(\underline{r}, \underline{L}) \hookrightarrow M^s_\theta(\underline{r}, \underline{L})$$

This allows us to formulate following rationality criterion.

Corollary 3.4.4. *Let $\underline{t} = (\underline{r}, \underline{d})$ satisfy*

- $\underline{r} = (n, n, \ldots, n)$.
- $d_{i-1} - d_i > n(2g-2)$ *for all i*
- $\underline{\theta}$ *be any stability parameter satisfying $\theta_i - \theta_{i-1} > d_{i-1} - d_i$.*

Then for generic choice of $\underline{L}$, $M^{\theta-ss}(\underline{n}, \underline{L})$ contains an irreducible component which is a rational variety. In particular, the irreducibility of $M^{\theta-ss}(\underline{n}, \underline{L})$ implies rationality of $M^{\theta-ss}(\underline{n}, \underline{L})$.

Proof. Under above assumptions, $N^s(\underline{r}, \underline{L})$ is rational as it is an iterated $\mathbb{P}^N$-bundle over a rational variety as in 3.3.6. It is also irreducible being a $\mathbb{P}^N$-bundle over an irreducible variety. Thus the closure of $N^s_\theta(\underline{r}, \underline{L})$ in $M^s_\theta(\underline{r}, \underline{L})$ is a irreducible rational variety. By 3.3.4, the dimension of this rational subvariety is the same as the dimension of $M^s_\theta(\underline{r}, \underline{L})$. Thus this is a non-empty irreducible component in $M^s_\theta(\underline{r}, \underline{L})$. The second claim follows immediately. □

Remark 3.4.5. *This further motivates the question if the analogue of the irreducibility result in [BGPGH18] holds for the fixed determinant case i.e for $M^{\theta-ss}(\underline{n}, \underline{L})$.*

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