

Light Bending Under General Relativity: A Study of Pulsar–Black Hole Binaries

By

Jyotijwal Debnath

PHYS10201804005

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

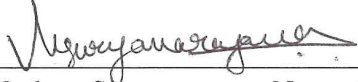
HOMI BHABHA NATIONAL INSTITUTE



Homi Bhabha National Institute

Recommendations of the Viva Voce Committee

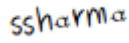
As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Jyotijwal Debnath titled "Light Bending Under General Relativity: A Study of Pulsar-Black Hole Binaries" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.


Chairman - Venkata Suryanarayana Nemani
Date: 7/01/2026


Guide/Convenor - Manjari Bagchi
Date: 7/01/2026


Examiner - Manoneeta Chakraborty
Date: 7/01/2026


Member 1 - D. Indumathi
Date: 7/01/2026


Member 2 - Sayantan Sharma
Date: 7/01/2026


Member 3 - Pinaki Chaudhuri
Date: 7/01/2026


Member 4 - K. G. Arun
Date: 7/01/2026

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I hereby certify that I have read this thesis prepared under my direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: 7/01/2026

Place: Chennai



Guide

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.



Jyotijwal Debnath

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me.
The work is original and has not been submitted earlier as a whole or in part for a degree
/ diploma at this or any other Institution / University.

Jyotijwal Debnath

Jyotijwal Debnath

CERTIFICATION ON ACADEMIC INTEGRITY

Undertaking by the Student

1. I, **Jyotijwal Debnath**, HBNI Enrolment No. **PHYS10201804005**, hereby undertake that the thesis titled "**Light Bending Under General Relativity: A Study of Pulsar-Black Hole Binaries**" has been prepared by me and is my original work.
2. I also hereby undertake that this document has been duly checked using a plagiarism detection tool and found to be plagiarism-free as per the guidelines of the Institute/UGC.
3. I am aware and undertake that if plagiarism is detected in my thesis at any stage in the future, appropriate penalties will be imposed as per the guidelines of the Institute/UGC.

Jyotijwal Debnath 07/01/2026

Signature of the Student with Date

Endorsed by the Thesis Supervisor:

I certify that the thesis written by the student is plagiarism-free, as affirmed above.

Signature of the Thesis Supervisor with Date *Manjari Bagchi* 7/01/2026

Name : Manjari Bagchi
 Designation : Professor-H
 Department/Centre : Theoretical Physics
 Name of CI/OCC : The Institute of Mathematical Sciences (IMSc)

List of Publications Arising From the Thesis:

- **Published**

1. A study of the light bending phenomenon under full general relativity for a pulsar in a binary with a Schwarzschild black hole.

Jyotijwal Debnath, Manjari Bagchi and Avishek Basu, Monthly Notices of the Royal Astronomical Society, October 2023, Volume 524, Pages 5411–5431, <https://doi.org/10.1093/mnras/stad2147>

2. On the effect of light bending phenomenon for a pulsar in a binary with a Kerr black hole.

Jyotijwal Debnath and Majnari Bagchi, Volume 541, Issue 2, August 2025, Pages 821–846, <https://doi.org/10.1093/mnras/staf974>

List of Publications Not Arising From the Thesis:

1. Multiband extension of the wideband timing technique

A. K. Paladi et al. Monthly Notices of the Royal Astronomical Society, January 2024, Volume 527, Pages 213–231, <https://doi.org/10.1093/mnras/stad3122>

2. Improving DM estimates using low-frequency scatter-broadening estimates

J. Singha et al. Monthly Notices of the Royal Astronomical Society, November 2024, Volume 535, Issue 1, Pages 1184–1192, <https://doi.org/10.1093/mnras/stae2405>

List of Presentations and Participations at Conferences

• Contributed presentations

1. Presented poster on *Light bending phenomenon under full general relativity for a pulsar in a binary with a Schwarzschild black hole* in conference “National Conference on REcent Trends in the study of Compact Objects (RETCO-V)” (3-5 April) at Kodaikanal Solar Observatory, India.
2. Gave a talk on *Light bending phenomenon under full general relativity for a pulsar in a binary with a Schwarzschild black hole* in international conference “10th International Conference on Gravitation and Cosmology: New Horizons and Singularities in Gravity (ICGC 2023)” at IIT Guwahati (6 Dec to 9 Dec 2023), India.
3. Gave a talk on *Light bending phenomenon under full general relativity for a pulsar in a binary with a Schwarzschild black hole* in conference “Exploring the Cosmos: A Conference on Relativistic Universe, 2024” (7 to 8 March 2024) at North Bengal University, Siliguri, India.

• Conferences attended

1. “Neutron Stars: The celestial clocks that probe extreme physics” (1-8 Feb 2023) IMSc, Chennai, India.
2. “Gravitational waves meet effective field theories” (20 - 23 August 2023) Benasque, Spain.

To

My late grandfather, whose memory continues to inspire me.

ACKNOWLEDGEMENTS

This thesis reflects not only the academic journey I have undertaken over the past few years but also the memorable experiences and meaningful moments I shared at Matscience. I am truly grateful for the opportunity to be part of such a stimulating and supportive environment, where I was able to carry out my doctoral research.

I would like to express my heartfelt gratitude to my supervisor, Prof. Manjari Bagchi, for her unwavering support, guidance, and constant encouragement throughout this journey. Her profound understanding of the subject and her thoughtful suggestions have played a pivotal role in shaping both the direction and the quality of this thesis. I feel fortunate to have had her as my advisor and mentor, and I remain deeply thankful for the confidence she instilled in me and the impact she has had on my academic development.

I also wish to thank the members of my doctoral committee — Prof. Venkata Suryanarayana Nemani, Prof. Manjari Bagchi, Prof. D. Indumathi, Prof. Sayantan Sharma, Prof. Pinaki Chaudhuri, and Prof. K. G. Arun — for their valuable feedback and constructive suggestions. Their continued support and encouragement have been vital throughout my academic journey.

I am grateful to Prof. Maura McLaughlin, Prof. A. Gopakumar, and Prof. Bhal Chandra Joshi for their insightful discussions and valuable suggestions, which greatly enriched the progress and depth of my research.

I also thank Dr. Avishek Basu for the early discussions that helped shape the research problems and motivated me to pursue this field. I am especially thankful for his collaborative efforts and contributions to the research work.

I would like to express my appreciation to the administrative staff at IMSc, particularly Ms. R. Indra, Ms. Prema, Ms. Geetha M. Johnson, Mr. Shankar K. P., Ms. Usha Othreeswaran, and Mr. R. Baskaran. Their prompt assistance ensured that any administrative

issues I faced were quickly and smoothly resolved. I am also very grateful to the technical staff, especially Mr. G. Srinivasan, for their timely technical support.

The years I spent at Matscience were made truly memorable by the many wonderful friends I was fortunate to meet. I am especially thankful to Sushovan, Apurba, Tanmay Saha, Tanmoy Bera, Manas, Ankur, Vinod, Sidhu, Anupam da, Hitesh, Tanmoy Sengupta, Sabiar da, Pranendu da, Sujoy da, Soumya da, Pritam da, Rupam da, Sahil da, Sunil, Satish, Akhil, Toshali, Subashri, Himangshu, Prateek, Prabhat, Nishant, Gopal, Pavan, Arpan da, Pratik da, Srimoy da, Abhranil da, Debabrata da, Mrinal da, Ratan da, Shaswata da, Sarbartha, Swagatam, Sayak, Sumit Shaw, Aritra, Sourav, Arup, Vishwa-jeet, Sashikanta, Pavitra, Piyasa, Sadhana, and many others whose names I may have unintentionally missed.

Thank you all for the joyful memories — the endless gossips, Sunday cooking sessions, institute trips, and spirited games on the badminton court. I will always cherish our evening gatherings at the tea shop, surrounded by such a lively and warm group of friends.

I also extend my heartfelt thanks to my friends from the 2018 Physics batch — Tanmay, Apurba, Sushovan, Vinod, Amit, Gopal, Toshali, Subashri, Prateek, Pavan, Akhil, Prabhat, Himangshu, Nishant, and Mahaveer — for their camaraderie and generous help, especially during assignment deadlines.

The time spent at RSA-04 will always hold a special place in my memory. I am especially grateful to Dipanjan da, Sagnik da, and Sanjoy da for their thoughtful guidance and kind advice during the early phase of my PhD.

In later years, I had the pleasure of sharing an office with Apurba, Sushovan, and Tanmay. The laughter, shared music, casual banter, and light-hearted conversations amidst our research made those days truly unforgettable.

Contents

Summary	1
List of Figures	4
List of Tables	20
1 Introduction	22
1.1 Overview and structure of the thesis	22
1.2 Emission Mechanism of Pulsars	24
1.2.1 Rotation-Powered Emission	24
1.2.2 Curvature Radiation and Synchrotron Radiation From Pulsars . .	26
1.3 Pulsar emission geometry	29
1.3.1 The Lighthouse Model of Pulsar Emission	29
1.3.2 Pulsar beam model	32
1.4 Use of the Models Described in This Section	39
2 Binary Pulsars, Bending Delays, and Pulsar Timing	40

2.1	Formation of Binary Pulsars	40
2.1.1	Recycled Pulsars	42
2.2	Co-ordinate geometry relevant for binary pulsars	44
2.3	Modelling the evolution of the orbit and spins	47
2.4	Pulsar Timing	54
2.4.1	Effect of the Solar System's Gravitational Field	55
2.4.2	Effect of the Interstellar Medium (ISM)	57
2.4.3	Effect of companion's gravitational field for a binary pulsar . . .	58
2.5	Bending Delay	60
2.5.1	Previous Studies on Bending Delay in Binary Pulsars	62
3	A Study of the Light Bending Phenomenon: Pulsar in a Binary With a Schwarzschild Black Hole	64
3.1	General Relativistic Formalism for Bending Delay	65
3.2	Distortion of the beam and the pulse shape	69
3.3	Numerical study of bending delay	71
3.3.1	Features of the bending delay: demonstration for the Double Pulsar	71
3.3.2	Bending delay in pulsar–black hole binaries	79
3.3.3	Distortion of the beam and the change in the pulse profile due to the bending	90
3.4	Conclusions	98
4	Exploring the Light Bending Phenomenon for a Pulsar in a Binary With a	

Rotating Black Hole	99
4.1 The effect of the spin of the companion on the bending delay: A full general relativistic study	99
4.1.1 Impact of Companion Spin on Bending Delay	103
4.2 Numerical study	104
4.3 Exploring the bending delay when the companion is a supermassive black hole	124
4.4 Conclusion	133
5 Conclusion	134
A Rotational Transformations	139
A.1 Conversion from the I-frame to b-frame	140
A.1.1 Conversion from the I' -frame to the s-frame	140
A.1.2 Conversion from the s-frame to b-frame	142
A.1.3 Conversion from I-frame to b-frame	143
A.2 Conversion from the L-frame to T-frame	144
A.3 Conversion from the T-frame to b-frame	146
A.4 Conversion from the m-frame to I-frame	148
A.5 Conversion relation between the bh' -frame and the s-frame	149
A.6 Conversion between the bh' -frame and the b-frame	151
A.7 Conversion between bh' -frame and T-frame	152

B Details of Calculations for the Schwarzschild Spacetime 153

B.1	Equations of motion of light rays in the Schwarzschild spacetime	153
B.2	Geodesics in the Schwarzschild space time	154
B.2.1	Impact parameter D	157
B.2.2	Geodesics for $D = 0$	158
B.2.3	Null Geodesics for $D = 0$	158
B.2.4	Null geodesics for $D \neq 0$	159
B.3	Initial Direction of the light rays	162
B.3.1	The light ray initially travelling away from the companion (u initially decreasing):	164
B.3.2	The light ray initially travelling towards the companion (u initially increasing):	166
B.4	Solving the geodesic equation for $D > \frac{3\sqrt{3}GM_c}{c^2}$ (solid blue curve in Fig. B.1), regardless of the initial direction of the light ray	168

C Details of Calculations for the Kerr Spacetime 171

C.1	Equations of motion of light rays in the Kerr spacetime	171
C.2	Null geodesics	178
C.2.1	θ integrals	179
C.2.2	r integrals	198
C.3	Elliptical integrals in the form of algebraic integration	205
C.4	Derivations of useful expressions needed for θ integrations	207

C.4.1	Derivation of Eq. (C.43)	207
C.4.2	Derivation of Eq. (C.53)	207
C.4.3	Derivation of Eq. (C.59a), i.e., the relation between Ψ_j and Γ_J . .	210
C.5	Calculation methods of p_r/p_t , p_θ/p_t and p_ϕ/p_t using the initial position and the initial direction of the light ray	211
C.5.1	Conditions satisfied by $ D_q $	218
D	Geometrical Explanation for The Pulse Profile Distortion Near The Superior Conjunction	225
D.1	Configurations Other Than Superior Conjunction	234
D.2	Superior Conjunction Configuration with a Stellar-Mass Black Hole . . .	235
	Bibliography	237

SUMMARY

Pulsars are extremely precise cosmic clocks, and their timing signals can be used to test general relativity. In binary systems, the orbital motion of the pulsar, together with the gravitational field of its companion, gives rise to several distinct delays in the propagation of the pulsar signal. Because of the pulsar's motion in its orbit, signals emitted at different orbital positions arrive at different times; this produces the Rømer delay. The gravitational potential of the companion further affects the propagation time of the signal and gives rise to the Shapiro delay. The combined influence of gravitational redshift and time dilation, as measured by standard laboratory clocks, introduces an additional correction known as the Einstein delay. Moreover, the gravitational field of the companion bends the trajectory of the light ray (i.e., gravitational light bending), causing the signal to follow a slightly longer path and thus introducing an additional geometric delay.

Beyond these well-known effects, light bending also modifies the observed rotational phase of the pulsar owing to the geometry of its emission beam (the lighthouse model). These effects are referred to as the longitudinal and latitudinal bending delays [1, 2]. In addition to introducing timing delays, this bending phenomenon can also distort the observed pulse shape.

Earlier studies, such as Schneider [3] and Doroshenko et al. [1], analyzed these bending delays in the weak-field limit and provided approximate expressions. Schneider [3] referred to this effect as the rotational time delay, while Doroshenko et al. [1] termed it the bending delay. Later, Rafikov et al. [2] derived more accurate formulas applicable even when the companion is another neutron star and the pulsar passes very close to it. They referred to this effect as the lens-rotational delay. In this thesis, we adopt the term bending delay to describe this phenomenon.

These bending delays are strongest near superior conjunction (when the pulsar is directly behind the companion relative to Earth), as in this configuration, the light from the pulsar

comes very close to the companion.

As pulsar–black hole binaries are expected to be the best natural laboratories to test theories of gravity, in this thesis we investigate the effect of the light bending phenomenon for such binaries. We first compute the bending delays for model pulsar–black hole (stellar-mass) binaries using parameter values such as eccentricity, orbital period, component masses, etc., from Chattopadhyay et al. [4]. Our results show good agreement with the approximate expressions given by Rafikov et al. [2], unless the pulsar is away from the superior conjunction where the approximate expression is not valid. The longitudinal delays from our method also match those from Doroshenko et al. [1] across the orbit, although Doroshenko et al. [1] did not provide a corresponding expression for the latitudinal delay.

We consider both non–rotating and rotating black hole companions. For the first case, the spacetime around the black hole is described by the Schwarzschild geometry, and for the second case, it is described by the Kerr geometry. We see no noticeable impact of the spin of the black hole on the values of the bending delays. To visualize the impact of the companion’s spin, we define longitudinal frame dragging bending (longitudinal FD delay) and latitudinal frame dragging bending (latitudinal FD delay), which are of the order of ns for a stellar–mass black hole companion. We also notice a discontinuity in both longitudinal and latitudinal FD delays at the orbital phases where the pulsar’s position vector lies in the plane defined by the black hole spin axis and the line of sight. At these orbital phases, the direction of the light ray relative to the black hole’s spin reverses, leading to the observed discontinuity.

We also explore the effect of different parameters like eccentricity, orbital period, companion mass, companion spin, etc., on the values of the bending delay.

We also investigate the distortion of the pulsar beam and change in pulse profile due to light bending using the core–double-cone model of Rankin [5]. We see that for the stellar–mass black hole companion, the profile strength increases but the shape remains unaltered. This enhancement is stronger for the configurations closer to the superior con-

junction.

Finally, we explore what happens when the bending is very strong, such as in the case where the companion is a supermassive black hole. Here too, the effect of spin is negligible. Interestingly, the shape and strength of the pulse profile change drastically when the pulsar is near superior conjunction. This is due to the Einstein ring being larger than the beam size, which can be explained using simple geometry based on lensing theory that is valid near superior conjunction.

List of Figures

- 1.1 The beam of a pulsar and the Line of Sight (LoS). The meanings of various vectors and angles are explained in the text. 31

- 1.2 The surface probability density of light rays on the $X_m Y_m$ plane, which is the cross-section of the beam from an isolated pulsar. The red circle marks a radius of $\mathcal{R}_{1.4,10}^{\text{out}}$. The pulsar parameters used are $P_s = 1.67$ s and $\dot{P}_s = 1.38 \times 10^{-15}$ ss $^{-1}$. The diamonds ($\diamond$) in magenta indicate the Line of Sight (LoS) positions across the beam for $\alpha = \zeta_L$ at different moments in time. The central point corresponds to time t_{mid} when the LoS passes through the center of the beam. The three points above this represent the LoS at $t_{\text{mid}} - 0.0274$ s, $t_{\text{mid}} - 0.0218$ s, and $t_{\text{mid}} - 0.0174$ s (from top to bottom). Similarly, the three points below the center are for $t_{\text{mid}} + 0.0174$ s, $t_{\text{mid}} + 0.0218$ s, and $t_{\text{mid}} + 0.0274$ s (from top to bottom). 38

- 1.3 Pulse profile for an isolated pulsar based on the core-double cone model. . 39

- 2.1 The diagram illustrates the orbital geometry of a pulsar (marked by a $\star$) in a binary system. The relevant coordinate axes and angles are described in the text. The sky plane, denoted by the $X_s Y_s$ axes, is shown in light red, while the orbital plane, represented by the $X_b Y_b$ axes, appears in green. The light blue plane corresponds to the $X'_I Y'_I$ axes, which are parallel to the $X_I Y_I$ plane introduced in Sec. 1.3.1. The unit vector $\hat{S}_p$ indicates the spin axis of the pulsar. Although this is physically along the Z_I -axis, it is displayed along the Z'_I -axis here, due to a parallel shift to the center of the $X_b Y_b Z_b$ frame. Similarly, $\hat{S}_{bh}$ denotes the spin axis of the companion black hole. While it lies along the Z_{bh} -axis in reality, it is shown along the shifted Z'_{bh} -axis for clarity. The locations of the ascending and descending nodes are marked as AN and DN, respectively. 48
- 3.1 A schematic diagram illustrating the bending of a light ray as it travels toward a gravitating object at the time of emission. Here, the source of the light ray is the pulsar and it is represented by a black star, while the gravitating object i.e., the binary companion is shown as a black filled circle. The coordinate axes and angles shown are described in the text. . 67
- 3.2 Bending delays in the signal of PSR J0737–3039A calculated using different methods. The right panel shows the latitudinal bending delay while the left panel shows the longitudinal bending delay. To calculate the values of the latitudinal bending delay in each formalism, we have calculated $\Delta\Phi_0$ using Eqs. (2.56) and (2.55). In the figure legends, RL06 stands for Rafikov et al. [2] and DK95 stands for Doroshenko et al. [1]. 72

3.3 The schematic diagram illustrating the bending of light rays in the signal from a pulsar in a binary. The companion is marked as a black filled circle, and the three possible positions of the pulsar are shown with black stars ($\star$). These positions are labeled as O_{-1} , O_0 , and O_1 . The position O_0 corresponds to the position of the pulsar at the superior conjunction. Magenta arrows represent the initial directions of some light rays, while the green arrows represent the directions of the same light rays after bending. Among these, only the rays shown by solid green lines reach to the observer, the rays represented by green dashed lines do not reach the observer. When the pulsar is at O_0 , it is clear that all the light rays from the circular region shown in blue (edge of a three dimensional hollow cone) reach to the observer although only two such light rays are shown. Each of these light rays undergoes a different amount of bending as their initial directions are different. On the other hand, when the pulsar is at positions O_1 and O_{-1} , which are away from O_0 , there is only one specific direction from which the observer can receive the signal from the pulsar. D_0 denotes the impact parameter of a light ray emitted from O_0 toward the companion, which reaches the observer after undergoing bending. Conversely, D_1 corresponds to a light ray emitted from O_1 away from the companion, which still reaches the observer. It is obvious that the value of D_1 is much larger than the value of D_0 . That is why we consider only the situations where the value of the impact parameter is greater than $3\sqrt{3}GM_c/c^2$, even when the light ray was initially moving away from the companion, as mentioned in Sec. 3.1, ... 75

3.4	Bending delays obtained using our method for the signal of PSR J0737–3039A, assuming an inclination angle of $i = 90.14^\circ$. The left panel represents the longitudinal bending delay and the right panel represents the latitudinal bending delay. Both delay curves become irregular between the range $\omega + A_T \simeq 89.8^\circ$ and $\omega + A_T \simeq 90.2^\circ$. The irregular regions are displayed in gray. Here, the bending delay values are calculated using our formalism, and the value of $\Delta\Phi_0$ is obtained using Eqs. (2.56) and (2.55).	76
3.5	The longitudinal and latitudinal bending delays in the signal of a system similar to PSR J0737–3039A, evaluated for two different values of α and fixed parameters $i = 87.5^\circ$, $\zeta_L = 50^\circ$, and $\eta_p = 45^\circ$. These delays are calculated using our formalism, and the value of $\Delta\Phi_0$ has been calculated using Eqs. (2.56) and (2.55).	78
3.6	Comparison of the bending delay values calculated using our formalism and the expressions of Rafikov et al. [2] over a complete orbital cycle for a hypothetical pulsar–black hole binary. Here we set $i = 87.5^\circ$, and all other parameters are the same as those listed in Table 3.1. The left panel displays the longitudinal bending delay and the right panel illustrates the latitudinal bending delay. For the longitudinal bending delay, we also include the expression provided by Doroshenko et al. [1]. While calculating the latitudinal bending delays using our method, the value of $\Delta\Phi_0$ obtained from Eqs. (2.56) and (2.55). In the figure legends, RL06 stands for Rafikov et al. [2] and DK95 stands for Doroshenko et al. [1]	82
3.7	Comparison of latitudinal bending delay values estimated using three different methods, for varying orbital inclination angles i , in a hypothetical pulsar–black hole binary. All other parameter values are the same as those listed in Table 3.1.	86

3.8	Longitudinal bending delay curves for hypothetical pulsar–black hole binaries, where one parameter is varied in each panel while all other parameters are kept fixed as listed in Table 3.1. The inclination angle is set to $i = 84^\circ$ for all panels except panel (d), where i is varied.	87
3.9	Latitudinal bending delay curves for hypothetical pulsar–black hole binaries, where one parameter is varied in each panel while all other parameters are fixed as given in Table 3.1. The inclination angle is set to $i = 84^\circ$ in all cases except panel (d), where i is varied.	88
3.10	Surface probability density of light rays on the cross-section of the pulsar beam in the $X_m Y_m$ plane, accounting for gravitational bending in a hypothetical pulsar–black hole binary. The top panel (a) shows results for $i = 90.0^\circ$, and the bottom panel (b) for $i = 89.8^\circ$. In both cases, we set $\omega + A_T = 90^\circ$, with all other parameters taken from Table 3.1. Each panel displays the intensity distribution at nine different times. For each instant, the location where the LoS intersects the beam cross-section is marked by a magenta diamond ($\diamond$). The red circle in each plot denotes the theoretical boundary of the beam cross-section without bending, i.e., a circle of radius $\mathcal{R}_{1.4,10}^{\text{out}}$	93

- 3.11 Surface probability density of light rays on the $X_m Y_m$ plane—i.e., the cross-section of the pulsar beam under gravitational bending in a hypothetical pulsar–black hole binary. The top panel (a) corresponds to $\omega + A_T = 89^\circ$, and the bottom panel (b) to $\omega + A_T = 91^\circ$. In both cases, we set $i = 90^\circ$, and all other parameters are adopted from Table 3.1. Each panel presents the intensity distribution at nine different times. For each time, in addition to the surface density of the light rays, the intersection point of the LoS with the beam cross-section is marked by a magenta diamond ($\diamond$). The red circle in each plot indicates the theoretical beam boundary without bending, represented by a circle of radius $\mathcal{R}_{1.4,10}^{\text{out}}$ 94
- 3.12 Pulse profiles at four different orbital inclinations for a hypothetical pulsar–black hole binary, comparing cases with (solid orange lines) and without (blue dashed lines) gravitational bending. In each panel, the intensity refers to the number of light rays within a circular region of radius $\mathcal{R}_{1.4,10}^{\text{out}}/50$ centered on the line of sight (LoS). For all cases, the pulsar is positioned at the orbital phase $\omega + A_T = 90^\circ$, and the remaining parameters are taken from Table 3.1. 96
- 3.13 Pulse profiles for a hypothetical pulsar–black hole binary, comparing cases with (solid orange) and without (dashed blue) the influence of bending. Panel (a) shows results for $\omega + A_T = 89^\circ$ and panel (b) for $\omega + A_T = 91^\circ$, both with an inclination of $i = 90^\circ$. All remaining parameters follow Table 3.1. Here, intensity refers to the number of rays passing through a circular area of radius $\mathcal{R}_{1.4,10}^{\text{out}}/50$ centered at the LoS. 97

- 4.1 Differences in bending delays computed using two distinct formulations of null geodesics around a non-rotating black hole: one using the Schwarzschild solution and the other using the Kerr solution with the spin parameter set to $\tilde{a} = 0$. For both calculations, we adopt $\lambda_{\text{bh}} = i = 87.5^\circ$, and all other relevant parameter values are taken from Table 4.1. 107
- 4.2 Bending delay variations across a full orbit for a hypothetical pulsar–black hole system, comparing the cases with spin parameters $\tilde{a} = 0$ (black dashed curve) and $\tilde{a} = 0.9$ (blue triangles). The system assumes $\lambda_{\text{bh}} = i = 87.5^\circ$, with all other parameters taken from Table 4.1. The left panel illustrates the delay due to latitudinal bending, while the right panel depicts the longitudinal component. A vertical orange line marks $A_T = 0^\circ$, which is taken as the zero point in time ($t = 0$). 108
- 4.3 A schematic diagram depicting how the directions of light rays (shown in green) that align with the line of sight (LoS, red line) change depending on the spin of the black hole, indicated by a magenta dashed circle with arrows. The pulsar’s orbital path (light-blue plane) includes several positions labeled as Q_1 through Q_6 , marked by black stars. The equatorial plane of the black hole is represented by the light-grey plane, and the black hole itself is partially illustrated as a filled black semicircle (not to scale). At positions Q_1 , Q_2 , and Q_3 , the light rays travel in the same direction as the black hole’s spin, whereas at Q_4 , Q_5 , and Q_6 , their directions are opposite to the spin. The purple dashed line shows the projection of the LoS onto the equatorial plane of the black hole, while the green dashed line represents its projection onto the orbital plane. The points Q_1 , Q_2 , and Q_3 lie on one side of the green dashed line, and Q_4 , Q_5 , and Q_6 lie on the other. 109

- 4.4 Frame-dragging (FD) bending delays for $\tilde{a} = 0.9$ calculated over a complete orbit for a hypothetical pulsar–black hole binary with $\lambda_{\text{bh}} = i = 87.5^\circ$. All other relevant parameters are taken from Table 4.1. The left panel illustrates the latitudinal component of the FD bending delay, while the right panel displays the longitudinal component. In both panels, the grey dashed vertical line within the zoomed inset highlights the position of the discontinuity. 110
- 4.5 Frame-dragging (FD) bending delays computed for $\tilde{a} = 0.9$ across a complete orbital cycle in a hypothetical pulsar–black hole binary system. The parameters used here are $\eta_{\text{bh}} = -40^\circ$ and $\lambda_{\text{bh}} = i = 87.5^\circ$, with all other values taken from Table 4.1. The left panel displays the latitudinal FD bending delay, while the right panel shows the longitudinal FD bending delay. In both panels, the grey dashed vertical line marks the position of the discontinuity. 110
- 4.6 Bending delays and the frame-dragging (FD) bending delays for a hypothetical pulsar-black hole binary. The top left panel (panel a) shows the latitudinal bending delay and the top right panel (panel b) shows the longitudinal bending delay. The bottom left panel (panel c) shows the latitudinal FD bending delay and the bottom right panel (panel d) shows the longitudinal FD bending delay. We have chosen spin parameter $\tilde{a} = 0.5$, $\lambda_{\text{bh}} = i = 90^\circ$, with all other system parameters taken from Table 4.1. Irregularities appear in the delay curves between $A_T + \omega = 89.7^\circ$ and $A_T + \omega = 90.3^\circ$. The irregular regions are shown in gray. 112

- 4.7 Latitudinal frame-dragging (FD) bending delay curves for hypothetical pulsar–black hole binaries. In each panel, one parameter is varied while all others are held constant as specified in Table 4.1. The spin parameter is fixed at $\tilde{a} = 0.5$, except in panel (f) where it is varied. The inclination $\lambda_{\text{bh}} = i = 87.5^\circ$ is used throughout, except in panel (c), where only i is varied while keeping λ_{bh} fixed at 87.5° . Vertical dashed lines mark the positions of the discontinuities in the delay curves. 115
- 4.8 Latitudinal frame-dragging (FD) bending delay curves for hypothetical pulsar–black hole binaries. In each panel, a single parameter is varied while all others are held fixed as specified in Table 4.1. The spin parameter is set to $\tilde{a} = 0.5$ in all panels except (f), where it is varied. Likewise, the inclination angles are taken as $\lambda_{\text{bh}} = i = 87.5^\circ$, except in panel (c), where only i is varied while λ_{bh} remains fixed at 87.5° . The vertical dashed lines indicate the positions of discontinuities in the delay curves. 116
- 4.9 Latitudinal (panel a) and longitudinal (panel b) frame-dragging (FD) bending delay curves for hypothetical pulsar–black hole binaries, shown for different values of the spin parameter $\tilde{a}$. The inclination angles are fixed at $\lambda_{\text{bh}} = i = 87.5^\circ$, and all other relevant parameters are adopted from Table 4.1. Vertical grey dashed lines indicate the positions of discontinuities in the delay curves. 117

4.10	Effects of varying η_{bh} and λ_{bh} on the latitudinal and longitudinal bending delays for hypothetical pulsar–black hole binaries with $\tilde{a} = 0.9$ and $i = 90^\circ$. Each panel shows the delay differences over a restricted orbital phase range (in degrees) where the variations are most prominent. Panel (a) (top right) displays $\Delta\tau_{\text{lat}\eta,\eta_1-0}$, panel (b) (top left) shows $\Delta\tau_{\text{long}\eta,\eta_1-0}$, panel (c) (bottom right) presents $\Delta\tau_{\text{lat}\lambda,\lambda_1-0}$, and panel (d) (bottom left) shows $\Delta\tau_{\text{long}\lambda,\lambda_1-0}$. These quantities are defined in the main text. All relevant parameters are taken from Table 4.1, except for the ones being varied. For panels (a) and (b), λ_{bh} is set equal to i	119
4.11	Latitudinal FD bending delays for hypothetical pulsar–black hole binaries with $\tilde{a} = 0.5$, shown for different orientations of the black hole’s spin axis. In each panel, only one parameter is varied while all others are kept fixed, with $i = 87.5^\circ$ and remaining values taken from Table 4.1. In panels (a) and (b), where η_{bh} is varied, we set $\lambda_{\text{bh}} = i$	120
4.12	The longitudinal FD bending delay curves for hypothetical pulsar-black hole binaries, where each panel illustrates the effect of varying a single parameter. All other parameters are fixed to the values listed in Table 4.1, with $\tilde{a} = 0.5$ and $i = 87.5^\circ$, unless stated otherwise. In panels (a) and (b), η_{bh} is varied while keeping $\lambda_{\text{bh}} = i$	121

- 4.13 Photon distribution across the beam cross-section of a pulsar orbiting a black hole, assuming $\lambda_{\text{bh}} = i = 90^\circ$. The top-left panel shows the undistorted beam without gravitational bending. The top-right, bottom-left, and bottom-right panels display the distorted photon distributions due to bending for spin parameters $\tilde{a} = 0, 0.5$, and 0.9 , respectively. All other parameters are taken from Table 4.1. In each case, the intensity profile corresponds to the moment when the line of sight (LoS) passes through the center of the beam. In the panels with $\tilde{a} \neq 0$, the black solid line indicates the projection of the black hole's spin axis onto the beam cross-section. 122
- 4.14 Pulse profiles of a pulsar in a binary system with a black hole companion, assuming $\lambda_{\text{bh}} = i = 90^\circ$ and orbital phase $A_T + \omega = 90^\circ$. The black dashed curve represents the profile without gravitational bending. The profiles including bending are shown for different spin parameters of the black hole, e.g., $\tilde{a} = 0$ (solid orange line), $\tilde{a} = 0.5$ (magenta unfilled circles), and $\tilde{a} = 0.9$ (sea-green crosses). All other parameters are consistent with those listed in Table 4.1. 123
- 4.15 Comparison of pulse profiles for a pulsar orbiting a black hole with $\lambda_{\text{bh}} = i = 90^\circ$. The black dashed curve represents the profile without gravitational bending, while the profiles including bending are shown for different values of the black hole spin parameter: $\tilde{a} = 0$ (solid orange), $\tilde{a} = 0.5$ (magenta open circles), and $\tilde{a} = 0.9$ (sea-green crosses). All other relevant parameters are as listed in Table 4.1, except for the companion mass, which is set to $M_c = 10^6 M_\odot$ 128

- 4.16 Pulse profile comparison for a pulsar orbiting a black hole with $\lambda_{\text{bh}} = i = 85^\circ$ and orbital phase $A_T + \omega = 90^\circ$. The black dashed line shows the profile without gravitational bending, while the profiles accounting for bending are displayed for different black hole spin parameters: $\tilde{a} = 0$ (solid orange), $\tilde{a} = 0.5$ (magenta open circles), and $\tilde{a} = 0.9$ (sea-green crosses). All other relevant parameters are the same as those in Table 4.1, except for the companion mass, which is set to $M_c = 10^6 M_\odot$ 128
- 4.17 Pulse profile comparison for a pulsar in orbit around a black hole companion, with $\lambda_{\text{bh}} = i = 80^\circ$. The black dashed curve represents the profile without gravitational bending, while the effects of bending are shown for different black hole spin parameters: $\tilde{a} = 0$ (solid orange), $\tilde{a} = 0.5$ (magenta open circles), and $\tilde{a} = 0.9$ (sea-green crosses). All other relevant parameters are as listed in Table 4.1, except for the companion mass, which is set to $M_c = 10^6 M_\odot$ 129
- 4.18 Bending delays across a full orbital cycle for a hypothetical pulsar–supermassive black hole binary system with $\lambda_{\text{bh}} = i = 80^\circ$. The left panel displays the latitudinal bending delay, while the right panel shows the longitudinal bending delay. Results are shown for different spin parameters of the black hole: $\tilde{a} = 0$ (solid black line), $\tilde{a} = 0.1$ (magenta circles), $\tilde{a} = 0.5$ (green squares), and $\tilde{a} = 0.9$ (blue triangles). All other relevant parameters follow Table 4.1, except for the black hole mass, which is set to $M_c = 10^6 M_\odot$. The vertical orange line in each panel marks $A_T = 0^\circ$, corresponding to time $t = 0$. Phases to the left of this line represent orbital positions beyond 360° 131

4.19 FD bending delays over a complete orbital cycle for a hypothetical pulsar–supermassive black hole binary with $\lambda_{\text{bh}} = i = 80^\circ$. The left panel presents the latitudinal FD bending delays, while the right panel shows the longitudinal ones. Each curve corresponds to a different black hole spin parameter: $\tilde{a} = 0$ (solid black), $\tilde{a} = 0.1$ (magenta circles), $\tilde{a} = 0.5$ (green squares), and $\tilde{a} = 0.9$ (blue triangles). All other relevant parameters are taken from Table 4.1, except the companion mass, which is set to $M_c = 10^6 M_\odot$. The vertical orange line in both panels marks $t = 0$ (i.e., $A_T = 0^\circ$); orbital phases to its left correspond to positions beyond 360° . . 132

A.1 The transformation from the s-frame to the I-frame. The green plane is the $Z_s X_s$ plane and the red plane is the $X_s Y_s$ plane. 141

A.2 The transformation from the s-frame to the b-frame. The red plane is the $X_s Y_s$ plane (the sky plane), and the blue plane is the $X_b Y_b$ plane (the orbital plane). The X_s -axis is the line of the node, that is along the intersection of the sky-plane and the orbital plane. 142

A.3 The wrong transformation from s to b frame. The red plane is the $X_s Y_s$ plane (the sky plane), and the blue plane is the $X_b Y_b$ plane (the orbital plane). The Y_s -axis is the line of the node, that is along the intersection of the sky-plane and the orbital plane, which contradicts our choice. . . . 144

A.4 The transformation from the L-frame to the T-frame. The blue plane is the $X_L Y_L$ plane. The left figure is for the case when $\psi_{L,\text{in}}$ is negative and the right figure is for the case when $\psi_{L,\text{in}}$ is positive. 145

A.5 The transformation from the b-frame to the T-frame. The green plane is the $Z_b X_b$ plane and the red plane is the $X_b Y_b$ plane or the orbital plane. . . 146

A.6 The transformation from the I-frame to the m-frame. The green plane is the $Z_I X_I$ plane and the red plane is the $X_I Y_I$ plane. 148

A.7	The transformation from the s-frame to the bh-frame. The green plane is the $Z_s X_s$ plane and the red plane is the $X_s Y_s$ plane.	150
B.1	The plot of $f(u)$ vs u for 3 different values of the impact parameter D in terms of $3\sqrt{3} \frac{GM_c}{c^2}$. The numerical values of u and $f(u)$ depend on the value of M_c . Here we have used $M_c = 1.0 M_\odot$. We have checked that the behaviour of $f(u)$ remain unchanged for a different choice of M_c , only the values of $f(u)$ and roots change.	161
B.2	Schematic diagrams of the light bending phenomenon in the $X_L Y_L$ plane. The left diagram is for the case when the light ray initially moves towards the gravitating body. The right diagram is for the case when the light ray initially moves away from the gravitating body. In both of the cases, the source of the light ray is shown with a black star and the gravitating body is shown with a black filled circle. Here, $\hat{n}_{L,in}$ denotes the initial direction of the light ray, while $\hat{n}_{L,\infty}$ represents its final direction after bending. The quantity $\psi_{L,in}$ is the initial azimuthal angle of the light ray (at the time of emission) in the L-frame, i.e., the angle between the X_L axis and the position vector $\vec{r}_L$. Similarly, $\psi_{L,\infty}$ is the azimuthal angle corresponding to the final direction of the light ray, i.e., the angle between the X_L axis and $\hat{n}_{L,\infty}$	163
C.1	The function $\Theta(\theta)$ for four different cases depending on value of D_q as discussed in the text. Here we choose $GM_c/c^2 = 1000$ km (corresponding to $M_c = 677.09 M_\odot$) and $D_\lambda = 100$ km. Panel-a shows Case-1, Case-2, and Case-3, each for $\tilde{a} = 0.5$, but for different values of D_q . Panel-b is for Case-4 where $\tilde{a} = 0$ and $D_q > 0$	182

C.2	Four possible cases that can arise for the function $\mathcal{R}(r)$ depending on the value of D_q and D_λ are shown in the figures. In each case, we plot $\mathcal{R}(r)$ along the abscissa and r along the ordinate. We choose $GM_c/c^2 = 1000$ km (corresponding to $M_c = 677.09 M_\odot$) and $\tilde{a} = 0.5$. The outer event horizon is located at the value of r_+ as 1866 km and is shown with a red dot. The inner event horizon is located at the value of r_- as 134 km and is shown with a green dot in each of the figures. The upper left figure is for Case-1, the upper right figure is for Case-2, the lower left figure is for Case-3 and the lower right figure is for Case-4. See text for classifications of the cases. In Case-2, Case-3, and Case-4, the outer event horizon is outside the range of r shown in the figures. The regions containing the roots have zoomed in for Case-1, Case-2, Case-3.	200
C.3	The T-frame and the b-frame. The orbital plane is shown in light blue whereas the $X_T Y_T$ plane is shown in light green. The position of the pulsar in its orbit is marked with a star.	217
C.4	A hypothetical function $f(x)$, which has two negative roots x^+ and x^- , and $x^+ > x^-$. The function has a minimum value at $x^- < x_{\min} < x^+$. Then, from the schematic diagram, it is clear that $f(x) > 0$ possible only for $x > x^+$ and $x < x^-$	219
D.1	A schematic diagram representation of the bending and propagation of light rays at the superior conjunction configuration.	226

D.2 Understanding strong bending for a pulsar - super massive black hole binary system when the pulsar is near the superior conjunction for different combinations of i and $A_T + \omega$ in the observer's frame, where the centre of the beam moves along the red line ' $B'B''$ ' that passes through the Line-of-Sight ('LoS'). The arrows on $B'B''$ denotes the direction of motion of the beam. Three positions of beam B_1 , B_2 , and B_3 have been shown in each panel. The magenta ring marked J is the ring on which the light rays from the rings K_1 and K_2 can come due to strong bending as explained in the text. As the magenta ring is chosen to pass through LoS, the observer can receive light rays from the arcs MN on K_1 and ST on K_2 , or more specifically from the segments of those arcs that fall on the beam. The direction of the motion of the segments have been shown with small arrows of the same colour used for that segment. The area of the beam traversed by such a segment results the pulse profile. 230

List of Tables

- 3.1 Parameters for a canonical pulsar—black hole binary. These are (from top to bottom), the spin period of the pulsar (P_s), the rate of change of the spin period of the pulsar ($\dot{P}_s$), the orbital period of the binary (P_b), the eccentricity (e), inclination of the orbit (i), the mass of the pulsar (M_p), the mass of the black hole (M_c), the longitude of the periastron (ω), the angle between the spin axis and the magnetic axis (α), the initial value of the angle between the X_s -axis and the projection of the spin axis in the sky plane (η_p), and the initial value of the angle between the Z_s -axis and the spin axis (λ_p). The values of P_s , $\dot{P}_s$, P_b , e , M_p , M_c , are taken from Chattopadhyay et al. [4]. The values of other parameters are chosen arbitrarily, but satisfying the visibility condition $|180^\circ + \mathscr{W}_{1.4,10}^{\text{out}} - \alpha| \geq |\lambda_p| \geq |180^\circ - \mathscr{W}_{1.4,10}^{\text{out}} - \alpha|$. Moreover, the values of the time-dependent parameters are the values at $t = 0$ (when $A_T = 0$). 81

4.1 This table summarizes the parameters used for a representative pulsar–black hole binary system. Listed from top to bottom are: pulsar spin period (P_s), spin-down rate ($\dot{P}_s$), orbital period (P_b), orbital eccentricity (e), inclination of the orbit i , the pulsar mass (M_p), black hole mass (M_c), longitude of periastron (ω), orbital inclination relative to the sky plane (i), the angle between the pulsar’s spin and magnetic axes (α), the angle between the X_s -axis and the projected pulsar spin axis on the sky plane (η_p), the angle between the pulsar spin axis and the Z_s -axis (λ_p), and similarly, η_{bh} is the angle between the X_s -axis and the projected spin axis of the black hole on the sky plane and λ_{bh} is the angle between the black hole spin axis and the Z_s -axis. $\tilde{a}$ is the dimensionless spin parameter of the black hole. Values for P_s , $\dot{P}_s$, and P_b , e , M_c , M_p are taken from Chattopadhyay et al. [4], while the remaining parameters are selected to satisfy the visibility condition $|180^\circ + \mathscr{W}_{1.4,10}^{\text{out}} - \alpha| \leq |\lambda_p| \leq |180^\circ - \mathscr{W}_{1.4,10}^{\text{out}} - \alpha|$, where $\mathscr{W}_{1.4,10}^{\text{out}}$ is the outer cone half-opening angle at 1.4 GHz, defined by the angular distance at which intensity falls to 10% of its peak, following Chapter 1. Time-dependent parameters are quoted at $t = 0$ (i.e., when $A_T = 0$), and in some cases, parameter values are varied to explore their effects, with two values listed where applicable—the first being used more frequently. Unless stated otherwise, we typically assume $\lambda_{bh} = i$. Exact parameter values for each figure are specified in the corresponding captions. 105

Chapter 1

Introduction

1.1 Overview and structure of the thesis

Pulsars, first discovered in 1967 by Jocelyn Bell Burnell and Antony Hewish, have captivated the astrophysical community with their unique properties and profound implications for understanding the universe. These rapidly rotating neutron stars, remnants of massive stars that have undergone supernova explosions [6], emit beams of electromagnetic radiation from their magnetic poles. As these beams sweep across the Earth, they create a characteristic pulsing signal, which has provided a wealth of information about the extreme environments in which pulsars exist.

The study of pulsars has advanced our knowledge in several key areas of astrophysics, including the physics of neutron stars, the behavior of matter under extreme conditions, and the nature of gravitational waves. Pulsars serve as highly precise cosmic clocks, with some millisecond pulsars exhibiting timing stability rivaling that of atomic clocks. This precision has made them invaluable tools for testing the general theory of relativity, probing the interstellar medium, and potentially detecting low-frequency gravitational waves.

In recent years, the study of pulsar–black hole binaries has gained significant attention due to their potential to provide new insights into both pulsar physics and the nature of black holes. The interaction between a pulsar’s radiation and the intense gravitational field of its black hole companion gives rise to a range of relativistic effects that can be probed through high-precision pulsar timing. The timing models commonly employed are generally based on post-Newtonian (PN) approximations. However, recent studies suggest that PN approximations may break down when the pulsar is in a binary with a supermassive or intermediate-mass black hole companion [7]. In such extreme mass ratio binaries (EMRBs), the orbital dynamics and spin evolution of the pulsar are also strongly influenced by relativistic effects [8]. Kimpson et al. [9] examined the interplay between pulsar spin dynamics, orbital motion, and the propagation path of the emitted signal in EMRBs. Hackmann and Dhani [10] and Ben-Salem and Hackmann [11] investigated the exact propagation delays predicted by General Relativity (GR) for pulsars orbiting supermassive black holes and compared these results with PN approximations. More recently, Della Monica et al. [12] and Della Monica and de Martino [13] developed a numerical framework to compute the exact propagation delay in Schwarzschild spacetime for pulsars in binaries with supermassive black holes. Their results are in good agreement with those of Hackmann and Dhani [10]. However, none of these studies considered the bending delay, which arises from the combined effects of gravitational light bending and the emission geometry of the pulsar. In the present thesis, we have studied the bending delay for pulsar - black hole binaries, and for the sake of simplicity, we have taken wide enough orbits so that PN approximation is still valid.

Understanding the bending delay in pulsar-black hole binaries is crucial for several reasons. First, it offers a unique opportunity to test general relativity in the strong-field regime, where deviations from Newtonian gravity are most pronounced. Second, precise measurements of the bending delay can provide constraints on the mass and spin of the black hole, offering insights into its formation and evolution. Finally, studying these delays contributes to our broader understanding of binary systems, including their dynamics

and evolution, and can aid in the detection of gravitational waves. This thesis focuses on investigating the bending delay in pulsar-black hole binaries, aiming to develop an accurate model for predicting bending delays with the help of full general relativistic theory.

This thesis is organised as follows. In the current chapter, we have discussed the pulsar emission mechanism, emission geometry, and beam model for an isolated pulsar. Chapter 2 presents the coordinate geometry used to study binary pulsars, along with a detailed discussion on pulsar timing and an introduction to the concept of the bending delay. In Chapter 3, we present a detailed numerical study of the bending delay for a non-rotating companion. Chapter 4 extends this to the case of a rotating black hole companion. Finally, we conclude in Chapter 5.

1.2 Emission Mechanism of Pulsars

Pulsars are generally assumed to be rotating magnetic dipoles with their spin axis and magnetic axis misaligned. This misalignment between the two axes results in the emission of electromagnetic radiation. The radiation is a consequence of the pulsar losing rotational energy, leading to a gradual spin-down. Additionally, due to the pulsar's rotation, the magnetic field lines near the magnetic poles remain open, allowing charged particles to accelerate to nearly the speed of light. These charged particles emit highly polarized radiation that can span a wide range of frequencies, often ranging from radio waves to gamma rays.

1.2.1 Rotation-Powered Emission

The rotating magnetic dipole, which is misaligned with the rotation axis, emits electromagnetic radiation at its rotational frequency [14]. This mechanism is responsible for the majority of a pulsar's rotational energy loss over its lifetime, causing the pulsar to grad-

ually spin down and increase its rotational period. If $\vec{m}$ denotes the magnetic moment of the dipole, then the rate at which energy is radiated away is given by [15]:

$$\dot{E}_{\text{dipole}} = \frac{2}{3c^3} |\vec{m}|^2 \omega_s^4 \sin^2 \alpha, \quad (1.1)$$

where $\omega_s = 2\pi/P_s$ is the angular frequency of the pulsar, P_s is the rotational period, and α is the angle between the spin axis and the magnetic axis. The rate of loss of rotational energy is given by:

$$\dot{E}_{\text{rotational}} = -\frac{d}{dt} \left(\frac{1}{2} I \omega_s^2 \right) = 4\pi^2 I \dot{P}_s P_s^{-3}, \quad (1.2)$$

where I is the moment of inertia of the pulsar, and $\dot{P}_s$ denotes the rate at which the spin period is increasing. Approximately more than 90% of the total energy radiated from the pulsar is due to this mechanism. Hence, for simplicity, we can equate Eq. (1.1) and Eq. (1.2) to obtain:

$$\dot{P}_s = K P_s^{2-n}, \quad (1.3)$$

where $K = \frac{8\pi^2 \vec{m}^2 \sin^2 \alpha}{3Ic^3}$ and n is called the braking index. For ideal dipole radiation, $n = 3$. From this expression, we can estimate the age of a pulsar and its surface magnetic field. Integrating Eq. (1.3) from $t = 0$ to $t = T$, we get:

$$T = \frac{P_{s,T}}{(n-1)\dot{P}_{s,T}} \left[1 - \frac{P_{s,0}}{P_{s,T}} \right], \quad (1.4)$$

where $P_{s,0}$ and $P_{s,T}$ are the spin periods of the pulsar at $t = 0$ and $t = T$, respectively, and $\dot{P}_{s,T}$ is the rate of decrease of the spin period at $t = T$. Assuming $P_{s,0} \ll P_{s,T}$ and using $n = 3$ for ideal dipolar emission, we get:

$$\tau_c = \frac{P_{s,T}}{2\dot{P}_{s,T}} \approx 15.8 \times 10^6 \text{ Myr} \left(\frac{P_{s,T}}{\text{sec}} \right) \left(\frac{\dot{P}_{s,T}}{10^{-15}} \right)^{-1}. \quad (1.5)$$

This is the characteristic age of a pulsar. However, this formula is not applicable to recycled pulsars, which have undergone binary evolution. Now for pulsars the magnetic field strength can be written as $|\vec{B}| \approx \frac{|\vec{m}|}{r^3}$, where r is the radial distance from the center of the pulsar. Hence the surface magnetic field strength can be written as (from Eq. (1.3)):

$$B_{\text{surface}} = \sqrt{\left(\frac{3Ic^3}{8\pi^2 \sin^2 \alpha R_s^6} \right) \frac{\dot{P}_s}{P_s^{2-n}}}, \quad (1.6)$$

where R_s is the radius of the pulsar. For a canonical neutron star, we can assume $R_s = 10\text{km}$, $I = 10^{45} \text{ g cm}^2$, and using $n = 3$, we obtain:

$$B_{\text{surface}} = 3.2 \times 10^{19} \times \sin^2 \alpha \sqrt{P_s \dot{P}_s} \text{ G}, \quad (1.7)$$

where P_s is the rotational period in s and $\dot{P}_s$ is the rate of change of rotational period in ss^{-1} . Eq. 1.7 implies the maximum possible surface magnetic field is,

$$B_{\text{surface, max}} = 3.2 \times 10^{19} \times \sqrt{P_s \dot{P}_s} \text{ G}. \quad (1.8)$$

This is the characteristic magnetic field of a pulsar, indicating that the surface of the pulsar possesses a very strong magnetic field.

1.2.2 Curvature Radiation and Synchrotron Radiation From Pulsars

Most of the energy radiated by a pulsar is carried away by dipole emission. However, this emission is not directly observed as the pulse profile in pulsar observations. The observed pulse profile is generated by different mechanisms, primarily occurring at the magnetic poles of the pulsar.

The vacuum solution for a rotating conducting sphere with a dipole magnetic field, where the magnetic moment is aligned with the spin axis (aligned rotator), produces a high electric field around the sphere [16]. Charged particles from the conducting sphere ex-

perience a force due to this electric field and come out of the surface. These charged particles arrange themselves in such a way that the net Lorentz force on each particle is zero. Consequently, the charged particles align themselves along the magnetic field lines, forming a magnetosphere that corotates with the sphere.

By analogy, we conclude that a pulsar is surrounded by a large number of charged particles aligned along the magnetic field lines, corotating with the star [17]. This corotation is only possible up to the field lines for which the corotation velocity of charged particles is less than the speed of light (c). This limit defines a hypothetical cylindrical surface known as the light cylinder, with a radius referred to as the light cylinder radius ($R_{LC} = \frac{cP_s}{2\pi}$). The field lines which are closed within the light cylinder are called closed field lines and in the closed field line region charged particles co-rotate with the pulsar. The field lines originated from near the magnetic poles of the pulsar do not close within the light cylinder boundary and cause the charged particles to flow out. These field lines are called open field lines. The region on the surface of the pulsar from which the field lines remain open is called the polar cap region. This region forms a closed area on the surface of the pulsar, with its boundary defined as (R_s, θ_p) , where θ_p is the polar position of the last open field line at the surface of the pulsar. For a dipolar field line, the relationship $\sin^2 \theta / r = \text{constant}$ holds, where r is the distance from the center of the dipole and θ is the polar angle measured from the axis of the dipole. The boundary of the polar cap at the surface of the pulsar is given by:

$$\frac{\sin^2 \theta}{r} = \frac{1}{R_{LC}} = \frac{\sin^2 \theta_p}{R_s} \quad (1.9a)$$

$$\sin^2 \theta_p = \frac{2\pi R_s}{cP_s} \quad (1.9b)$$

As the charged particles in the open field lines region cannot corotate with the pulsar, a strong electric field near the polar cap ejects them from the surface with large Lorentz factors, reaching up to several million [18, 19]. While these charged particles attempt to

move in the direction of the electric field, the strong magnetic field exerts a force on the perpendicular component of their velocity, causing them to gyrate around the magnetic field lines and emit synchrotron radiation. However, near the surface of the pulsar, the magnetic field is so strong that the lifetime of synchrotron emission is extremely short, or almost negligible. Consequently, the charged particles are forced to move along the magnetic field lines.

Since the magnetic field lines themselves are curved, the charged particles moving along these curved paths emit synchrotron radiation due to this curvature. This emission mechanism is known as curvature radiation. However, charged particles with high Lorentz factors primarily produce high-energy gamma rays [20]. This raises the question: how is radio emission produced?

The intense magnetic field of the pulsar facilitates pair production from these high-energy gamma rays [21]. Pair production can occur in multiple stages, creating cascades of electron-positron pairs. This process reduces both the Lorentz factor of the particles and the energy of the emitted radiation, shifting it into the radio frequency range [22]. This suggests that high-frequency radio emission originates closer to the surface of the pulsar, which helps explain the profile broadening of pulsar emission at radio frequencies—that is, the pulse width decreases with increasing observation frequency [23].

An expression for the radio emission height ($h_{v,em}$) as a function of the emission frequency is given by Kijak and Gil [24, 25]:

$$h_{v,em} = 400 \text{ km} \left(\frac{\nu}{10^9} \right)^{-0.26} \left(\frac{\dot{P}_s}{10^{-15}} \right)^{0.07} \left(\frac{P_s}{s} \right)^{0.30}, \quad (1.10)$$

where, ν is measured in Hz. However, as charged particles approach the light cylinder, the magnetic field strength decreases and is no longer sufficient to constrain the particles strictly along the field lines. In this region, particles can acquire significant transverse motion and emit synchrotron radiation. It is believed that synchrotron emission from

near the light cylinder contributes to the observed high-energy radiation from pulsars, including gamma rays, optical emission, and, in some cases, X-rays [26].

This model of pulsars highlights the importance of the avalanche process of pair production for a pulsar to be radio-active. This avalanche process depends on both the strength of the magnetic field and the rotation period. Therefore, for a neutron star to function as a pulsar, it must possess a certain magnetic field strength and a specific spin period. This leads to the theoretical model of the pulsar death line [27].

1.3 Pulsar emission geometry

1.3.1 The Lighthouse Model of Pulsar Emission

Soon after the discovery of pulsars, Radhakrishnan and Cooke [28] first proposed an emission model for pulsars that is widely accepted to this day. This model is known as the lighthouse model of pulsars. In this model, charged particles moving along the magnetic field lines emit radiation tangential to the field lines due to their high Lorentz factor. Given the dipolar structure of the magnetic field lines, the entire emission from the pulsar emerges in the shape of a cone, with the magnetic axis serving as the axis of this cone. Due to the high polarization of the emitted radiation from charged particles, only the emission aligned with the observer's line of sight (LoS) can be detected.

Fig. 1.1 shows a schematic diagram of the geometry of a pulsar and its beam at an arbitrary time t . The coordinates X_I, Y_I, Z_I (known as the 'I-frame') define a Cartesian coordinate system where the pulsar's spin axis aligns with the Z_I direction, and the magnetic axis rotates around it with a period P_s , making a constant angle α . As illustrated in Fig. 1.1, $\hat{m}_I$ is a unit vector along the magnetic axis, which also serves as the axis of the beam with a half-opening angle of $\mathscr{W}$. The trajectory of $\hat{m}_I$ around the Z_I -axis is indicated by the dashed line marked as B in the figure. All light rays within the beam rotate around

the Z_I axis with the period P_s . An arbitrary light ray with a unit vector $\hat{n}_I$ is shown in Fig. 1.1, and its trajectory is denoted as A. The azimuthal angle of the magnetic axis, i.e., the angle between the projection of $\hat{m}_I$ on the $X_I Y_I$ plane and the X_I axis, is denoted as Φ_m . The azimuthal angle is also referred to as the ‘longitude’ or sometimes as the ‘phase’. For numerical calculations, we assume that at $t = 0$, $\hat{m}_I$ lies in the $X_I Z_I$ plane. Thus, we can express $\Phi_m(t) = \frac{2\pi}{P_s}t$.

The unit vector along the Line-of-Sight (LoS) is denoted by $\hat{N}_I$. This vector lies in the $Z_I X_I$ plane and makes an angle ζ_L with the Z_I axis (co-latitude), as shown in the figure. We define the positive direction of the LoS as pointing from the pulsar to the observer. The angle between the LoS and $\hat{m}_I$ is denoted by Γ . For a particular pulsar, the angles α and $\mathscr{W}$ remain constant over time, while Γ and Φ_m vary. The angle ζ_L can vary very slowly due to the free precession of the pulsar caused by its ellipticity [29]. However, this effect is ignored in this context. Therefore, the magnetic axis vector $\hat{m}_I$, the LoS vector $\hat{N}_I$, and an arbitrary light ray $\hat{n}_I$ in the I-frame can be expressed as:

$$\hat{m}_I = [\cos \Phi_m \sin \alpha, \sin \Phi_m \sin \alpha, \cos \alpha] \quad (1.11a)$$

$$\hat{N}_I = [\sin \zeta_L, 0, \cos \zeta_L] \quad (1.11b)$$

$$\hat{n}_I = [\cos \Phi_p \sin \zeta_p, \sin \Phi_p \sin \zeta_p, \cos \zeta_p] , \quad (1.11c)$$

where Φ_p and ζ_p represent the azimuthal angle and co-latitude of an arbitrary light ray in the beam, as shown in Fig. 1.1.

From Fig. 1.1, it is evident that the pulsar will be observable only if some light rays within the beam satisfy the condition $|\alpha - \mathscr{W}| \leq |\zeta_L| \leq |\alpha + \mathscr{W}|$. The duration of observability, or the width of the pulse, is determined by the time for which the LoS remains within the beam, which is given by the condition $\Gamma \leq \mathscr{W}$.

An additional frame is shown in Fig. 1.1, namely, the $X_m Y_m Z_m$ frame, or the ‘m-frame’.

1.3.2 Pulsar beam model

Each pulsar has a unique pulse profile, much like a fingerprint, and these profiles can consist of multiple components. In the dipolar model of a pulsar, the curvature of the magnetic field lines decreases as one approaches the magnetic poles. As a result, the curvature radiation is likely to be more intense at the outer edge of the conal emission, farther from the magnetic axis. This observation led to the concept of the hollow cone model of the pulsar beam [31], which was initially proposed by Radhakrishnan and Cooke [28]. The hollow cone model can explain single and double-component pulse profiles depending on the trajectory of the line of sight (LoS) through the beam. However, many pulsars exhibit pulse profiles with more than two components.

To account for triple-component profiles, Backer [32] introduced a pencil beam at the center of the pulsar beam. Nevertheless, some pulsars display pulse profiles with four or even five components. To explain these complex profiles, Rankin [5] proposed the core-double cone model. In this model, there is a core component centered around the magnetic axis, while the conal component splits into two parts: the inner cone and the outer cone. However, this simple core-double cone model fails to reproduce more complex profiles, which can be generated using alternative models such as the patchy-beam model [33] or the fan-beam model [34]. Despite this, to date, no beam model can fully explain the beam structure of a pulsar, and the mechanism of radio emission is not yet fully understood.

A beam toy model

In this work, we adopt the simple core-double cone model. To construct these components, it is necessary to determine the half-opening angles at two key locations for each component: typically at the point where the intensity is at its maximum and at a point where the intensity is a fraction of this maximum. These half-opening angles depend on the pulsar's spin period P_s (in seconds) and the observing frequency ν . Given that 1.4

GHz is one of the most commonly used frequencies for pulsar observations, we follow Gil et al. [35] and use the following expressions for the half-opening angles of different components at $\nu = 1.4$ GHz:

$$\mathscr{W}_{1.4,\text{peak}}^{\text{core}} = 0^\circ, \quad (1.13a)$$

$$\mathscr{W}_{1.4,\text{peak}}^{\text{in}} = 3.7^\circ P_s^{-0.5}, \quad (1.13b)$$

$$\mathscr{W}_{1.4,\text{peak}}^{\text{out}} = 4.6^\circ P_s^{-0.5}, \quad (1.13c)$$

$$\mathscr{W}_{1.4,50}^{\text{core}} = 1.2^\circ P_s^{-0.5}, \quad (1.13d)$$

$$\mathscr{W}_{1.4,10}^{\text{in}} = 4.42^\circ P_s^{-0.5}, \quad (1.13e)$$

$$\mathscr{W}_{1.4,10}^{\text{out}} = 5.83^\circ P_s^{-0.5}, \quad (1.13f)$$

Here, $\mathscr{W}_{1.4,\text{peak}}^{\text{core}}$, $\mathscr{W}_{1.4,\text{peak}}^{\text{in}}$, and $\mathscr{W}_{1.4,\text{peak}}^{\text{out}}$ are the half-opening angles corresponding to the locations of maximum intensity for the core, inner cone, and outer cone, respectively. Similarly, $\mathscr{W}_{1.4,10}^{\text{in}}$ and $\mathscr{W}_{1.4,10}^{\text{out}}$ are the half-opening angles corresponding to the locations where the intensity is 10% of the maximum value for the inner and outer cones, respectively. $\mathscr{W}_{1.4,50}^{\text{core}}$ is the half-opening angle corresponding to the location where the intensity is 50% of its maximum value for the core.

The numerical factors and exponents in the above expressions are either taken directly from Gil et al. [35] or slightly adjusted (within error bars) to produce a realistic model of the beam that can result in the observed pulse profiles for various pulsars.

For any given half-opening angle $\mathscr{W}$, the corresponding radius $\mathscr{R}$ on the cross section of the beam at a height h from the centre of the pulsar is given by:

$$\mathscr{R} = h \tan \mathscr{W}. \quad (1.14)$$

For simplicity we assume that the intensity distribution for each of component of the beam is Gaussian on the cross-section of the beam, given by:

$$I(r) = I_0 \exp \left(-\frac{(r-\mu)^2}{2\sigma^2} \right), \quad (1.15)$$

where μ is the mean and σ is the standard deviation of the Gaussian distribution, r is the radial coordinate on the cross-section of the beam, and I_0 is a multiplicative factor, whose unit determines the unit of the intensity. The values of μ for a particular component is chosen equal to the value of the radius where the intensity is the maximum, obtained from Eqs. (1.13) and (1.14). Hence, the value of μ for the core component is zero. The value of σ for a particular component is chosen in such a way that the value of r at which the intensity would fall $\beta\%$ of the maximum intensity would match with the value $\mathcal{R}_\beta$ obtained from Eqs. (1.13) and (1.14). The values of I_0 can be chosen the same or different for the three components. This will lead to different shapes of the pulse profile. After getting the analytical intensity distributions of all three components, we generated a number of light rays on the cross-section of the beam whose surface density match with these distributions. The details of this numerical construction has been discussed below.

We first calculate the emission height $h_{1.4,\text{em}}$ at $\nu = 1.4$ GHz using Eq. (1.10) and the values of $P_s = 1.67$ s and $\dot{P}_s = 1.38 \times 10^{-15}$ ss $^{-1}$. The $X_m Y_m$ plane, on which we create the intensity distributions, is defined at $h_{1.4,\text{em}}$. Then, using Eq. (1.14), we calculate the values of $\mathcal{R}_{1.4,\text{peak}}^{\text{core}} = 0$, $\mathcal{R}_{1.4,\text{peak}}^{\text{in}}$, $\mathcal{R}_{1.4,\text{peak}}^{\text{out}}$, $\mathcal{R}_{1.4,50}^{\text{core}}$, $\mathcal{R}_{1.4,10}^{\text{in}}$, and $\mathcal{R}_{1.4,10}^{\text{out}}$ with the help of the values of the corresponding half-opening angles obtained from Eqs. (1.13) and $h_{1.4,\text{em}}$. Here $\mathcal{R}_{1.4,\text{peak}}^{\text{core}}$, $\mathcal{R}_{1.4,\text{peak}}^{\text{in}}$, and $\mathcal{R}_{1.4,\text{peak}}^{\text{out}}$ are the radii of the beam corresponding to the peak of core, the inner cone, and the outer cone, respectively. Similarly, $\mathcal{R}_{1.4,50}^{\text{core}}$ is the radius of the beam corresponding to the 50% of the maximum intensity of the core, $\mathcal{R}_{1.4,10}^{\text{in}}$ and $\mathcal{R}_{1.4,10}^{\text{out}}$ are the radii of the beam corresponding to the 10% of the maximum intensities for the inner and the outer cones, respectively.

Now, for each component of the beam, we select the mean and the standard deviation of

the Gaussian intensity distribution (see Eq. (1.15)) in such a way that $\mu_{\text{core}} = \mathcal{R}_{1.4,\text{peak}}^{\text{core}} = 0$, $\mu_{\text{in}} = \mathcal{R}_{1.4,\text{peak}}^{\text{in}}$ and $\mu_{\text{out}} = \mathcal{R}_{1.4,\text{peak}}^{\text{out}}$. Here μ_{core} , μ_{in} , and μ_{out} represent the mean of the intensity distributions of the core, the inner cone, and the outer cone, respectively. The standard deviations of the intensity distributions of these components are denoted by σ_{core} , σ_{in} , and σ_{out} , respectively. We choose the values of $\sigma_{\text{core}} = 6.018$, $\sigma_{\text{in}} = 1.987$, and $\sigma_{\text{out}} = 3.401$ in such a way that the values of $\mathcal{R}_{1.4,50}^{\text{core}}$, $\mathcal{R}_{1.4,10}^{\text{in}}$, and $\mathcal{R}_{1.4,10}^{\text{out}}$ of the intensity distributions match with those obtained using Eqs. (1.13), (1.10), and (1.14).

The choice of the values of the multiplicative constant in Eq. (1.15) gives the relative strengths of different components. In this work, we choose $I_0^{\text{core}} = 300 \text{ km}^{-2}$, $I_0^{\text{in}} = 500 \text{ km}^{-2}$, and $I_0^{\text{out}} = 700 \text{ km}^{-2}$, where I_0^{core} , I_0^{in} , and I_0^{out} denote the constant for the core, the inner cone, and the outer cone, respectively. Instead of a physical unit for the intensity, we have chosen its unit as area inverse so that the multiplication of the intensity by an area will give a dimensionless quantity that can be taken proportional to the number of the light rays in that area. We have not modelled the scattering tail and the noise in the profiles, as getting realistic profiles is not our aim, we want to demonstrate the effect of the light bending using model profiles.

In our model, the intensity of the core falls off very rapidly, it becomes 10% of the maximum value at the radius of 12.91 km on the $X_m Y_m$ plane, before the intensity of the inner cone becomes significant - the intensity of the inner cone becomes 10% of the maximum value only at the radius of 17.6 km, then reaching the peak at the radius of 21.86 km, it again falls off to 10% of the maximum value at the radius of 26.12 km. However, the intensity of the outer cone becomes significant even before this, it reaches 10% of the maximum value at the radius of 19.89 km. Hence, there is a significantly large region where the intensities of both of the cone components are significantly large and it becomes difficult to identify the components. To avoid this, we truncate the inner core outwards and the outer cone inwards at the radius of 25.22 km where the intensities of these two components are equal.

After getting the intensity distributions for all three components of the beam as described above, we generate a number of starting points of the light rays on the $X_m Y_m$ plane in such a way that the surface densities of these points match the intensity distributions of the three components. To achieve this, first, near the centre of the cross-section, where only the core component is important, the number of light rays (N_{cent}) in a small area of radius $\mathcal{R}_{\text{cent}}$ around the centre is taken as the value of $I_0^{\text{core}} \pi \mathcal{R}_{\text{cent}}^2$, rounded off to the nearest integer. We choose the value of $\mathcal{R}_{\text{cent}}$ as 0.05 km. We assume that in this area, these N_{cent} light rays are uniformly distributed in r and θ , where r and θ are the polar coordinates on the $X_m Y_m$ plane. Outside this central region, we proceed as follows. At a radial position r , we calculate the intensity $I(r)$ for each of the components of the beam following the distributions discussed earlier, and choose a small annular region of width $dr = 0.01$ km around r . The area of this annular region is $\mathcal{A} = 2\pi r dr$ and the number of light rays N_{ray} in that annular region is obtained by rounding off $I(r)\mathcal{A}$ to the nearest integer. We calculate the number of light rays (or more specifically, the starting point of the light rays) for each of the components separately and distribute those over θ with a uniform distribution. We note that for each value of r , there is always a non-zero value of $I(r)$ for all three components of the beam. However, when the value of $I(r)$ is so small that the value of $I(r)\mathcal{A}$ of a particular component is less than 0.5, we don't have any contribution from that component in the ray distribution. We also emphasise the fact that, these numbers are not the actual numbers of the light rays that the beam has in reality. The surface density of the light rays generated in this way just corresponds to the intensity distribution on the cross-section of the beam as per the model we consider.

To visualise the intensity distribution on the cross-section of the beam, instead of using the values of $I(r)$ obtained from Eq. (1.15), we use the surface probability density of the starting points of the light rays. This is because, after bending, the intensity distribution on the cross section of the beam would be different, which we do not know a priori. To get the surface probability density of the starting points of the light rays, we first convert the polar coordinates r, θ of each of these points to Cartesian coordinates on the $X_m Y_m$ plane.

Then, we calculate the normalised surface density or the surface probability density of the light rays in this plane. To do this, we use a two-dimensional histogram with 100 bins in each dimension and calculate the probability density of the light rays in each bin as (number of rays in the bin)/(total number of rays $\times$ bin area). In Fig. 1.2, the colour map represent the surface probability density obtained in this manner and the abscissa and the ordinate correspond to the values of the Cartesian coordinates in the $X_m Y_m$ plane. This beam model, along with the choice of $\alpha = \zeta_L$ (which ensures that the LoS cuts the beam through its midpoint), produces five-component pulse profiles in which the outer components are the strongest and the core component is the weakest. These profiles would be close to the observed profiles of PSR B1237+25, PSR J1831–04, PSR 2224+65, and PSR B1737–30, respectively, as reported by Seiradakis et al. [36]. We have not modelled the scattering tail and the noise in the profiles, as getting realistic profiles is not our aim, we want to demonstrate the effect of the light bending using model profiles. In Fig. 1.2, the magenta diamonds show the trajectory of the LoS across the beam and the red line is the circle of radius $\mathcal{R}_{1.4,10}^{\text{out}}$. We note that in the schematic diagram of Fig. 1.1, we showed $\mathcal{R}_{1.4,10}^{\text{out}}$ as $\mathcal{R}_{\text{beam}}$.

Pulse profile

As previously mentioned, pulsar emission is highly polarized, and as a result, we do not observe the full beam. Instead, we only see the intensity distribution along the trajectory of the line of sight (LoS) as it crosses the beam, which we receive as a pulse profile. However, the signal we receive from a pulsar is not 100% polarized. To obtain a pulse profile from the distribution of light rays on the cross-section of the beam in our model, we employ a simple technique.

We divide the pulsar spin period P_s into 20,000 time-steps. For each time-step, we first determine the orientation of the beam in the inertial frame (I-frame), in which the LoS remains fixed. We then consider a very small circular region of radius $\mathcal{R}_{1.4,10}^{\text{out}}/50$ around

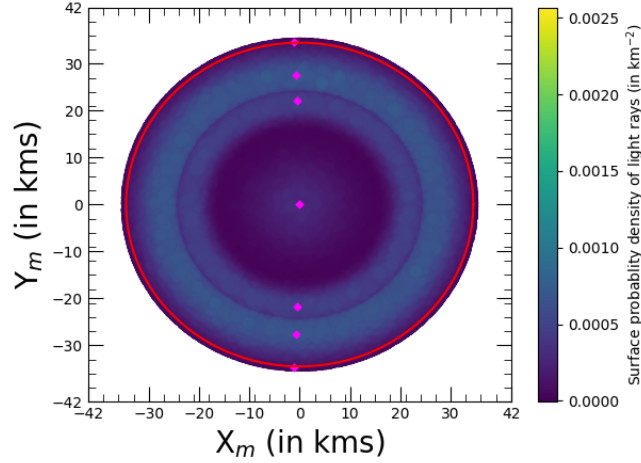


Figure 1.2: The surface probability density of light rays on the $X_m Y_m$ plane, which is the cross-section of the beam from an isolated pulsar. The red circle marks a radius of $\mathcal{R}_{1.4,10}^{\text{out}}$. The pulsar parameters used are $P_s = 1.67$ s and $\dot{P}_s = 1.38 \times 10^{-15}$ ss $^{-1}$. The diamonds ($\diamond$) in magenta indicate the Line of Sight (LoS) positions across the beam for $\alpha = \zeta_L$ at different moments in time. The central point corresponds to time t_{mid} when the LoS passes through the center of the beam. The three points above this represent the LoS at $t_{\text{mid}} - 0.0274$ s, $t_{\text{mid}} - 0.0218$ s, and $t_{\text{mid}} - 0.0174$ s (from top to bottom). Similarly, the three points below the center are for $t_{\text{mid}} + 0.0174$ s, $t_{\text{mid}} + 0.0218$ s, and $t_{\text{mid}} + 0.0274$ s (from top to bottom).

the LoS on the $X_m Y_m$ plane and check whether any light rays fall within that region. When we detect a non-zero value, it indicates that the LoS has entered the beam. The number of light rays inside that circular region is used as a proxy for the intensity. It is important to note that this intensity is distinct from the surface probability density of the light rays mentioned earlier.

In Fig. 1.3, this intensity is plotted along the ordinate, while the pulse phase is plotted along the abscissa. The pulse phase is defined to vary from 0 to 1, covering a full rotation of the magnetic axis. For visualization purposes, we define the phase so that the peak of the core component of the profile for an isolated pulsar occurs at a phase of 0.5. Instead of plotting the full range of the pulse phase, we display only the range over which the LoS is within the beam.

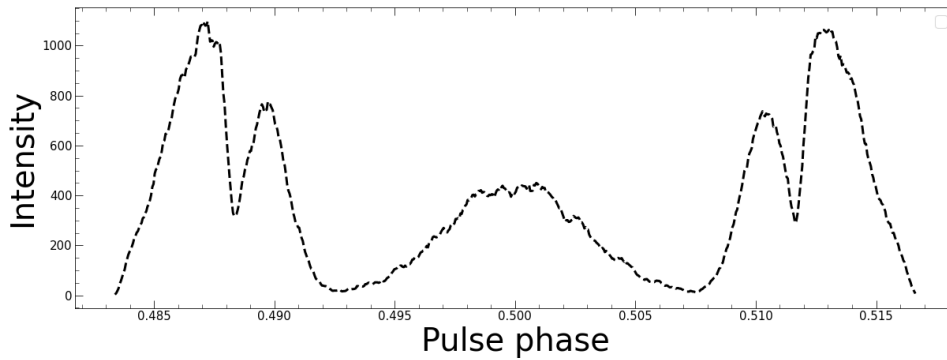


Figure 1.3: Pulse profile for an isolated pulsar based on the core-double cone model.

1.4 Use of the Models Described in This Section

We use the lighthouse model and the core–double-cone structure of the pulsar beam to investigate how different light rays are affected by gravitational bending. This bending introduces what we refer to as the *bending delay*, which will be defined later. These models also help us understand how the intrinsic shape of the observed pulse profile is modified due to the bending of light rays, a topic that will be discussed in the following chapters.

Chapter 2

Binary Pulsars, Bending Delays, and Pulsar Timing

2.1 Formation of Binary Pulsars

To date, nearly 3,724 pulsars are known, and approximately 11% of these are found in binary systems, where at least one component is a pulsar. These are referred to as binary pulsars. The companion star in such systems can be a main sequence star, a white dwarf, a neutron star, or even a black hole. While black hole pulsar binaries are yet to be observed, theoretical studies suggest that such binaries should exist within our Galaxy. Binary pulsars generally exhibit different values for parameters such as magnetic field strength and spin period compared to solitary pulsars, leading to the question of their formation.

Consider a binary system consisting of two massive main sequence stars with masses M_1 and M_2 , where $M_1 \gg M_2$. The more massive star (the primary) will have a shorter main sequence lifetime and eventually expand into a red giant. During this phase, it will fill its Roche lobe, leading to mass transfer from M_1 to M_2 . Assuming a conservative mass transfer, where the mass lost by M_1 is gained by M_2 , let M'_1 and M'_2 be the masses of the

stars after the transfer. The conservation of angular momentum yields:

$$\frac{M_1 M_2}{M'_1 M'_2} = \left(\frac{P_{b2}}{P_{b1}} \right)^{1/3}, \quad (2.1)$$

where P_{b1} and P_{b2} are the orbital periods of the binary before and after the mass transfer, respectively. According to Eq. (2.1), as mass is transferred from the more massive star M_1 to M_2 , the orbit of the binary initially shrinks. If the mass transfer is significant enough such that $M'_1 < M'_2$, the orbit will then begin to expand. After this phase, the system will consist of a binary where one star, with a helium core of mass M'_1 , and the companion star, still in the main sequence, has a mass $M'_2 > M'_1$. This scenario explains the Algol paradox. If M'_1 is sufficiently large, it may eventually become a neutron star (NS) or a black hole (BH) through a supernova explosion. For example, the Galactic high-mass X-ray binary Cyg X-1 represents a black hole—main sequence binary and might eventually evolve into a wide binary [37].

For a spherically symmetric supernova explosion, if we assume the mass loss is instantaneous, the binary can survive if the mass loss is less than half of the total mass of the binary [38], i.e.,

$$\Delta M < \frac{1}{2} (M'_1 + M'_2). \quad (2.2)$$

If the explosion is asymmetric, the kick imparted to the newly formed neutron star or black hole can either disrupt the binary or tighten it, depending on the direction and magnitude of the kick relative to the orbit. If M'_1 is significantly smaller than M'_2 (i.e., most of the mass has been transferred to the star with mass M_2), the binary is more likely to survive the first supernova explosion [39], provided the kick from the supernova is sufficiently small. However, this kick can make the binary wider and more eccentric [40].

Following this, the secondary star, now more massive, will evolve off the main sequence into a giant star. This star will eventually fill its Roche lobe, leading to mass transfer from the secondary star to the first-born neutron star or black hole. Despite the high

magnetic field of the first-born neutron star or black hole, which may drive some matter away, the mass transfer rate is usually high enough for matter to accrete onto the neutron star or black hole. We note that during this accretion process if the end state of the primary star is a neutron star at this stage and accumulates enough matter to exceed the Tolman–Oppenheimer–Volkoff (TOV) limit, it can become a black hole. However, the secondary star will then undergo a supernova explosion. If this explosion is instantaneous and spherically symmetric, the binary can only survive if Eq. (2.2) is satisfied with the current masses of the binary objects. In most cases, the binary will not survive the second supernova, and the secondary star may end as a white dwarf, neutron star, or black hole. Both objects will then separate and move away with high velocities. However, if the supernova is asymmetric with minimal mass loss and a small kick, a binary system with one candidate being the first-born neutron star or black hole and the other being a white dwarf, neutron star, or black hole could form. If any one of the binary components which act as a pulsar, then we call it a binary pulsar.

The aforementioned discussion provides a general overview of binary pulsar formation. Detailed calculations and simulations are required for a comprehensive understanding. The rarity of neutron star–black hole binaries explains why they are challenging to detect. Nonetheless, theoretical studies predict their existence in the Galaxy [41, 42, 43, 44, 45, 46, 4]. The detection of gravitational waves from the merger of a neutron star and a black hole in a binary supports this prediction [47]. With the advent of sensitive radio telescopes like MeerKAT [48, 49] and FAST [50, 51], as well as future telescopes like SKA [52], the discovery of pulsar-black hole binaries is anticipated. This underscores the importance of understanding the effects of strong field gravity on pulsar data in such systems.

2.1.1 Recycled Pulsars

In the binary formation process, suppose the primary star becomes a neutron star and acts as a pulsar while the secondary star is still in main sequence star. Initially, the spin period

and surface magnetic field strength of this pulsar are expected to be similar to those of an ordinary pulsar, with a spin period on the order of seconds and a surface magnetic field in the range of $10^{12} - 10^{13}$ G. As the binary evolves, the secondary star eventually fills its Roche lobe, leading to mass transfer onto the pulsar. This infalling matter interacts with the pulsar's magnetic field, emitting X-rays. The pulsar's high magnetic field repels the incoming matter, causing the pulsar to spin down, i.e., its spin period increases. The X-ray pulsar GX 1+4 is currently in this state and exhibits rotational spin-down [53]. However, this spin-down phase does not last indefinitely. The accreting matter possesses significant angular momentum, which eventually forces the pulsar to co-rotate with the angular momentum of the accreting matter, leading to a spin-up of the pulsar (i.e., a decrease in its spin period to millisecond timescales). The X-ray binary Cen X-3 (Centaurus X-3), one of the first discovered X-ray pulsars, provides evidence of this angular momentum transfer during the accretion process, as observed in its spin-up behavior [54].

At the core of the pulsar, superconducting protons and superfluid neutrons produce superconducting vortices and Feynman vortices, respectively, which are pinned together. During the initial spin-down phase, as matter accretes from the companion, the Feynman vortices expel the superconducting vortices from the crust, leading to a decay of the magnetic field due to ohmic dissipation [27]. However, this magnetic field decay does not persist indefinitely. As the pulsar enters the spin-up phase, the Feynman vortices begin to regenerate, drawing the remaining superconducting vortices back towards the core. This process explains the lower magnetic field strength of a recycled pulsar compared to an ordinary pulsar.

However, if the secondary star becomes a pulsar after the supernova explosion, it must initially be an ordinary pulsar, characterized by a high magnetic field and a long spin period. The Double Pulsar system PSR J0737-3039 A/B supports this claim. It is important to note that recycled pulsars do not necessarily have to remain in a binary system. As discussed, the probability of a binary surviving the second supernova explosion is quite

low, often leading to the disruption of the binary. Consequently, recycled pulsars may be ejected into space at high velocities.

Recycled pulsars are classified into two categories based on their spin period and magnetic field. If the secondary star is sufficiently massive (leading to a neutron star or black hole as the end product), the mass transfer period is brief, resulting in the transfer of a smaller amount of angular momentum and a lesser degree of magnetic field decay. This scenario leads to the formation of mildly recycled pulsars, with spin periods ranging from 10 to 100 ms and magnetic fields on the order of 10^{10} G. In the case of the Double Pulsar system PSR J0737-3039 A/B, the fast pulsar (pulsar A) is a mildly recycled pulsar with $P_s = 22.7$ ms and the slow pulsar (pulsar B) is a normal pulsar with $P_s = 2.77$ s.

In contrast, if the secondary star is less massive (resulting in a white dwarf as the end product), the mass transfer period is extended, leading to a greater degree of angular momentum transfer and more significant magnetic field decay. This process results in fully recycled pulsars, which have spin periods in the range of 1 to 10 ms and magnetic field strengths on the order of 10^8 G. An example of a fully recycled pulsar is PSR J0437-4715, which is in a binary system with a low-mass white dwarf. This pulsar has a spin period of 5.75 ms.

2.2 Co-ordinate geometry relevant for binary pulsars

In this section, we describe various frames that are useful for studying binary pulsars and, consequently, the bending delay. The transformation relations between these frames are discussed in Appendix A. In Fig. 2.1, $X_s Y_s Z_s$ is a reference frame centered at the barycentre of the binary system, with the Z_s -axis oriented along the negative direction of the line of sight (LoS), i.e., from the observer to the barycentre¹. Hence, the unit vector

¹For a single pulsar, the line of sight (LoS) is defined from the pulsar to the observer. For a binary pulsar, the LoS is defined from the barycentre of the binary system to the observer. However, these two directions are practically the same because, in any reasonable scenario, the observer is located at a distance

along the LoS in the $X_s Y_s Z_s$ frame can be represented as,

$$\hat{N}_s = [0, 0, -1]. \quad (2.3)$$

The X_s -axis is aligned with the line of the ascending node (AN), and the $X_s Y_s$ plane corresponds to the plane of the sky. This $X_s Y_s Z_s$ frame is referred to as the ‘sky-frame’ or the ‘s-frame’.

The $X_b Y_b Z_b$ frame is another reference frame in which the pulsar orbits its companion in the $X_b Y_b$ plane, with the X_b -axis pointing towards the periastron of the orbit. Consequently, the Z_b -axis is aligned with the orbital angular momentum. This $X_b Y_b Z_b$ frame is also known as the ‘b-frame’. If ω and i represent the longitude of the periastron and the inclination of the orbit, respectively, then the angle between the Z_s -axis and the Z_b -axis is i , and the angle between the X_s -axis and the X_b -axis is ω , as illustrated in Fig. 2.1. In Fig. 2.1, the sky plane is depicted in light red, and the orbital plane is shown in light blue.

There are the $X_p Y_p Z_p$ and $X_c Y_c Z_c$ frames (not shown in the figure), centered at the pulsar and its companion, respectively. We refer to these frames as the ‘p-frame’ and the ‘c-frame’. These two frames are connected to the b-frame through parallel shifts along the line joining the pulsar and the companion. Since the b-frame, p-frame, and c-frame are parallel to each other, any direction vector calculated in one frame will be the same in all other frames. Moreover, as already explained in footnote 1, the direction of the LoS is the same in all three frames.

We also assume that the spin axis of the pulsar, which is along the Z_I -axis, makes an angle λ_p with the Z_s -axis, and the projection of the spin axis in the sky plane makes an angle η_p with the X_s -axis. The angle λ_p is related to ζ_L defined in the earlier section as $\lambda_p = 180^\circ - \zeta_L$. The unit vector along the spin axis is denoted by $\hat{S}$, which is physically along the Z_I -axis but can be mathematically shifted parallelly and taken along the Z_I' -axis, as shown in Fig. 2.1. The true anomaly A_T of the pulsar is also depicted in Fig. 2.1. The

much greater than the size of the orbit.

angle $A_T + \omega$ represents the orbital phase of the pulsar. The pulsar is marked with a black star in the figure.

We now require an additional frame to specify the direction of the companion's spin axis (if it exists), similar to the I-frame of the pulsar. We call this frame the bh-frame and centered at the companion². In this frame, the spin axis of the companion lies along the Z_{bh} axis. This frame can be translated to the barycenter of the orbit along the line joining the pulsar and the companion, giving $X'_{\text{bh}} Y'_{\text{bh}} Z'_{\text{bh}}$.

We assume that the spin axis of the companion forms an angle λ_{bh} with the Z_s -axis and that the projection of the spin axis on the sky plane forms an angle η_{bh} with the X_s -axis, as illustrated in Fig. 2.1. These two angles specify the orientation of the X_{bh} and Y_{bh} axes and thus the X'_{bh} and Y'_{bh} axes with respect to the sky frame. From Fig. 2.1, we can write the spin axis of the companion black hole in sky frame (see Appendix A):

$$\hat{S}_{\text{bh}} = [\cos \eta_{\text{bh}} \sin \lambda_{\text{bh}}, \sin \eta_{\text{bh}} \sin \lambda_{\text{bh}}, \cos \lambda_{\text{bh}}], \quad (2.4)$$

We note that the c-frame and bh-frame are both centered at the companion. The Z_{bh} axis aligns with the spin of the companion, while the Z_c axis aligns with the orbital angular momentum of the companion.

To model the orbital evolution as discussed in Sec. 2.3, we need the angle between the spin-axis of the pulsar and the orbital angular momentum, which we define as ξ_p . With the help of geometry, we find a relation between ξ_p with other angles mentioned earlier as:

$$\xi_p = \cos^{-1} (\cos \lambda_p \cos i - \sin \lambda_p \sin \eta_p \sin i). \quad (2.5)$$

²Although the geometry described in this section is generic for a binary pulsar, regardless of the nature of the companion object, the term bh-frame is justified since, throughout this thesis, we consider the companion to be a black hole.

To study the effect of bending, we need to define an additional reference frame. This frame is defined with the help of $\hat{n}_b$, i.e., the unit vector along the initial direction of a light ray (before bending) in the b-frame. We called this frame $X_T Y_T Z_T$ or T-frame. The T-frame is defined in such a way that the Y_T -axis is chosen along the radial position of the pulsar in b-frame, i.e, along $\hat{r}_b$ while the Z_T -axis is oriented perpendicular to the plane formed by $\hat{n}_b$ and $\hat{r}_b$, i.e,

$$\hat{y}_T = \hat{r}_b , \quad (2.6)$$

and

$$\hat{z}_T = (\hat{r}_b \times \hat{n}_b) / |\hat{r}_b \times \hat{n}_b| . \quad (2.7)$$

The construction of the T-frame depends on the initial direction of the light ray and is unique to each of the light rays. The position vector of the pulsar at any given time in the c-frame is denoted by $\vec{r}_c$ and in the b-frame by $\vec{r}_b$. $\hat{r}_c$ and $\hat{r}_b$ are unit vectors along these vectors. Hence, $\hat{r}_c = \hat{r}_b$ as the c-frame and the b-frame are parallel to each other. .

2.3 Modelling the evolution of the orbit and spins

When the gravitational field is not too strong between the binary companions then we can use Keplerian parameters to trace the trajectory of the pulsar in the binary system. In the Keplerian formalism, the binary system with a pulsar of mass M_p and a companion of mass M_c will move in a plane ($X_b Y_b$) around the barycenter of the system in a closed elliptical orbit with eccentricity e and orbital period P_b . Let the pulsar be along the X_b axis at $t = 0$ then the radial distance of the pulsar (r_b) and the companion (r_{co}) from the barycenter at any time t is given by [55]:

$$r_b = a_p \frac{(1 - e^2)}{1 + e \cos A_T [E(t)]} , \quad (2.8a)$$

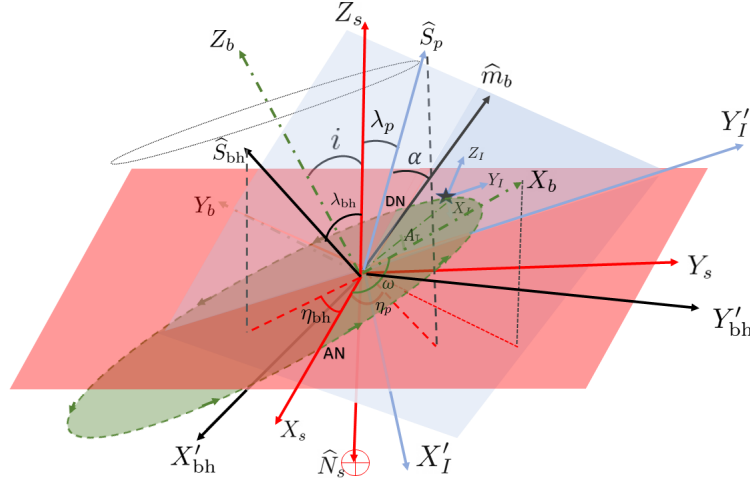


Figure 2.1: The diagram illustrates the orbital geometry of a pulsar (marked by a $\star$) in a binary system. The relevant coordinate axes and angles are described in the text. The sky plane, denoted by the $X_s Y_s$ axes, is shown in light red, while the orbital plane, represented by the $X_b Y_b$ axes, appears in green. The light blue plane corresponds to the $X'_I Y'_I$ axes, which are parallel to the $X_I Y_I$ plane introduced in Sec. 1.3.1. The unit vector $\hat{S}_p$ indicates the spin axis of the pulsar. Although this is physically along the Z_I -axis, it is displayed along the Z'_I -axis here, due to a parallel shift to the center of the $X_b Y_b Z_b$ frame. Similarly, $\hat{S}_{bh}$ denotes the spin axis of the companion black hole. While it lies along the Z_{bh} -axis in reality, it is shown along the shifted Z'_{bh} -axis for clarity. The locations of the ascending and descending nodes are marked as AN and DN, respectively.

$$r_{co} = r_b \frac{M_p}{M_c}, \quad (2.8b)$$

where, a_p is the semimajor axis of the orbit of the pulsar and it depends on the masses of the binary objects given by:

$$a_p = P_b^2 \left(\frac{G(M_p + M_c)}{4\pi^2} \right)^{(1/3)} \times \frac{M_c}{(M_p + M_c)}, \quad (2.9)$$

where G is the Newtonian gravitational constant and $A_T[E(t)]$ is the true anomaly of the pulsar, i.e the angle between the X_b axis and the radius vector $\hat{r}_b$. The true anomaly of the orbit is a function of the eccentric anomaly of the orbit ($E(t)$). The equation that relates the true anomaly and eccentric anomaly with time t is given by [15]:

$$E(t) - e \sin E(t) = \frac{2\pi}{P_b} t, \quad (2.10a)$$

$$A_T[E(t)] = 2 \arctan \left[\sqrt{\frac{1+e}{1-e}} \tan \frac{E(t)}{2} \right]. \quad (2.10b)$$

The distance between the pulsar and the companion is given by:

$$r_c = r_b + r_{co} \quad (2.11)$$

Hence, we write the position vector of the pulsar in the b-frame ($\vec{r}_b$) and c-frame ($\vec{r}_c$), as discussed in Section 2.2, as:

$$\vec{r}_b = r_b [\cos A_T, \sin A_T, 0], \quad (2.12a)$$

$$\vec{r}_c = r_c [\cos A_T, \sin A_T, 0], \quad (2.12b)$$

The position vector of the pulsar in the bh-frame, $\vec{r}_{bh}$ (if the companion has any spin), can be obtained by a coordinate transformation as described in Appendix A:

$$\vec{r}_{bh} = R_y(-\lambda_{bh}) R_z(-\eta_{bh}) R_x(i) R_z(\omega) \vec{r}_c. \quad (2.13)$$

Moreover, the position vector of the pulsar in the s-frame is given by:

$$\vec{r}_s = R_x(i) R_z(\omega) \vec{r}_b, \quad (2.14)$$

where $\vec{r}_s$ is the position vector of the pulsar in the s-frame.

However, the above Keplerian description is insufficient if the gravitational field around the binary is strong. In such cases, the effects of general relativity (GR) manifest as deviations from the Keplerian description. For moderately strong field gravity, as in the cases of binary pulsars, these general relativistic effects can be treated within post-Newtonian (PN) approximation, resulting in five main post-Keplerian (PK) parameters in the timing models, which are, the rate of change of the orbital period ($\dot{P}_b$), the precession of the pe-

riastron ($\dot{\omega}$), the Einstein delay parameter (γ), and the two Shapiro delay parameters (r , s). Among these PK parameters, $\dot{P}_b$ and $\dot{\omega}$ need to be considered while calculating the orbital position of the pulsar in addition to the Keplerian parameters described above.

A relativistic binary emits gravitational waves leading to a decrease in both the orbital size and the eccentricity. We model the rate of change of the orbital period ($\dot{P}_b$) and the rate of change of the eccentricity ($\dot{e}$) up to the first Post-Newtonian order following the standard expressions [56, 15] that are independent of the spin of the pulsar as well as the spin of the companion.

$$\dot{P}_b = -\frac{192\pi}{5} \left(\frac{P_b}{2\pi} \right)^{-5/3} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right) (1-e^2)^{-7/2} T_\odot^{5/3} \frac{M_p M_c}{(M_p + M_c)^{1/3}} \quad (2.15)$$

$$\dot{e} = -\frac{304}{15} \frac{G^3 M_p M_c (M_p + M_c)}{c^5 a^4 (1-e^2)^{5/2}} e \left(1 + \frac{121}{304}e^2 \right) \quad (2.16)$$

Additionally, GR also predicts the precession of the rotational axis of the pulsar and the companion, as well as the Laplace-Runge-Lunze vector as a combined effect of mass and spin (Lens-Thirring or LT effect) of the members of the binary. Either the spin of both of the objects, or only one contribute to the LT effect. For a system like the Double pulsar, the main contribution comes from the spin of the fast pulsar. For a pulsar–black hole binary, the dominant contribution would come from the spin of the black hole. Below we present the discussion for a pulsar–black hole binary.

The precession of the orbit and the spin axes of the pulsar and the companion black hole are affected by their masses, spins, the orbital separation and the eccentricity of the orbit. In the present work, when we include the spin of the companion, we have assumed that the spin vector of the companion is aligned with the orbital angular momentum vector in most cases and hence the spin-precession of the companion black hole can be ignored.

The precession of the spin vector of the pulsar around the orbital angular momentum is $\dot{\vec{\mathcal{J}}}_p = \vec{\Omega}_{\mathcal{J}_p} \times \vec{S}_p$ where $\vec{\mathcal{J}}_p$ is the spin vector of the pulsar and $\vec{\Omega}_{\mathcal{J}_p}$ is the veloc-

ity of precession of the spin vector which is the summation of the PN terms and the Lense-Thirring term, i.e., $\vec{\Omega}_{\mathcal{S}_p} = \vec{\Omega}_{1\text{PN}_p} + \vec{\Omega}_{\text{LT}_p}$ (up to the 1PN order). From Barker and O'Connell [57], the 1PN term can be written as:

$$\vec{\Omega}_{1\text{PN}_p} = A_{1\text{PN}_p} \hat{L}, \quad (2.17)$$

with

$$A_{1\text{PN}_p} = \left(\frac{G}{c^3} \right)^{2/3} \frac{\sigma_b^{2/5}}{(1-e^2)} \frac{M_c(4M_p + 3M_c)}{2(M_p + M_c)^{4/3}}, \quad (2.18)$$

where c is the speed of light in vacuum and $\sigma_b = 2\pi P_b^{-1}$. The Lense-Thirring term is given by:

$$\vec{\Omega}_{\text{LT}_p} = A_{\text{LT}_p} \left[\hat{\mathcal{S}}_{\text{bh}} - 3(\hat{L} \cdot \hat{\mathcal{S}}_{\text{bh}}) \hat{L} \right] = -2A_{\text{LT}_p} \hat{L}, \quad (2.19)$$

where $\hat{L}$ is the unit vector along the orbital angular momentum and $\hat{\mathcal{S}}_{\text{bh}}$ is the unit vector along the spin-axis of the black hole. In the second step of Eq. (2.19), we have used the fact that $\hat{\mathcal{S}}_{\text{bh}} \parallel \hat{L}$. We also know that,

$$A_{\text{LT}_p} = \frac{G}{c^3} \tilde{a} \frac{\sigma_b^2}{(1-e^2)^{3/2}} \frac{M_c^2}{2(M_p + M_c)}. \quad (2.20)$$

where $\tilde{a}$ is the spin parameter of the companion black hole related to its spin angular momentum $\mathcal{S}_{\text{bh}}$ by the relation $\mathcal{S}_{\text{bh}} = \tilde{a} GM_c^2/c$. $\tilde{a}$ takes a positive value for the prograde rotation of the companion (with respect to its orbital rotation) and a negative value for the retrograde rotation of the black hole in such a way that $|\tilde{a}|$ remains in the range 0 to 1.

To investigate the precession of the orbit, we calculate the rate of change of the longitude of the periastron, i.e., $\dot{\omega}$, using the PN term (up to the 2nd order) as well as the Lense-Thirring terms due to the spin of the pulsar as well as the spin of the black hole using the expressions given in Bagchi [58] as $\dot{\omega} = \dot{\omega}_{\text{PN}} + \dot{\omega}_{\text{LT},p} + \dot{\omega}_{\text{LT,bh}}$. The PN term is given by:

$$\dot{\omega}_{\text{PN}} = \frac{3\beta_0^2 \sigma_b}{1-e^2} [1 + \beta_0^2 f_0], \quad (2.21)$$

where

$$\beta_0 = \frac{(GM_T \sigma_b)^{1/3}}{c}, \quad (2.22)$$

with $M_T = M_p + M_c$, and

$$f_0 = \frac{1}{1-e^2} \left(\frac{39}{4} \mathcal{X}_p^2 + \frac{27}{4} \mathcal{X}_c^2 + 15 \mathcal{X}_p \mathcal{X}_c \right) - \left(\frac{13}{4} \mathcal{X}_p^2 + \frac{1}{4} \mathcal{X}_c^2 + \frac{13}{3} \mathcal{X}_p \mathcal{X}_c \right), \quad (2.23)$$

where $\mathcal{X}_p = M_p/M_T$. The Lense-Thirring terms are given by

$$\dot{\omega}_{\text{LT,p}} = -\frac{3\beta_0^3 \sigma_b}{1-e^2} [\beta_{sp} g_{sp}], \quad (2.24a)$$

and

$$\dot{\omega}_{\text{LT,c}} = -\frac{3\beta_0^3 \sigma_b}{1-e^2} [\tilde{a} g_{sc}]. \quad (2.24b)$$

In Eq. (2.24a), we have,

$$\beta_{sp} = \frac{c \mathcal{S}_p}{GM_p^2} = \frac{2\pi c I_p}{GM_p^2}, \quad (2.25)$$

where $\mathcal{S}_p$ is the magnitude of the spin vector of the pulsar, $\vec{\mathcal{S}}_p = 2\pi P_s^{-1} I_p \widehat{\mathcal{S}}_p$ where I_p is the moment of inertia of the pulsar, which depends on its mass, spin and the Equation of State of the matter [59, and references therein] and can be taken as $1.8 \times 10^{45} \text{ g cm}^2$ for a neutron star of mass $1.8 M_\odot$. We also know [58] that,

$$g_{sp} = \frac{\mathcal{X}_p (4\mathcal{X}_p + 3\mathcal{X}_c)}{6(1-e^2)^{1/2} \sin^2 i} \times [(3 \sin^2 i - 1) \cos \xi_p + \cos i \cos \lambda_p]. \quad (2.26)$$

The angles ξ_p and λ_p are defined in Sec. 2.2. With our choice of the spin of the black hole being aligned with the orbital angular momentum, we can write,

$$g_{sc} = \frac{\mathcal{X}_c (4\mathcal{X}_c + 3\mathcal{X}_p)}{3(1-e^2)^{1/2}}. \quad (2.27)$$

The Lense-Thirring effect also causes a change in the orbital inclination angle. As we have taken the spin of the black hole to be perpendicular to the orbital plane, it would not contribute to the change in the orbital inclination. However, the spin of the pulsar would contribute. Following Damour and Schafer [60], we can write,

$$\frac{di}{dt} = \frac{\sigma_b^2}{2(M_p + M_c)c^2} \sum_{j=p, \text{bh}} \vec{\mathcal{J}}_j \cdot \hat{\mathbf{Y}} = \frac{\pi I_p \sigma_b^2}{(M_p + M_c)c^2} \vec{\mathcal{J}}_p \cdot \hat{\mathbf{Y}}, \quad (2.28)$$

where $\hat{\mathbf{Y}}$ is the unit vector pointing toward the ascending node, i.e., along the X_s -axis (see Fig. 2.1). The change in i will result in the change in $\hat{L}$ about which $\hat{S}_p$ precesses.

There are more recent works on the precession of the spin and the orbit, which are more appropriate when a millisecond pulsar orbits a supermassive black hole in a very tight binary. We will discuss those works when we present numerical results of a test case of pulsar - supermassive black hole binary.

The precession of the spin of the pulsar induces changes in the values of the angles λ_p and η_p . The change in the values of these angles produces change in the value of ξ_p through Eq. (2.5), and the changes in λ_p and ξ_p affect the value of $\dot{\omega}_{\text{LT},p}$ through Eqs. (2.24a) and (2.26).

Additionally, the change in the value of λ_p affects the observability condition that the LoS must pass through the cross-section of the beam once for every rotation for a neutron star to be visible as a pulsar. We use the half-opening angle where the intensity of the outer cone becomes 10 per cent of the maximum intensity of this component (at 1.4 GHz), i.e., $\mathcal{W}_{1.4,10}^{\text{out}}$, in the observability condition:

$$|\alpha - \mathcal{W}_{1.4,10}^{\text{out}}| \leq |\zeta_L| \leq |\alpha + \mathcal{W}_{1.4,10}^{\text{out}}|, \quad (2.29)$$

which can also be written as

$$180^\circ - |\mathcal{W}_{1.4,10}^{\text{out}} + \alpha| \leq |\lambda_p| \leq 180^\circ - |\mathcal{W}_{1.4,10}^{\text{out}} - \alpha|, \quad (2.30)$$

as $|\lambda_p| = 180^\circ - |\zeta_L|$.

If we choose the spin of the black hole to be along the orbital angular momentum, there would be no precession of the spin of the black hole and the values of the angles λ_{bh} and η_{bh} would remain constant. This is the configuration we use when we study the effect of the light bending phenomenon for a pulsar with a supermassive black hole companion. This is also the configuration we use in most of our investigations on the effect of the light bending phenomenon for a pulsar with a stellar-mass black hole companion, except a few cases where we investigate the impact of the change in the values of either λ_{bh} or η_{bh} . However, in such cases, the precession of the spin of the black hole is negligibly small.

We ignored the non-relativistic terms in $\dot{P}_b$, $\dot{e}$, $\dot{\omega}$, $\dot{\vec{\mathcal{J}}}_p$, and di/dt . A comprehensive discussions about these terms these can be found in Lorimer and Kramer [15].

2.4 Pulsar Timing

The spinning stability of pulsars, especially, millisecond pulsars with spin periods of 1 to 30 ms and spin period derivatives of about $10^{-19} \text{ s s}^{-1}$ or less, that have been recycled by accreting matter in the past, are extremely stable rotators, i.e., their rotational properties can be predicted precisely. These stable rotators, commonly found in binaries, can be used as tools to study of gravitational physics. An inconsistency in the pulse Times of Arrivals (ToAs) can provide valuable information about the signal propagation path, the orbital motion, and the spin evolution of the pulsar. Some of the most common causes of delays in ToAs are discussed here in short. First, the passage of the light ray through the gravitational field of the companion and the spin and the orbital motion of the pulsar and the companion introduces a delay in the propagation time that we have already discussed. Next, the light ray moves through the Interstellar Medium (ISM) for which we have to introduce Dispersion Measure (DM) correction to the signal. Next, the gravitational field of the Sun and massive planets, as well as the spin and the orbital motion of the Earth,

introduce various delays in the propagation time. Beside these delays, the geometry of pulsar emission, i.e, the lighthouse model of pulsars introduces delays such as the aberration delay and the bending delay that affects ToAs. Each of these effects is discussed in detail below without going into the derivation (see Maggiore [61] and Lorimer and Kramer [15]).

2.4.1 Effect of the Solar System's Gravitational Field

The gravitational field of the Sun and other bodies within the solar system introduces a delay in the propagation time of pulsar signals. However, the dominant effect arises from the Sun's gravitational field, allowing us to approximate by considering only the Sun's contribution. For a nearly static and weak Newtonian source, the spacetime interval ds can be expressed to linear order in terms of a metric perturbation ϕ as:

$$ds^2 = -[1 + 2\phi(\mathbf{x})]c^2 dt^2 + [1 - 2\phi(\mathbf{x})]d\mathbf{x}^2, \quad (2.31)$$

where $(ct, \mathbf{x})$ represent the time and spatial coordinates. Light travels along a path where $ds^2 = 0$, which, to the lowest order in $\phi(\mathbf{x})$, gives:

$$c dt = \pm[1 - 2\phi(\mathbf{x})]|d\mathbf{x}|. \quad (2.32)$$

Let $\vec{R}_p$ denote the fixed position of the pulsar (considered isolated), $\vec{R}_b$ the position of the Solar System Barycenter (SSB), t_e the emission time, and $\vec{R}_{\text{obs}}$ the observer's position at the reception time t_a . Using Eq. (2.32), the time of flight for the signal is:

$$c(t_a - t_e) = \int_{\vec{R}_{\text{obs}}}^{\vec{R}_p} |d\mathbf{x}| [1 - 2\phi(\mathbf{x})]. \quad (2.33)$$

Let $\vec{R}_{\text{ob}}$ be the vector from the observer to the SSB. Then, we can express t_a as:

$$t_a \simeq \left(t_e + \frac{|\vec{R}_p - \vec{R}_b|}{c} \right) + \frac{\vec{R}_{\text{ob}} \cdot \hat{n}}{c} - \frac{2}{c} \int_{\vec{R}_{\text{obs}}}^{\vec{R}_p} |d\mathbf{x}| \phi(\mathbf{x}), \quad (2.34)$$

where $\hat{n} = \frac{\vec{R}_p - \vec{R}_b}{|\vec{R}_p - \vec{R}_b|}$ is the unit vector from the SSB to the pulsar. The term $t_e + \frac{|\vec{R}_p - \vec{R}_b|}{c}$ represents the arrival time of the pulse at the SSB in the absence of the solar system's gravitational field, known as the barycentric arrival time t_{SSB} . Thus,

$$t_{\text{SSB}} = t_a - \frac{\vec{R}_{\text{ob}} \cdot \hat{n}}{c} + \frac{2}{c} \int_{\vec{R}_{\text{obs}}}^{\vec{R}_p} |d\mathbf{x}| \phi(\mathbf{x}). \quad (2.35)$$

The first correction term, $\Delta R_{\odot} = -\frac{\vec{R}_{\text{ob}} \cdot \hat{n}}{c}$, is known as the solar system Rømer delay and arises from the Earth's orbital motion. The second correction, $\Delta s_{\odot} = \frac{2}{c} \int_{\vec{R}_{\text{obs}}}^{\vec{R}_p} |d\mathbf{x}| \phi(\mathbf{x})$, represents the solar system Shapiro delay due to light propagation through the Sun's gravitational field.

For a point at distance r from the Sun, we can approximate $\phi = -\frac{GM_{\odot}}{c^2 r}$. Inserting this into the expression for $\Delta s_{\odot}$, we obtain:

$$\Delta s_{\odot} = -\frac{2GM_{\odot}}{c^3} \ln \left(\frac{2d}{R_{\text{se}}} \right) + \frac{2GM_{\odot}}{c^3} \ln(1 + \cos \theta), \quad (2.36)$$

where d is the Earth-pulsar distance, R_{se} is the Sun-Earth distance, and θ is the angle between the Sun-Earth vector $\vec{R}_{\text{se}}$ and the Earth-pulsar direction vector $\vec{R}_{\text{ep}}$.

The Rømer and Shapiro delays are calculated in the coordinate time t , but the observatory clock measures the pulse arrival in the proper time t_a . This proper arrival time t_a is related to the SSB arrival time t_{SSB} by a correction called the Einstein delay $\Delta \mathcal{E}_{\odot}$, which arises due to both Earth's transverse Doppler shift $-\frac{v_E^2}{2c^2}$ (where v_E is Earth's velocity) and the gravitational redshift produced by the field ϕ . The solar system Einstein delay is:

$$\Delta \mathcal{E}_{\odot} = \int_0^t dt' \left[\frac{v_E^2(t')}{2c^2} - \phi(\mathbf{x}_E(t')) \right], \quad (2.37)$$

where $\mathbf{x}_E$ represents Earth's position.

Thus, the barycentric arrival time t_{SSB} of a pulse measured at time t_a in the Earth's proper frame, considering the solar system gravitational effects, is given by:

$$t_{\text{SSB}} = t_a + \Delta R_{\odot} + \Delta s_{\odot} + \Delta \mathcal{C}_{\odot}. \quad (2.38)$$

2.4.2 Effect of the Interstellar Medium (ISM)

The pulsar signal reaches Earth through the interstellar medium (ISM), which is filled with electrons, dust, and clouds, and exhibits turbulence across a variety of length scales. Due to this medium, the signal's velocity becomes frequency-dependent, given by ([15]):

$$v_g \simeq c \left(1 - \frac{n_e e^2}{2\pi m_e \nu^2} \right), \quad (2.39)$$

where n_e is the electron density along the path, e is the electron charge, m_e is the electron mass, and ν is the signal frequency. This frequency dependence causes dispersion, introducing a delay in the pulsar signal given by:

$$\Delta t_{\text{ISM}} = \frac{\Delta D}{\nu^2}, \quad (2.40)$$

where ν is the frequency of the signal, $\Delta D = D \times \text{DM}$. Here, D and the dispersion measure (DM) are defined as:

$$D = \frac{e^2}{2\pi m_e c}, \quad (2.41a)$$

$$\text{DM} = \int_0^L n_e dl, \quad (2.41b)$$

where L represents the path length of the signal through the ISM.

Thus, after accounting for the DM correction and the corrections due to the solar system's

gravitational field, we can express the timing formula as:

$$t_{\text{SSB}} = t_a - \frac{\Delta D}{v^2} + \Delta R_{\odot} + \Delta s_{\odot} + \Delta \mathcal{E}_{\odot} . \quad (2.42)$$

2.4.3 Effect of companion's gravitational field for a binary pulsar

Similar to the effects of the solar system's gravitational field and the Earth's motion, the gravitational field of a companion star and the orbital motion of a pulsar also introduce delays analogous to the Rømer delay, Shapiro delay, and Einstein delay. In the geometry of the binary pulsar described in Sec. 2.1, we can express the Rømer delay at the Keplerian level and the Shapiro delay at 1PN order as follows [15]:

$$\Delta R = a_p \sin i \left[(\cos E - e) \sin \omega + \sqrt{1 - e^2} \sin E \cos \omega \right] , \quad (2.43)$$

and

$$\Delta s = -2r \ln \left[(1 - e \cos E) - s \left\{ \sin \omega (\cos E - e) + \sqrt{1 - e^2} \cos \omega \sin E \right\} \right] , \quad (2.44)$$

where E is the eccentric anomaly of the pulsar orbit, e is the orbital eccentricity, and $r = \frac{GM_c}{c^3}$ and $s = \sin i$ are the Shapiro shape and range parameters.

The Einstein clock correction is given by:

$$\Delta \mathcal{E} = \gamma \sin E , \quad (2.45)$$

where γ is given by

$$\gamma = e \left(\frac{P_b}{2\pi} \right)^{1/3} \frac{G^{2/3}}{c^2} \frac{M_c (M_p + 2M_c)}{(M_p + M_c)^{4/3}} . \quad (2.46)$$

If the companion's gravitational field is strong enough, these equations for the Rømer delay, Shapiro delay, and Einstein delay must be corrected to higher Post-Newtonian (PN) order. The companion's strong field causes light rays to bend along a curved path, known as light bending. This effect results in an extra travel path, introducing a delay known as the geometric delay, Δ_{geo} . To first order, the geometric delay is given by:

$$\Delta_{\text{geo}} = \frac{2GM_c}{c^3} \left[\frac{|\vec{R}_{\pm} - \vec{R}_s|}{R_E} \right]^2, \quad (2.47)$$

where R_E is the Einstein radius of the companion, defined as [62]:

$$R_E^2 = \frac{4GM_c}{c^2} |\vec{r}_s| \sin i, \quad (2.48a)$$

$$R_E = \left[\left(\frac{4GM_c}{c^2} \right) a_R |\sin i| \frac{1 - e^2}{1 + e \sin \omega} \right]^{1/2}, \quad (2.48b)$$

$\vec{R}_s$ is the projection of $\vec{r}_s$ (see Eq. 2.14) onto the sky plane ($X_s Y_s$), and $\vec{R}_{\pm}$ are the positions of the positive and negative images of the source on the sky plane, given by:

$$\vec{R}_{\pm} = \frac{\vec{R}_s}{2} \left(1 \pm \sqrt{1 + \frac{4R_E^2}{|\vec{R}_s|^2}} \right). \quad (2.49)$$

Next, we consider the aberration effect. Due to the pulsar's orbital motion, the direction of the received signal differs from the emission direction in the pulsar's rest frame. This results in a delay known as the aberration delay, given by:

$$\Delta_{\text{AB}} = A [\sin(\omega + A_T(E)) + e \sin \omega] + B [\cos(\omega + A_T(E)) + e \cos \omega], \quad (2.50)$$

where,

$$A = -\frac{G^{1/3}}{c(2\pi)^{2/3}} \frac{P_s}{P_b^{1/3} \sqrt{1 - e^2}} \frac{M_c}{(M_p + M_c)^{2/3}} \frac{\sin \eta}{\sin \lambda}, \quad (2.51a)$$

$$B = -\frac{G^{1/3}}{c(2\pi)^{2/3}} \frac{P_s}{P_b^{1/3} \sqrt{1-e^2}} \frac{M_c}{(M_p + M_c)^{2/3}} \frac{\cos \eta \cos i}{\sin \lambda}. \quad (2.51b)$$

The bending of the light ray also causes a difference in the angle ($\Delta\theta$) at which the signal is received and at which it is emitted in the pulsar rest frame. This delay is known as the bending delay τ_b . The difference $\Delta\theta$ along the longitudinal direction of the pulsar gives rise to a longitudinal bending delay τ_{long} whereas the difference $\Delta\theta$ along the latitudinal direction gives rise to a latitudinal bending delay τ_{lat} . We discuss it in detail in the next section.

Hence after taking all the effects that we discussed above, an accurate timing formula can be expressed as [15],

$$t_{\text{SSB}} = t_a - \frac{\Delta D}{v^2} + \Delta R_{\odot} + \Delta s_{\odot} + \Delta \mathcal{E}_{\odot} + \Delta R + \Delta s + \Delta \text{geo} + \Delta \mathcal{E} + \Delta \text{AB} + \tau_b. \quad (2.52)$$

Note that, Lorimer and Kramer [15] omitted the bending delay τ_b term in the pulsar timing formula.

2.5 Bending Delay

In binary pulsars, the spacetime curvature around the companion object impacts both the propagation time and the direction of light rays emitted from the pulsar. The alteration in light ray directions contributes to various delays, notably the ‘geometric delay’ and additional delays termed the ‘longitudinal bending delay’ and the ‘latitudinal bending delay’ [2]. These arise because, after bending, a ray’s longitude and latitude will differ from those in an unbent path. If unaccounted for, these bending-induced delays appear as orbital-phase-dependent variations in the timing residuals. Additionally, this bending distorts the observed pulse profile, a combined effect of both longitudinal and latitudinal changes.

The bending effect is strongest at the superior conjunction, where the pulsar is positioned behind the companion, aligning in a straight line with Earth. This occurs when both the orbital inclination angle (i) and the orbital phase ($\omega + A_T$) reach 90° . Nonetheless, bending is detectable across a range of i and $\omega + A_T$ values close to 90° .

Longitudinal Bending Delay

Due to bending, the beam's longitudinal positions shift. Thus, at a given time t , if the light ray along the line of sight (LoS) had longitude (phase) Φ (0 in our choice) in an unbent scenario, its phase will differ due to bending. A different light ray with phase (Φ_{without}) without bending will now be positioned along the LoS after bending. The resulting phase difference is $\Delta\Phi = \Phi_{\text{without}} - \Phi$, producing the longitudinal bending delay:

$$\tau_{\text{long}} = \Delta\Phi \times \frac{P_s}{2\pi} . \quad (2.53)$$

All light rays on the post-bending line-of-sight (LoS) path share the same $\Delta\phi$ value.

Latitudinal Bending Delay

Bending similarly alters ray co-latitudes. For instance, a light ray initially at co-latitude ζ_L may no longer be visible after bending, while another ray at an original co-latitude ζ_{without} now aligns with ζ_L . The change in co-latitude is therefore $\Delta\zeta = \zeta_{\text{without}} - \zeta_L$. Consequently, the beam's shape along the LoS path changes, affecting the half-width of the cut beam (Φ_0). If $\Delta\Phi_0$ represents this change, then the latitudinal bending delay is given by:

$$\tau_{\text{lat}} = \Delta\Phi_0 \times \frac{P_s}{2\pi} . \quad (2.54)$$

The definition of Φ_0 is provided in sec 1.3.1 and illustrated in Fig. 1.1. Rafikov et al. [2] presented $\Delta\Phi_0$ as:

$$\Delta\Phi_0 = \Delta\zeta \left(\frac{1}{\tan\Phi_0 \tan\zeta_L} - \frac{1}{\sin\Phi_0 \tan\alpha} \right), \quad (2.55)$$

and expressed Φ_0 as:

$$\cos\Phi_0 = \frac{\cos\mathscr{W} - \cos\zeta_L \cos\alpha}{\sin\zeta_L \sin\alpha}, \quad (2.56)$$

where $\mathscr{W}$ is the half-opening angle defining the outer boundary of the beam (here, $\mathscr{W} = \mathscr{W}_{1.4,10}^{\text{out}}$). These equations apply only if beam distortion from bending is minimal; otherwise, calculating $\Delta\Phi_0$ numerically is preferred.

2.5.1 Previous Studies on Bending Delay in Binary Pulsars

The first exploration of the bending delay in binary pulsars was performed by Schneider [3], examining edge-on binaries in a weak-field approximation. Schneider [3] referred to this effect as the rotational time delay. Later, Doroshenko et al. [1] extended this work, providing an approximate bending delay formula based on Keplerian and post-Keplerian parameters and the angles η_p and λ_p (see Fig. 2.1). Rafikov et al. [2] pointed out that the analysis of Doroshenko et al. [1] is effective only when the pulsar-companion separation, projected on the sky-plane, exceeds the Einstein radius. Rafikov et al. [2] overcomes this limitation and provides accurate expressions for both longitudinal and latitudinal delays, incorporating the companion mass and Keplerian orbital parameters. Rafikov et al. [2] referred to this effect as the lens-rotational delay. Hu et al. [63] attempted to measure this delay in the Double Pulsar system J0737–3039A/B, referring to it as the deflection delay. All these studies assume a gravitational lensing approach, where light rays initially travel linearly toward the lens plane, bending by an angle proportional to the companion mass and inversely proportional to the impact parameter. This approximation is generally valid near the superior conjunction and when the companion’s gravitational field is not

exceedingly strong. For systems like the Double Pulsar, this method yields timing delay estimates close to fully general relativistic results. For example, Kramer et al. [64] found that in the Double Pulsar (PSR J0737-3039 A/B), the difference between general relativistic and lensing-based longitudinal delay estimates by Doroshenko et al. [1] is around 50 nanoseconds near the superior conjunction—a negligible discrepancy given observational limits.

In this thesis, we study bending delays without these approximations, utilizing a full general relativistic framework for null geodesics, which disregards the motion of the companion during light travel through curved spacetime. Our primary focus is to analyze bending delays and pulse profile distortions for pulsar–nonspinning and pulsar–spinning black hole binaries, using comprehensive general relativistic calculations.

Chapter 3

A Study of the Light Bending

Phenomenon: Pulsar in a Binary With a Schwarzschild Black Hole

In this chapter, we investigate the phenomenon of light bending and the resulting impact on the pulsar's signal in a binary with a Schwarzschild (non-rotating) black hole, within the framework of full general relativity. Specifically, we solve the null geodesic equations in Schwarzschild spacetime to trace the paths of the emitted light rays.

We begin by outlining the method used to solve the geodesic equations in Schwarzschild geometry. It is important to note that all the relevant formalisms and models—such as the beam model, the spin-orbit evolution, and other associated frameworks—have been introduced in the previous chapters and will be employed here without further elaboration.

3.1 General Relativistic Formalism for Bending Delay

To investigate the effect of gravitational bending on pulsar emissions, it is necessary to define an additional reference frame. This frame is established using $\hat{n}_b$, the unit vector along the initial direction of a light ray (before bending) in the b-frame. Since $\hat{n}_b$ varies for each light ray, the associated reference frame is unique for every ray. In this section, we present the formalism for gravitational bending of a single light ray, which applies generally to all rays in the pulsar's emission beam.

In a Schwarzschild spacetime, light rays follow geodesics confined to a plane referred to as the plane of the light ray. That plane necessarily includes the pulsar (as the emission origin) and the companion (as the gravitating body causing the bending). Consequently, the plane of the light ray contains both the vector joining the pulsar and its companion, $\vec{r}_c$, and the initial direction of the ray, $\hat{n}_b$. For this reason, the first reference frame introduced here is the $X_L Y_L Z_L$ frame, centered on the companion. The $X_L Y_L$ plane coincides with the plane of the light ray, while the Z_L axis is specified as $\hat{r}_b \times \hat{n}_b$. This coordinate system, referred to as the L-frame, is illustrated in Fig. 3.1, where the companion is shown as a solid black circle and the pulsar as a solid star. The directions of the X_L and Y_L axes will be explained in a later section. The geodesic equation for a light ray in the Schwarzschild spacetime, expressed in the L-frame, is given by Chandrasekhar [65]:

$$\left(\frac{du}{d\psi}\right)^2 = \frac{2GM_c u^3}{c^2} - u^2 + \frac{1}{D^2}, \quad (3.1)$$

where u is the inverse of the radial distance from the companion, D represents the impact parameter, defined as the perpendicular distance between the initial path of the light ray and the gravitating object., and ψ is the azimuthal angle measured from the X_L -axis. Here, M_c is the companion's mass, G is the gravitational constant, and c is the speed of light in vacuum. Details of solving Eq. (3.1) are provided in Appendix B.

Following Poutanen [66], the impact parameter D can be expressed as:

$$D = \frac{r_L \sin(\chi)}{\sqrt{1 - \frac{2GM_c}{c^2 r_L}}}, \quad (3.2)$$

where $r_L = |\vec{r}_L|$, with $\vec{r}_L$ being the position vector of the pulsar in the L-frame. Since the origins of the L-frame and c-frame coincide, we have $|\vec{r}_L| = |\vec{r}_c|$.

The emission angle χ , measured between $\vec{r}_b$ and $\hat{n}_b$, defines two possible cases, the first being $\pi/2 < \chi \leq \pi$, where the ray initially moves toward the companion, and the second being $0 \leq \chi \leq \pi/2$, where the ray moves away from the companion.

The gravitational bending is significant only for the first case, i.e., when the light ray initially travels towards the companion. Chandrasekhar [65] showed that in such a case, one can get physically acceptable geodesics along which the light ray can travel from a large distance to a point near the gravitating body and then go to an infinite distance only if $D > 3\sqrt{3}GM_c/c^2$ (see Appendix. B). For this condition, he also provided the parametric forms of u and ψ that satisfy Eq. (3.1) in terms of the roots of the right hand side of Eq. (3.1) (see Eq. (B.35) and Eq. (B.36)). We adopt these parametric solutions, as they describe the bending of light rays from a pulsar in a binary system that eventually reach the observer.

It is obvious that the pulsar's position vector in the L-frame, i.e., $\vec{r}_L$, is also the initial (at the time of emission) radius vector of the light ray in the L-frame, i.e., $u_{L,\text{in}} = 1/|\vec{r}_L|$. Substituting $u = u_{L,\text{in}}$ into the parametric expressions of u and ψ , we obtain the initial value of the azimuthal angle $\psi_{L,\text{in}}$. Similarly, by setting $u = 0$ in those parametric expressions, we determine the final value of the azimuthal angle $\psi_{L,\infty}$ (see Eq. B.38).

In the second case, i.e., when the light rays initially travel away from the companion, they can escape the gravitational field of the companion for any value of the impact parameter. Obviously, for this to occur, when the gravitating body is a black hole, the emission point must lie outside the event horizon of the black hole, which is naturally satisfied in a

pulsar–black hole binary. Although the bending is very minimal if the light rays initially travel away from the companion, we still include these cases. However, in such a case, the light rays that reach the observer have $D \leq 3\sqrt{3}GM_c/c^2$ if the inclination orbit is very close to 90° and the pulsar is very close to inferior conjunction, i.e the pulsar companion and the observer lies in a line with pulsar is at the middle of the observer and companion. It is very unlikely that the light rays that reach the observer have $D \leq 3\sqrt{3}GM_c/c^2$. This point can be understood from Fig. 3.3.

Hence, in the present work, we consider only the cases when $D > 3\sqrt{3}GM_c/c^2$ regardless of the value of χ . We note that the older works of Doroshenko et al. [1] and Rafikov et al. [2] are valid only for $D \gg 3\sqrt{3}GM_c/c^2$ and the orbital phase near 90° .

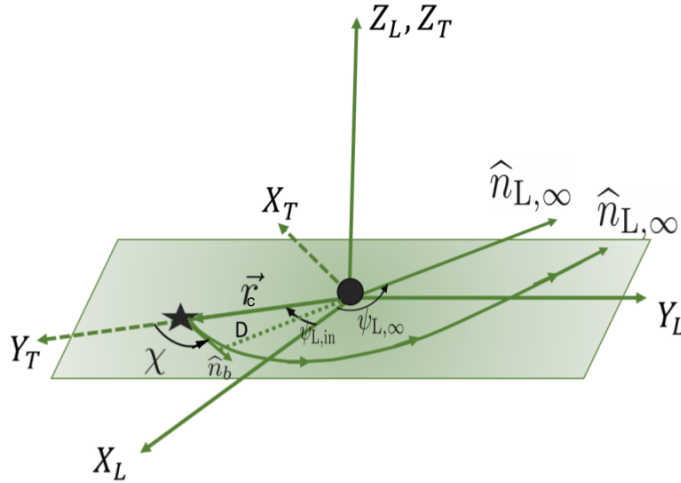


Figure 3.1: A schematic diagram illustrating the bending of a light ray as it travels toward a gravitating object at the time of emission. Here, the source of the light ray is the pulsar and it is represented by a black star, while the gravitating object i.e., the binary companion is shown as a black filled circle. The coordinate axes and angles shown are described in the text.

The final direction of the light ray in the L-frame can be written as $\hat{n}_{L,\infty} = [\cos \psi_{L,\infty}, \sin \psi_{L,\infty}, 0]$. When $\pi/2 < \chi \leq \pi$, we choose $\psi_{L,\text{in}}$ to be negative and when $0 \leq \chi \leq \pi/2$, we choose $\psi_{L,\text{in}}$ to be positive and $\psi_{L,\infty}$ must be positive always. Now, we need to transform $\hat{n}_{L,\infty}$ from the L-frame to the b-frame to estimate the effect of the bending. This transformation seems difficult as we do not know the directions of the X_L -axis and the Y_L -axis with respect to the b-frame or other related frames like the c-frame,

the p-frame, the s-frame, or the I-frame. However this transformation can be done with the help of the T-frame defined in the Sec. 2.2. We choose the Y_T axis along $\vec{r}_c$ and the Z_T -axis along the Z_L -axis.

As we know the directions of the Y_T -axis and the Z_T -axis (Eq. A.10 and A.11), we can find the direction of the X_T -axis, which is along $\hat{y}_T \times \hat{z}_T$, where $\hat{y}_T$ is the unit vector along the Y_T -axis and $\hat{z}_T$ is the unit vector along the Z_T -axis. It is obvious that the $X_L Y_L$ plane and the $X_T Y_T$ plane are the same plane, i.e., the plane of the light ray which contains $\hat{n}_b$ and $\hat{r}_b$ as shown in Fig. 3.1. Fig. 3.1 also shows that the $X_T Y_T Z_T$ frame can be obtained from the $X_L Y_L Z_L$ frame by a rotation around the Z_L -axis by an amount of $-(\pi/2 - \psi_{L,in})$ (we note that $\psi_{L,in}$ is negative in Fig. 3.1). Hence, $\hat{n}_{T,\infty} = R_z(\pi/2 - \psi_{L,in})\hat{n}_{L,\infty}$.

Next, we find a relation between the T-frame and the b-frame to transform $\hat{n}_{T,\infty}$ from the T-frame to the b-frame, i.e., to obtain $\hat{n}_{b,\infty}$. If the angle between the Z_T -axis and the Z_b -axis is θ_{Tb} and the angle between the Y_T -axis and the Y_b -axis is ϕ_{Tb} , then we can write

$$\theta_{Tb} = \cos^{-1}(\hat{z}_T \cdot \hat{z}_b) = \cos^{-1}(\hat{z}_T \cdot [0, 0, 1]) , \quad (3.3)$$

and

$$\phi_{Tb} = \cos^{-1}(\hat{y}_T \cdot \hat{y}_b) = \cos^{-1}(\hat{y}_T \cdot [0, 1, 0]) , \quad (3.4)$$

where the unit vectors $\hat{z}_b$ and $\hat{y}_b$ are represented by $[0, 0, 1]$ and $[0, 1, 0]$, respectively. Using Eqs. (A.14) and (A.15), we can get the values of θ_{Tb} and ϕ_{Tb} as $\hat{z}_T$ and $\hat{y}_T$ are known.

Using these angles, we get the final direction of the light ray in the b-frame as (see Eq. (A.18)):

$$\hat{n}_{b,\infty} = R_z(\phi_{Tb})R_y(\theta_{Tb})\hat{n}_{T,\infty} . \quad (3.5)$$

To estimate the values of the bending delays, first, we note a particular time, at which, in the absence of the bending, a light ray is aligned with $\hat{N}_I$, i.e., with the LoS. As obvious from Fig. 1.1, at this time, the phase of the light ray is zero and the co-latitude is ζ_L as it

is aligned with $\hat{N}_l$. Now, we identify the light ray which would have the phase zero and the co-latitude ζ_L at that particular time in the presence of bending. We do this with the help of the final directions of the light rays after bending, i.e., $\hat{n}_{b,\infty}$ as described above.

After this, we note what was the phase and the co-latitude of this particular light ray at that particular time when the bending was absent. If these values were $\Phi_{p,\text{old}}$ and ζ_{old} , then we can write $\Delta\Phi = \Phi_{p,\text{old}}$ and $\Delta\zeta = \zeta_{\text{old}} - \zeta_L$. We use this value of $\Delta\Phi$ in the expression of the longitudinal bending delay (Eq. (2.53)). The value of $\Delta\zeta$ can be used in Eq. (2.55) to get the value of $\Delta\Phi_0$ (in addition to Eq. (2.56)). This $\Delta\Phi_0$ can be used in Eq. (2.54) to estimate the value the latitudinal delay. However, as we mentioned earlier in the last paragraph of Sec. 2.5, we calculate $\Delta\Phi_0$ numerically whenever possible. The values of $\Delta\Phi_0$ found numerically match with the values obtained using Eqs. (2.55) and (2.56) unless the bending is very strong, i.e., near the superior conjunction.

We note that as the bending is not extremely strong in our case, the light ray that gets aligned with the LoS after bending is not very far from the light ray which gets aligned with the LoS without bending. Hence, to calculate the bending delays, we check only the light rays within a small area around the original ray.

3.2 Distortion of the beam and the pulse shape

As we mentioned in Sec. 1.3.2, we generate a number of starting points of the light rays on the $X_m Y_m$ plane following the core-double cone intensity distributions. From the coordinates of these starting points, we know their position vectors in the $X_m Y_m$ plane (see Fig. 1.1 and Section 1.3.1 for the definition of the $X_m Y_m$ plane). Let us consider for any particular light ray the position vector in the $X_m Y_m$ plane is given by $\vec{\mathcal{R}}_m$, then the unit vector along the light ray in the m-frame is given by:

$$\hat{n}_m = (h_{v,\text{em}}\hat{z}_m + \vec{\mathcal{R}}_m)(|h_{v,\text{em}}\hat{z}_m + \vec{\mathcal{R}}_m|)^{-1}. \quad (3.6)$$

The vector $\vec{\mathcal{R}}_m$ can be obtained if the coordinates of the light rays in the $X_m Y_m$ plane are known and vice versa. Using $\vec{\mathcal{R}}_m$, we find $\hat{n}_m$ using Eq. (3.6) and then from $\hat{n}_m$, we get $\hat{n}_I$ using the relation

$$\hat{n}_I = R_z(\Phi_m)R_y(\alpha)\hat{n}_m . \quad (3.7)$$

We note that in Eq. (3.6), $\hat{n}_m$ is defined at the centre of the pulsar (where $\hat{n}_I$ is also defined) not at the height $h_{v,em}$, where the light rays are generated.

Using Eq. (A.9), we get the initial direction $\hat{n}_b$ of each of the light rays as:

$$\hat{n}_b = R_z(-\omega)R_x(-i)R_z(\eta_p)R_y(\lambda_p)\hat{n}_I . \quad (3.8)$$

Then, using the formalism described in Sec. 3.1 , we get the final directions $\hat{n}_{b,\infty}$ for each of them. Then we convert each of these $\hat{n}_{b,\infty}$ to $\hat{n}_{m,\infty}$ using the inverse transformations of Eqs. (3.8) and (3.7). From $\hat{n}_{m,\infty}$, we get the position vectors and the coordinates on the $X_m Y_m$ plane of the apparent starting points of these bended light rays using Eq. (3.6). The surface density of these apparent starting points give the intensity distribution on the cross-section of the beam after bending.

We note that the initial direction of the light rays change continuously as the direction of the $X_m Y_m$ plane changes continuously with respect to the I-frame as the beam rotates about the Z_I -axis. As a result, the intensity distribution on the cross section changes rapidly within a spin period if the bending effect is strong enough (as near the superior conjunction).

3.3 Numerical study of bending delay

3.3.1 Features of the bending delay: demonstration for the Double Pulsar

To test our formalism, we first calculate the bending delays for the Double Pulsar, PSR J0737-3039A/B. Here we calculate the bending delay of the signal of the fast pulsar (pulsar A) due to the slow pulsar (pulsar B). We compare our results with the ones obtained by approximate equations of Doroshenko et al. [1] and Rafikov et al. [2]. In Fig. 3.2, we plot the bending delays in microseconds along the abscissa and the orbital phase $\omega + A_T$ in degrees along the ordinate. In this figure, the dashed lines are for the expression given by Doroshenko et al. [1], the solid lines are for the expressions given by Rafikov et al. [2] and the circles and triangles are the results obtained by our method. We note that Doroshenko et al. [1] provided the expression only for the longitudinal bending delay. Instead of the full orbit, we plot the bending delay only around the orbital phase of 90° . In this plot, instead of updated parameters, we have used the same values of the parameters for the Double Pulsar the same as those used by Rafikov et al. [2] for the sake of comparison. Like them, we have also used two different values of the inclination angle, $i = 90.56^\circ$ (magenta lines and circles) and $i = 90.28^\circ$ (green lines and triangles). We see exact agreements between the results obtained by our method and the ones obtained from the expressions of Rafikov et al. [2]. However, a difference between the results obtained by our method and the ones obtained from the expression of Doroshenko et al. [1] is visible for $i = 90.28^\circ$, and the difference is very small when $i = 90.56^\circ$. We have also checked (but not plotted) that if we use the updated parameters for the Double Pulsar as reported by [64], the results of all three expressions agree. The main reason is the fact that [64] obtained 89.35° (or 90.65°), which is further from 90° . These facts validate the correctness of our method, but the question remains whether the expressions of Rafikov et al. [2] are always applicable.

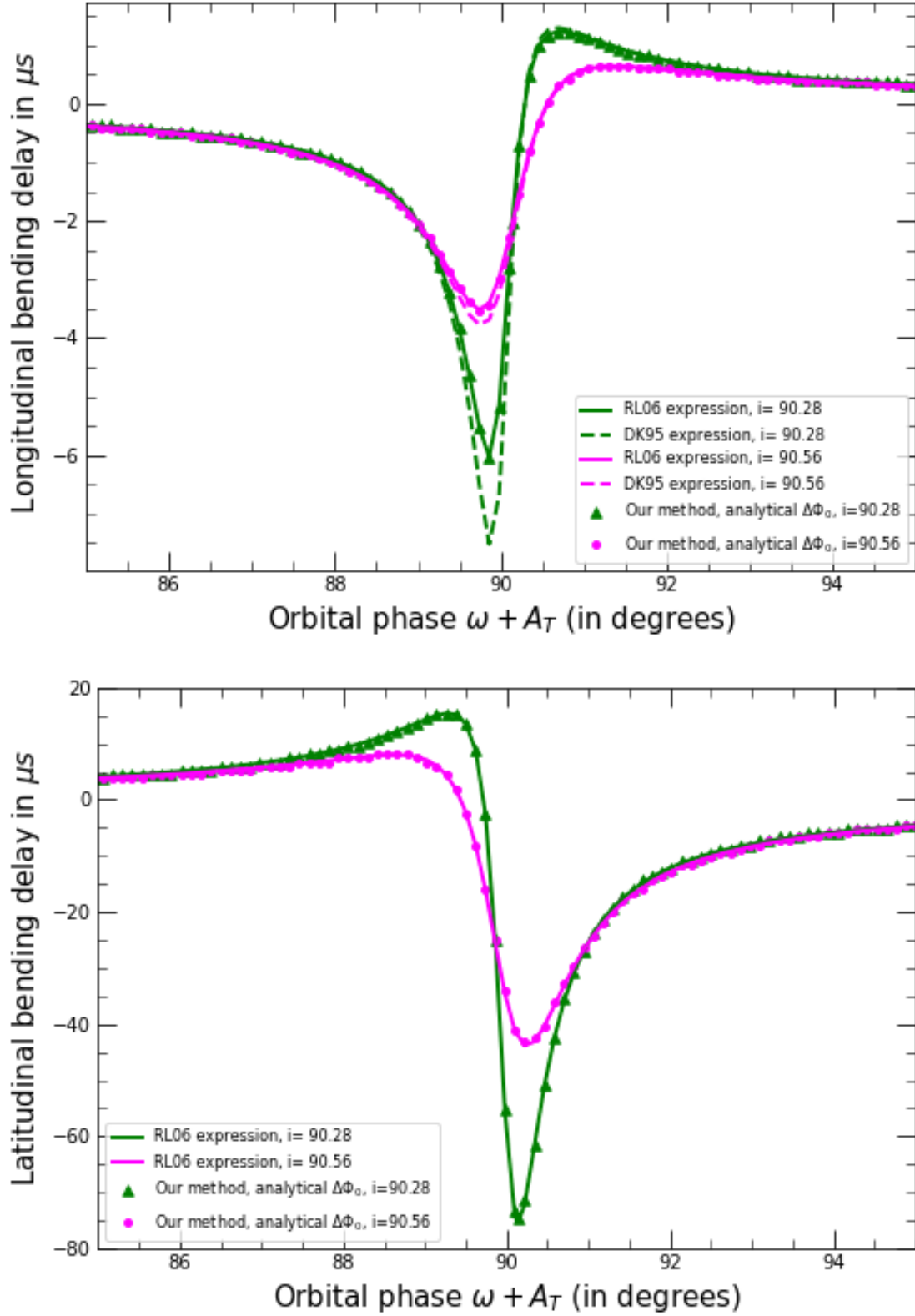


Figure 3.2: Bending delays in the signal of PSR J0737–3039A calculated using different methods. The right panel shows the latitudinal bending delay while the left panel shows the longitudinal bending delay. To calculate the values of the latitudinal bending delay in each formalism, we have calculated $\Delta\Phi_0$ using Eqs. (2.56) and (2.55). In the figure legends, RL06 stands for Rafikov et al. [2] and DK95 stands for Doroshenko et al. [1].

One interesting aspect of bending is revealed in our formalism. It is the fact that the longitudinal bending delay curve would have an irregular region when both the inclination angle and the orbital phase have their values close to 90° . In this irregular region, the bending delay has one or more sudden huge jumps or jumps in its value with the change in the value of the orbital phase.

The physical reason for this irregularity is the fact that in such a case, the rays emitted along different directions from a circular region can reach the observer as shown by the blue circle for the position O_0 of the pulsar in the schematic diagram in Fig. 3.3. This phenomenon makes the longitudinal bending delay curve irregular. We note that in Fig. 3.3, three possible orbital positions of the pulsar, O_{-1} , O_0 , and O_1 are marked. For each of these orbital positions, some of the rays emitted by the pulsar at a particular instant, i.e., at a particular rotational phase are shown with arrows. Here, we need to remember that at any given rotational phase, light rays from the full beam of the pulsar do not reach the observer, only one of the rays that is aligned with the LoS reaches the observer. In Fig. 3.3, initial directions of a few of the light rays are shown with magenta arrows and the green arrows are the directions of the corresponding light rays after bending. Among these, only the rays indicated by solid green lines reach the observer, the rays indicated by dashed green lines do not reach the observer. The latitudinal bending delay curve would also show an irregular region when both of the inclination angle and the orbital phase have their values close to 90° . However, the reason for that is the distortion of the shape of the beam as will be demonstrated in Sec. 3.3.3.

The exact value of $\omega + A_T$ at which the delay curves become irregular depends on the value of i . As an example, in Fig. 3.4, we plot the bending delay for PSR J0737–3039A by setting $i = 90.14^\circ$ and see that the bending delay curves become irregular between $\omega + A_T \simeq 89.9^\circ$ and $A_T + \omega \simeq 90.1^\circ$. Such features would be absent in the approximate formalism given by Rafikov et al. [2] or Doroshenko et al. [1].

If either of the inclination angle or the orbital phase (or both) is sufficiently different than

90° , the observer will see the light ray from only one particular direction of emission that depends on the values of these parameters. Hence, there would not be any irregular region in the bending delay curves as seen in Fig. 3.2. O_{-1} and O_1 are two such positions in Fig. 3.3. We note that at these two positions, the light rays that reach the observer were initially moving away from the companion, and hence those experience weak bending.

We need to remember the fact that, for the Double Pulsar, if we use the values of $\alpha = 4^\circ$ and $\zeta_L = 50^\circ$ as given by Rafikov et al. [2] and calculate the value of $\mathcal{W}_{1.4,10}^{\text{out}} = 38.69^\circ$ using Eq. (1.13f), the condition for the pulsar to be visible, $|\alpha - \mathcal{W}_{1.4,10}^{\text{out}}| \leq |\zeta_L| \leq |\alpha + \mathcal{W}_{1.4,10}^{\text{out}}|$ is not satisfied. However, using the value of $\Phi_0 = 115^\circ$ as given by Rafikov et al. [2], as well as ζ_L and α in Eq. (2.56), we get $\mathcal{W} = 51.73^\circ$, which satisfy $|\alpha - \mathcal{W}| \leq |\zeta_L| \leq |\alpha + \mathcal{W}|$ if we assume $\mathcal{W}_{1.4,10}^{\text{out}} = \mathcal{W}$. This means that either the above values of either or both of the angles α and ζ_L are not realistic or the beam of the pulsar A is different than that represented by Eqs. (1.13). Hence, for the Double Pulsar, we do not create the beam and do not calculate the value of $\Delta\Phi_0$ numerically. Instead, we calculate the values of $\Delta\Phi$ and $\Delta\zeta$ by creating light rays in small regions around the LoS. We use this value of $\Delta\Phi$ to calculate the longitudinal bending delay using Eq. (2.53). We use the value of $\Delta\zeta$ we obtain and the values of Φ_0 , ζ_L , and α as given by Rafikov et al. [2] in Eq. (2.55) to get the value of $\Delta\Phi_0$. We use this value of $\Delta\Phi_0$ in Eq. (2.54) to calculate the values of the latitudinal bending delay. However, Eqs. (2.55) and (2.56) are not valid when the bending is strong and the beam is distorted. Hence, the shape of the irregular region seen in the latitudinal bending delay curve (the right panel of Fig. 3.4) is not correct. If $\Delta\Phi_0$ were calculated numerically, the shape of the irregular region would be different - we would rather see a sudden but continuous jump in the latitudinal bending delay. However, when bending is comparatively weaker, the values of the latitudinal bending delay calculated using the numerically obtained value of $\Delta\Phi_0$ match exactly with the one obtained by using the value of $\Delta\Phi_0$ calculated using Eqs. (2.56) and (2.55). These facts will be demonstrated in the next section for a hypothetical pulsar–black hole binary.

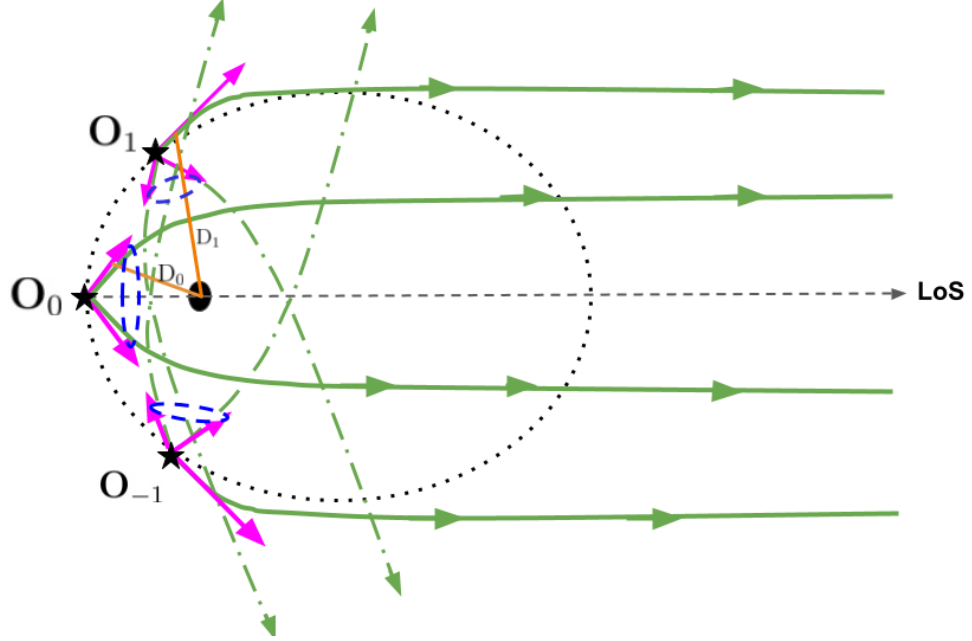


Figure 3.3: The schematic diagram illustrating the bending of light rays in the signal from a pulsar in a binary. The companion is marked as a black filled circle, and the three possible positions of the pulsar are shown with black stars ($\star$). These positions are labeled as O_{-1} , O_0 , and O_1 . The position O_0 corresponds to the position of the pulsar at the superior conjunction. Magenta arrows represent the initial directions of some light rays, while the green arrows represent the directions of the same light rays after bending. Among these, only the rays shown by solid green lines reach to the observer, the rays represented by green dashed lines do not reach the observer. When the pulsar is at O_0 , it is clear that all the light rays from the circular region shown in blue (edge of a three dimensional hollow cone) reach to the observer although only two such light rays are shown. Each of these light rays undergoes a different amount of bending as their initial directions are different. On the other hand, when the pulsar is at positions O_1 and O_{-1} , which are away from O_0 , there is only one specific direction from which the observer can receive the signal from the pulsar. D_0 denotes the impact parameter of a light ray emitted from O_0 toward the companion, which reaches the observer after undergoing bending. Conversely, D_1 corresponds to a light ray emitted from O_1 away from the companion, which still reaches the observer. It is obvious that the value of D_1 is much larger than the value of D_0 . That is why we consider only the situations where the value of the impact parameter is greater than $3\sqrt{3}GM_c/c^2$, even when the light ray was initially moving away from the companion, as mentioned in Sec. 3.1,

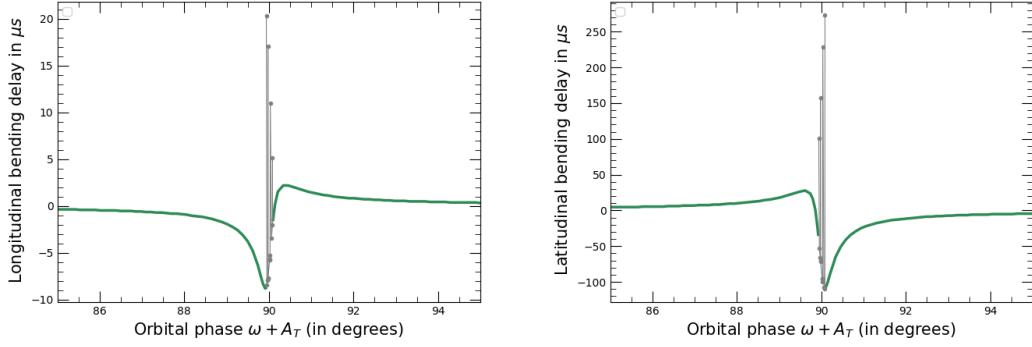


Figure 3.4: Bending delays obtained using our method for the signal of PSR J0737–3039A, assuming an inclination angle of $i = 90.14^\circ$. The left panel represents the longitudinal bending delay and the right panel represents the latitudinal bending delay. Both delay curves become irregular between the range $\omega + A_T \simeq 89.8^\circ$ and $\omega + A_T \simeq 90.2^\circ$. The irregular regions are displayed in gray. Here, the bending delay values are calculated using our formalism, and the value of $\Delta\Phi_0$ is obtained using Eqs. (2.56) and (2.55).

From Fig. 1.1, it is clear that when $\zeta_L = \alpha$, the trajectories A and B coincide and the LoS cuts the beam through the midpoint. On the other hand, when $\alpha \neq \zeta_L$, the trajectories A and B are different and the LoS cuts the beam away from its midpoint. This is the case in our calculations so far. Now it is time to explore what happens if the LoS goes close to the midpoint of the beam, which happens when the value of α is close to that of ζ_L . For this purpose, we change the value of α to 40° , and keep the values of $\zeta_L = 50^\circ$ and $\Phi_0 = 115^\circ$ unchanged. In Fig. 3.5, we notice the fact that the values of the latitudinal bending delay are much smaller when the LoS cuts the beam near the midpoint ($\alpha \simeq \zeta_L$) in comparison to those when the LoS cuts the beam away from it ($\alpha \neq \zeta_L$). However, the value of the longitudinal bending delay does not depend on the location of the trajectory of the LoS across the beam. For $\alpha \neq \zeta_L$, the values of the latitudinal bending delay are much larger than those of the longitudinal bending delay while for the $\alpha \simeq \zeta_L$, these two quantities are of the same order. In the next section, we will see that for $\alpha = \zeta_L$, the value of the latitudinal bending delay is very small, even smaller than that of the longitudinal bending delay. In that section, we will even see that a pulsar–black hole binary with $\alpha = \zeta_L$, has smaller values of the latitudinal bending delays than those for the Double

Pulsar with $\alpha \neq \zeta_L$. All these facts emphasise the impact of the location of the trajectory of the LoS across the beam on the value of the latitudinal bending delay, which is higher when the trajectory is far from the midpoint. This happens due to the enhanced distortion of the beam which will be clear in Sec. 3.3.3. We note that in this section, we kept the value of η_p fixed at 45° following Rafikov et al. [2]. The dependence of the values of the bending delays on η_p will be studied in the next section for hypothetical pulsar–black hole binaries.

In addition to the magnitude of the bending delays, we also notice that the shapes of the bending delay curves in Fig. 3.5 are different than those in Fig. 3.2. The reason for this is the fact that the values of i chosen in Fig. 3.2 are greater than 90° following Rafikov et al. [2], while in Fig. 3.5, we have $i < 90^\circ$. We have checked that if all other parameters are the same, then the bending delay curves for $i = 90^\circ + \Delta$ (where Δ is an arbitrary value of an angle) can be thought as a double reflection of the delay curve for $i = 90^\circ - \Delta$, first a vertical reflection followed by a horizontal reflection. This is the case for both the longitudinal and the latitudinal bending delays. As in a pulsar timing analysis, one cannot distinguish between two such values (equally spaced about 90°) of i , and it is common practise to use only acute angle values for i , for the rest of the thesis, we restrict ourselves to the values of $i \leq 90^\circ$.

The prescription we used, i.e., ignoring the motion of the companion of the pulsar is not physically accurate for a double neutron star system where both objects are of comparable masses. In such systems, the orbital motion of the companion can influence the propagation of the radio signal [63]. For the double pulsars system J0737–3039A/B, Rafikov and Lai [62] showed that the companion’s motion induces a small shift in the bending-delay curve (for the signal of the fast pulsar) along the orbital phase, of order 0.1° , while leaving the overall magnitude of the bending delay remain unchanged. In these cases, a more rigorous treatment, such as the Lorentz-covariant theory of light propagation in the gravitational field of moving bodies developed by Kopeikin and Schäfer [67], is required.

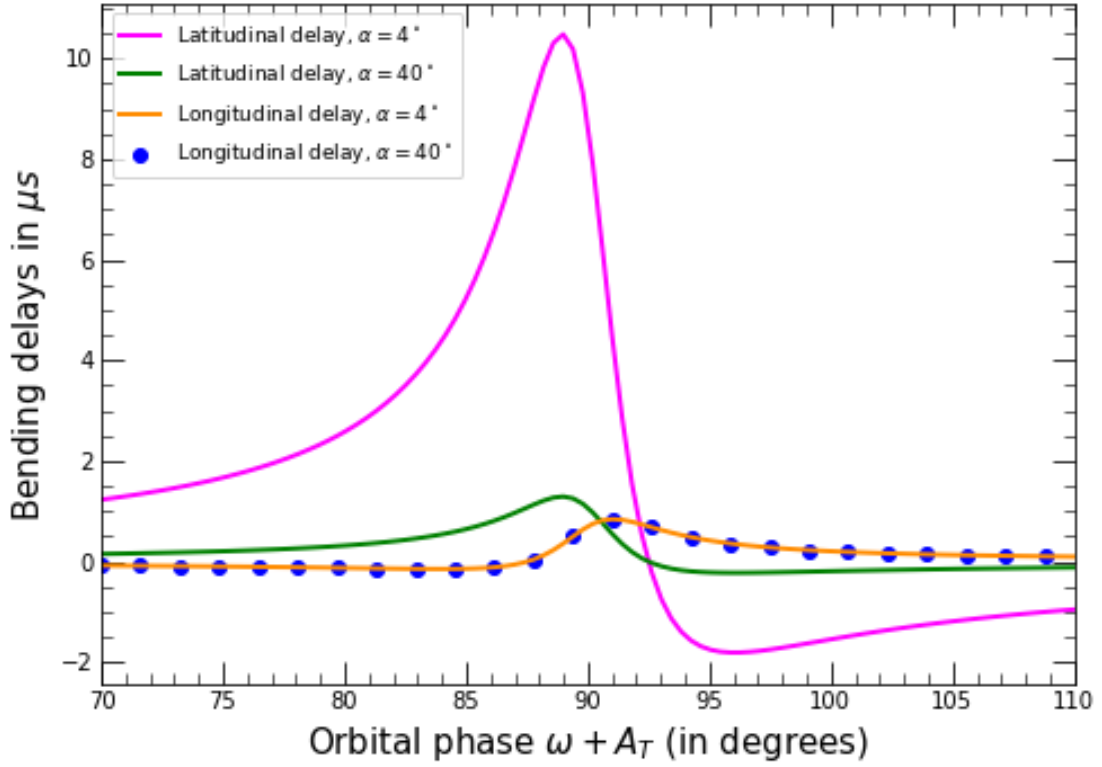


Figure 3.5: The longitudinal and latitudinal bending delays in the signal of a system similar to PSR J0737–3039A, evaluated for two different values of α and fixed parameters $i = 87.5^\circ$, $\zeta_L = 50^\circ$, and $\eta_p = 45^\circ$. These delays are calculated using our formalism, and the value of $\Delta\Phi_0$ has been calculated using Eqs. (2.56) and (2.55).

In this work, however, we focus on pulsar–black hole systems rather than double neutron star systems. For pulsar–supermassive black hole binaries, the motion of the black hole is entirely negligible, and the above approximation remains well justified. For pulsar–stellar mass black hole systems, on the other hand, the motion of the black hole is expected to produce an effect qualitatively similar to that predicted by Rafikov and Lai [62]. Since the bending delay is most prominent near an orbital phase of 90° , we verified that, for our adopted system parameters as mentioned in Table 3.1 with an inclination angle of 90° , the black hole moves by only 0.09° during the time taken by the signal to propagate between the pulsar and the black hole. This is comparable to the situation in the Double Pulsar system, where the slowly rotating companion pulsar moves by 0.13° during the signal propagation time from the fast-rotating pulsar. We therefore expect that the phase shift in the bending delay curve for the pulsar–stellar mass black hole system would be the same as that for the Double Pulsar system, without any significant modification to the overall shape of the delay curve. Such higher order corrections are not included in the present analysis and are left for a future work, where I plan to incorporate the motion of the companion. In this section, we have calculated the values of the bending delays for the Double Pulsar using our formalism for the sake of comparison with the results of Rafikov et al. [2] and Doroshenko et al. [1].

3.3.2 Bending delay in pulsar–black hole binaries

Next, we compute bending delays for hypothetical pulsar–black hole binaries. As there is no pulsar–black hole binary known at present, and different theoretical models predict different values of parameters for such binaries, we use some standard values for the parameters that are not too different than the theoretical prediction of one of the models by Chattopadhyay et al. [4], namely ZM-001. These parameters are the mass of the pulsar M_p , the mass of the companion M_c , P_s , $\dot{P}_s$, P_b , and e . The values of these parameters are given in Table 3.1. There are some other parameters whose values are needed for our

calculations, those are i , ω , η_p , λ_p , and α . We take $i = 87.5^\circ$ in most of our investigations. The values of α , η_p , λ_p are not observationally measurable. Unless otherwise mentioned, we choose the value of α as 50° , λ_p and η_p as 130° and 45° , respectively. The above choice of α and λ_p ensures the fact that the LoS cut the cross-section of the beam through its centre, which requires $\alpha = \zeta_L$.

We set the time $t = 0$ when the true anomaly (A_T) is zero, and assume that the values of the time-dependent parameters as given in Table 3.1 are the values at $t = 0$. With the above choice of parameters and with $i = 87.5^\circ$ and using the formalism mentioned in Sec. 2.3, we find that the rate of change of different parameters are very small and can be ignored over a few orbits. Specifically, ω changes by 0.00128° , η_p changes by -0.000324° , and λ_p changes by -0.000417° during one orbital revolution regardless of the spin of the black hole (the spin of the black hole affects the parameters in the lower decimal places). Similarly, P_b changes by -1.35 s, e changes by 7.05×10^{-15} , and i changes by $(8.17 \times 10^{-12})^\circ$ over one full orbit regardless of the spin of the black hole. The spin of the pulsar changes by $(6.56 \times 10^{-10})^\circ$ over one full orbital revolution. As the effect of the bending depends strongly on the value of i , we use various values of i for various studies as will be discussed due course.

We have mentioned in Sec. 1.3.1 that the pulsar is observable only if there are some light rays in the beam for which the angle ζ_L satisfies the condition $|\alpha - \mathcal{W}_{1.4,10}^{\text{out}}| \leq |\zeta_L| \leq |\alpha + \mathcal{W}_{1.4,10}^{\text{out}}|$. We also know that $\lambda_p = 180^\circ - \zeta_L$ (from Fig 2.1). Using Eq. (1.13f), we get $\mathcal{W}_{1.4,10}^{\text{out}} = 4.51^\circ$ for the value of P_s as given in Table 3.1. Hence, the range of λ_p for which the pulsar is visible is $184.51^\circ - \alpha \geq \lambda_p \geq 175.49^\circ - \alpha$ and for our chosen value of $\alpha = 50^\circ$, the range becomes $134.51^\circ \geq \lambda_p \geq 125.49^\circ$. Our choice of $\lambda_p = 130^\circ$ lies in the middle of this range. This choice of λ_p makes $\zeta_L = \alpha$. As we have discussed earlier, in such a case, the LoS cuts the beam through the midpoint. This fact can be seen in Figs. 1.2, 3.10, and 3.11 of Sec. 3.3.3.

We note that for a pulsar–black hole binary, we cannot choose a value of ζ_L that is

Table 3.1: Parameters for a canonical pulsar–black hole binary. These are (from top to bottom), the spin period of the pulsar (P_s), the rate of change of the spin period of the pulsar ($\dot{P}_s$), the orbital period of the binary (P_b), the eccentricity (e), inclination of the orbit (i), the mass of the pulsar (M_p), the mass of the black hole (M_c), the longitude of the periastron (ω), the angle between the spin axis and the magnetic axis (α), the initial value of the angle between the X_s -axis and the projection of the spin axis in the sky plane (η_p), and the initial value of the angle between the Z_s -axis and the spin axis (λ_p). The values of P_s , $\dot{P}_s$, P_b , e , M_p , M_c , are taken from Chattopadhyay et al. [4]. The values of other parameters are chosen arbitrarily, but satisfying the visibility condition $|180^\circ + \mathcal{W}_{1.4,10}^{\text{out}} - \alpha| \geq |\lambda_p| \geq |180^\circ - \mathcal{W}_{1.4,10}^{\text{out}} - \alpha|$. Moreover, the values of the time-dependent parameters are the values at $t = 0$ (when $A_T = 0$).

Parameters	Value
P_s [s]	1.67
$\dot{P}_s$ [ss ⁻¹]	1.38×10^{-15}
P_b [days]	5.5
e	0.35
i [deg]	87.5 (or 90.0)
M_p [$M_\odot$]	1.8
M_c [$M_\odot$]	14.5
ω [deg]	73.804
α [deg]	50
η_p [deg]	45
λ_p [deg]	130

significantly different than that of α which satisfies $|\alpha - \mathcal{W}_{1.4,10}^{\text{out}}| \leq |\zeta_L| \leq |\alpha + \mathcal{W}_{1.4,10}^{\text{out}}|$, as the value of $\mathcal{W}_{1.4,10}^{\text{out}}$ cannot be much larger than 5° for normal pulsars as predicted by Eq. (1.13f). For a millisecond pulsar, $\mathcal{W}_{1.4,10}^{\text{out}}$ would be significantly larger and the value of ζ_L can be different than that of α . Although the normal evolutionary scenario does not expect the pulsar in a pulsar–black hole binary to be a millisecond pulsar, it is possible for a dynamically formed binary in dense stellar environments like globular clusters. We do not study dynamically formed pulsar–black hole binaries in the present thesis.

In Fig. 3.6, we plot the bending delays in microseconds along the abscissa and the orbital phase in degrees along the ordinate over a full orbit for such a pulsar–black hole binary and compare the values of the bending delays estimated using our formalism (magenta triangles) with those estimated using the expressions given by Rafikov et al. [2] (green lines) and Doroshenko et al. [1] (orange lines) with $i = 87.5^\circ$ and the values of all other

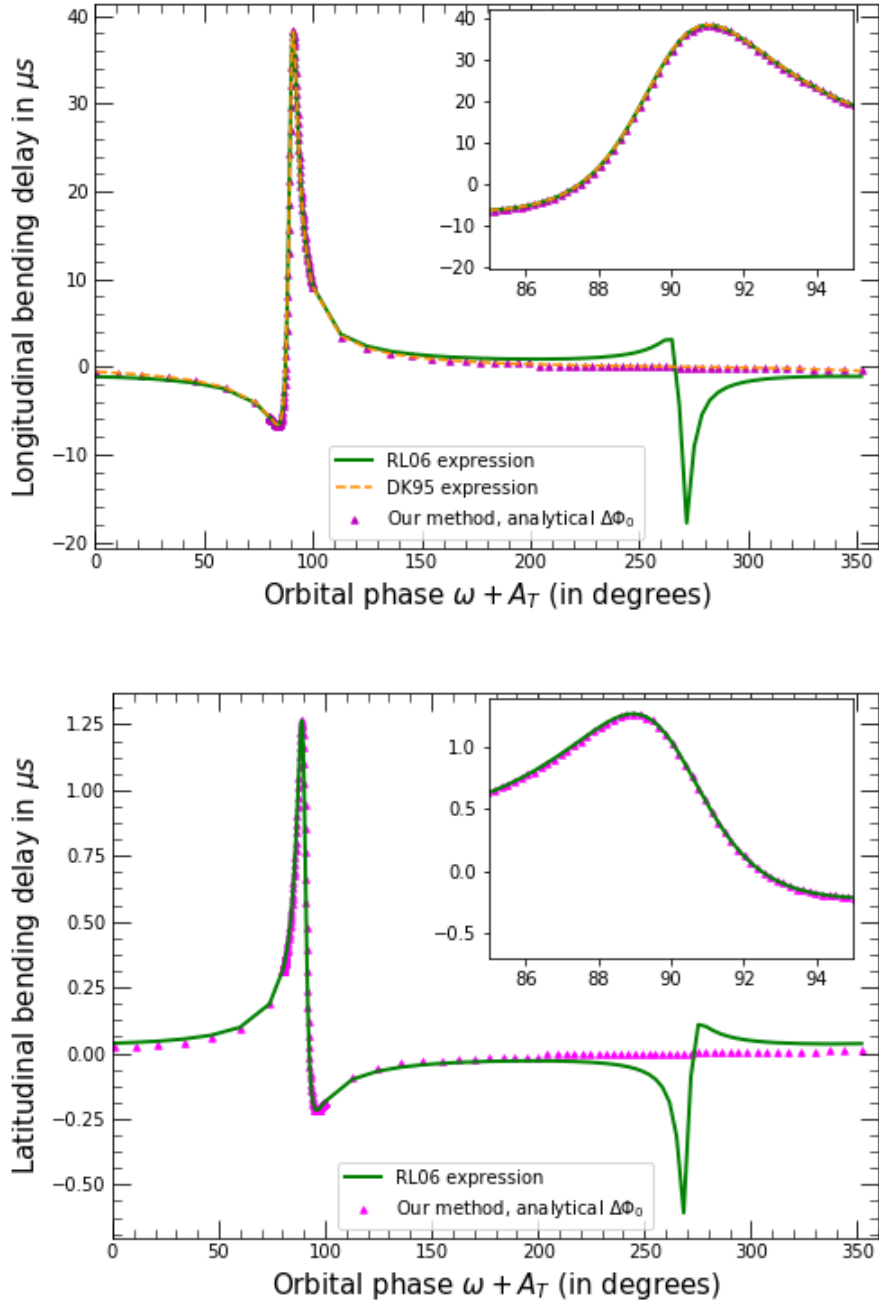


Figure 3.6: Comparison of the bending delay values calculated using our formalism and the expressions of Rafikov et al. [2] over a complete orbital cycle for a hypothetical pulsar–black hole binary. Here we set $i = 87.5^\circ$, and all other parameters are the same as those listed in Table 3.1. The left panel displays the longitudinal bending delay and the right panel illustrates the latitudinal bending delay. For the longitudinal bending delay, we also include the expression provided by Doroshenko et al. [1]. While calculating the latitudinal bending delays using our method, the value of $\Delta\Phi_0$ obtained from Eqs. (2.56) and (2.55). In the figure legends, RL06 stands for Rafikov et al. [2] and DK95 stands for Doroshenko et al. [1]

parameters are the same as those given in Table 3.1. Here, we used Eqs. (2.56) and (2.55) to calculate the value of $\Delta\Phi_0$. We see that the values of the bending delays obtained in our method match with those obtained using the expressions given by Rafikov et al. [2] except near the orbital phase of 270° . This is an artifact of the expressions of Rafikov et al. [2], which are valid only near the orbital phase of 90° , and hence should not be used for the values of the orbital phases away from 90° . On the other hand, we see that the values of the longitudinal bending delays obtained in our method match with those obtained using the expressions given by Doroshenko et al. [1] throughout the orbit. However, Doroshenko et al. [1] did not provide any expression for the latitudinal bending delay.

Next, we explore whether the values of the latitudinal bending delays would remain the same if we calculate $\Delta\Phi_0$ numerically. In Fig. 3.7, we compare the values of the latitudinal bending delay calculated using the two methods of calculating $\Delta\Phi_0$ as well as using the expression of Rafikov et al. [2] for different values of i . We see that all three methods agree unless $i \geq 88^\circ$. Even for the case of $i \geq 88^\circ$, the mismatch between the two methods of calculating $\Delta\Phi_0$ is significant only near the orbital phase of 90° . The higher is the value of i , the wider the range of the orbital phase where the two methods of calculating $\Delta\Phi_0$ give different results. Where the two methods differ, the value of $\Delta\Phi_0$ obtained numerically is much larger than the value predicted by Eqs. (2.56) and (2.55), resulting in a larger value of the latitudinal bending delay. This happens because of the increased size of the beam due to the bending as seen in Figs. 3.10 and 3.11. The results obtained in our method with the value of $\Delta\Phi_0$ estimated using Eqs. (2.56) and (2.55) disagrees with the expressions of Rafikov et al. [2] only when the values of both of the inclination angle and the orbital phase are very close to 90° as seen in the zoomed plot of Fig. 3.7d. This was also the case for the Double Pulsar too. As calculating $\Delta\Phi_0$ numerically is computationally very costly, it is preferable to calculate $\Delta\Phi_0$ analytically using Eqs. (2.56) and (2.55). As we have seen, for a pulsar–black hole binary with parameters the same as those given in Table 3.1, the analytical values of $\Delta\Phi_0$ are correct when $i \leq 87.5^\circ$. In Figs. 3.8 and 3.9, where we vary parameters, we use analytical values of $\Delta\Phi_0$ and $i = 84^\circ$ as when we

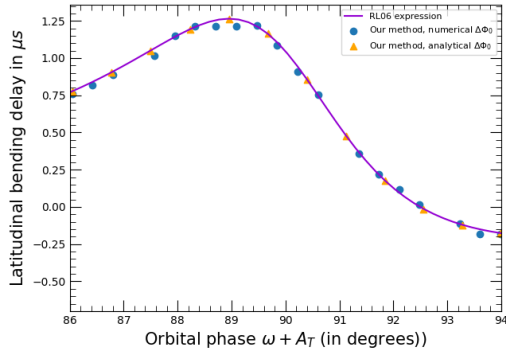
get the maximum latitudinal bending for $P_b = 1$ day, we found that the numerical and analytical values of the latitudinal bending delays match at $i = 84^\circ$ or smaller.

In Fig. 3.7d, we also see that when $i = 90^\circ$, the values of the latitudinal bending delay calculated using the numerically computed values of $\Delta\Phi_0$ are larger at the values of the orbital phase near 90° than that exactly at 90° . This can be understood from Fig. 3.11 where $i = 90^\circ$. Fig. 3.11a shows the structure of the beam with bending for $\omega + A_T = 89^\circ$. In the ninth panel of this figure, the LoS (denoted by a magenta diamond) exited the theoretical boundary of the beam depicted by the red circle, but due to the bending, it is still surrounded by light rays and hence inside the distorted beam. Similarly, Fig. 3.11b shows the structure of the beam with bending for $\omega + A_T = 91^\circ$. In the first panel of this figure, the LoS has not yet entered the theoretical boundary of the beam denoted by the red circle, but due to the bending, it is still surrounded by light rays and hence inside the distorted beam. These facts also lead to a tail on the right side of the pulse profile for $\omega + A_T = 89^\circ$ and $i = 90^\circ$ (Fig. 3.13a) and on the left side of the pulse profile for $\omega + A_T = 91^\circ$ and $i = 90^\circ$ (Fig. 3.13b). These tails are of very low intensity, about 1 to 5 in our units of intensity. On the other hand, in Fig. 3.10a, where the values of both of the inclination angle and orbital phase are 90° , there is no light ray around the LoS when it is outside the red circle (the first and the ninth panels). However, in this case there are some light rays near around the LoS when it is on the theoretical boundary (the second and the eighth panel). These also lead to large values of the latitudinal bending delay, although smaller than those at the orbital phase of 89° and 91° (all three are for $i = 90^\circ$). That is why the pulse profile with bending starts before and ends after the pulse profile without bending for these set of parameters, see Fig. 3.12d. More details about the beams shown in Figs. 3.10 and 3.11 and pulse profiles shown in Figs. 3.12 and 3.13 will be discussed in the next section.

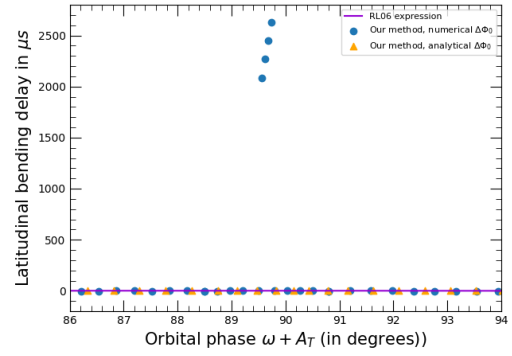
In Fig. 3.7c, when $i = 89.3^\circ$, we see that there is a small range of the orbital phase less than 89° where there are mild disagreements between the results obtained by the two

methods of estimating $\Delta\Phi_0$ whereas there is a comparatively wider range of the orbital phase greater than 89.5° where there are strong disagreements between the results obtained by the two methods of estimating $\Delta\Phi_0$. In Fig. 3.7b, where $i = 88^\circ$, we see the disagreement between the two methods of computing $\Delta\Phi_0$ only when the orbital phase is greater than 89.5° , and the disagreement is always large. This happens because for $i = 88^\circ$, for a range of orbital period larger than 89.5° , the LoS gets surrounded by the light rays before its entry to the theoretical boundary, whereas when the orbital phase is less than 89° , the LoS does not get surrounded by the light rays at all when it is outside the theoretical boundary of the beam. Similarly, for $i = 89.3^\circ$, for a range of orbital period larger than 89.5° , the LoS gets surrounded by the light rays before its entry to the theoretical boundary, and for a comparatively smaller range of orbital period less than 89° , the LoS gets surrounded by the light rays after its exit to the theoretical boundary.

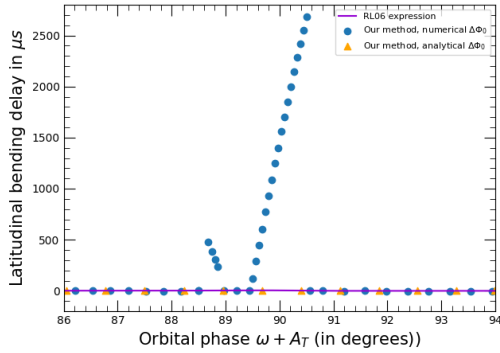
We now explore the dependence of the longitudinal and the latitudinal bending delays in our formalism on various parameters, e.g., M_c , P_b , e , i and η_p , one at a time, in Figs. 3.8 and 3.9, respectively. In both of the cases, when we vary other parameters, we keep the value of i fixed at 84° . When we vary i , we keep its value lower than 87.5° . The reason for making such choices for the value of i is the fact that among all the combinations of the values of various parameters, the maximum bending is obtained for $P_b = 1$ day. In this case, we have checked that the value of $\Delta\Phi_0$ calculated using Eqs. (2.56) and (2.55) matches with the value obtained numerically if $i \leq 84^\circ$. However, when we vary i and keep all other parameters the same as those given in Table 3.1, such an agreement can be obtained upto $i = 87.5^\circ$. As theoretically allowed values of the mass of a neutron star lie in the range of $\sim 1 - 3 M_\odot$ and the variation of M_p in this small range would not affect the value of the bending delays, we keep the value of M_p fixed as that mentioned in Table 3.1. We note that in all of the plots, instead of full orbits, we have chosen the range of the orbital phase (along the abscissa) between $80^\circ - 100^\circ$ to have the maximum delay around the middle of the plots.



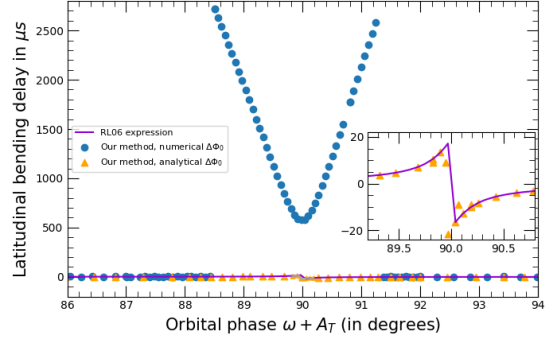
(a) $i = 87.5^\circ$



(b) $i = 88.0^\circ$

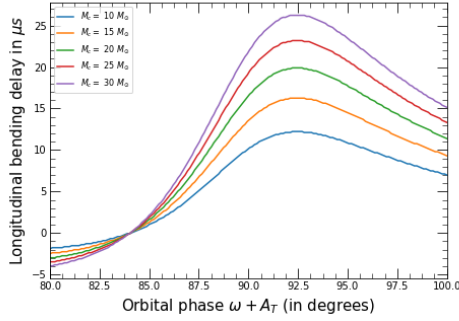


(c) $i = 89.3^\circ$

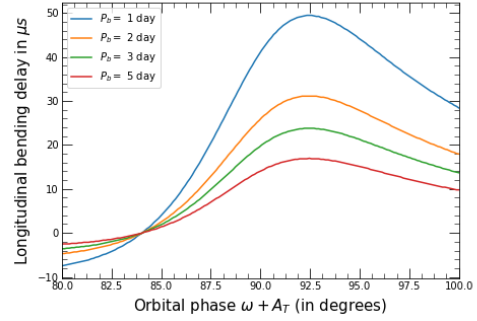


(d) $i = 90^\circ$

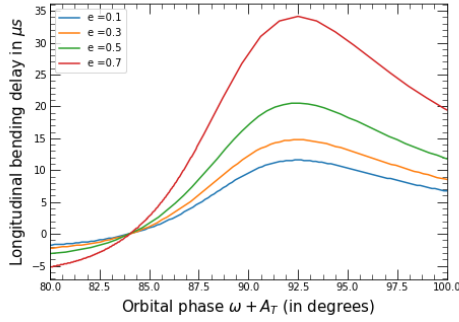
Figure 3.7: Comparison of latitudinal bending delay values estimated using three different methods, for varying orbital inclination angles i , in a hypothetical pulsar–black hole binary. All other parameter values are the same as those listed in Table 3.1.



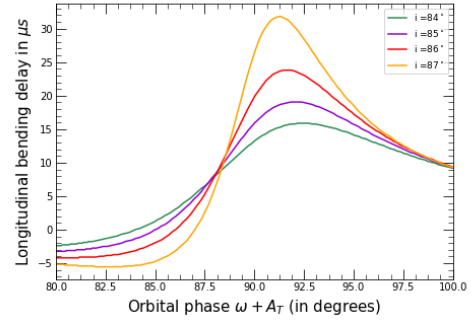
(a) Longitudinal delay curves for different values of M_c .



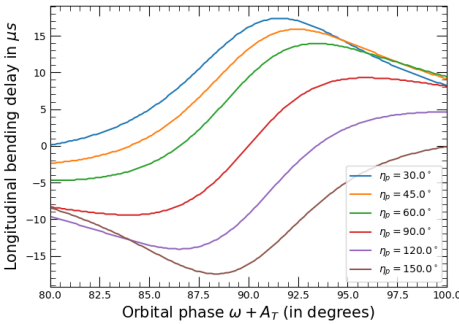
(b) Longitudinal delay curves for different values of P_b .



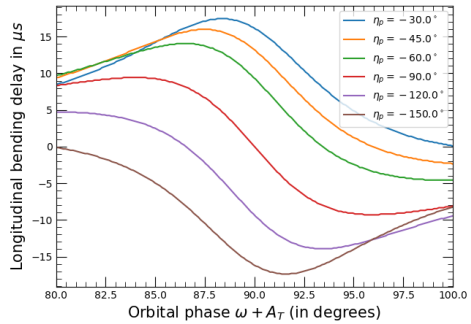
(c) Longitudinal delay curves for different values of e .



(d) Longitudinal delay curves for different values of i .

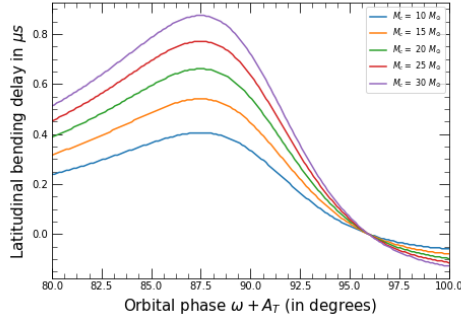


(e) Longitudinal delay curves for different positive values of η_p .

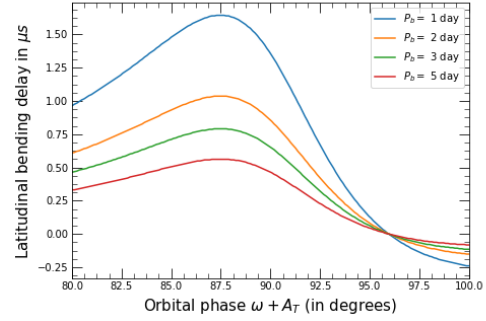


(f) Longitudinal delay curves for different negative values of η_p .

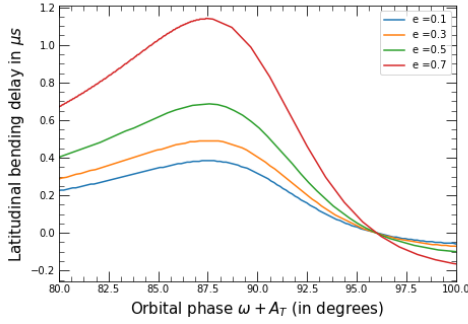
Figure 3.8: Longitudinal bending delay curves for hypothetical pulsar–black hole binaries, where one parameter is varied in each panel while all other parameters are kept fixed as listed in Table 3.1. The inclination angle is set to $i = 84^\circ$ for all panels except panel (d), where i is varied.



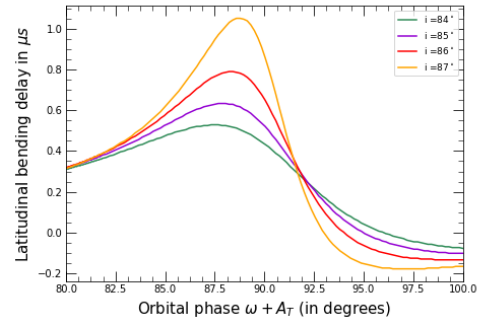
(a) Latitudinal delay curves for different values of M_c .



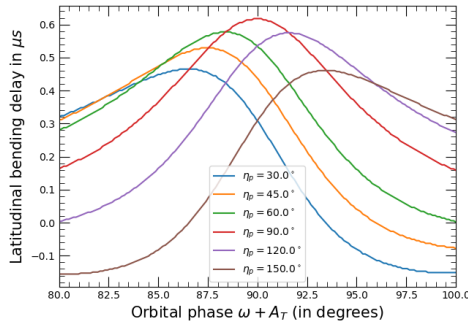
(b) Latitudinal delay curves for different values of the P_b .



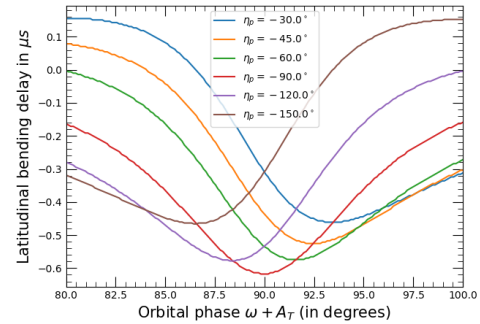
(c) Latitudinal delay curves for different values of e .



(d) Latitudinal delay curves for different values of i .



(e) Latitudinal delay curves for different positive values of η_p .



(f) Latitudinal delay curves for different negative values of η_p .

Figure 3.9: Latitudinal bending delay curves for hypothetical pulsar–black hole binaries, where one parameter is varied in each panel while all other parameters are fixed as given in Table 3.1. The inclination angle is set to $i = 84^\circ$ in all cases except panel (d), where i is varied.

Figs. 3.8a and 3.9a show the values of the longitudinal and the latitudinal bending delays, respectively, for the values of M_c as 10, 15, 20, 25, and 30 solar masses. As expected, the bending delays are larger for higher values of M_c . Figs. 3.8b and 3.9b show the values of the longitudinal and the latitudinal bending delays, respectively, for the values of P_b as 1, 2, 3, and 5 days. The bending effect becomes milder for larger values of P_b as those imply wider orbits since we have kept the masses of the objects fixed here. Figs. 3.8c and 3.9c show the values of the longitudinal and the latitudinal bending delays, respectively, for the values of e as 0.1, 0.3, 0.5, 0.7. We see that the maximum values of the bending delays are larger for higher values of the eccentricity. This happens because for a higher eccentricity, the same size of the orbit (i.e., the same values of P_b , M_p , and M_c) has a shorter distance between the pulsar and the companion near the periastron and nearby orbital phases. For our choice of $\omega = 73.804^\circ$, the orbital phase corresponding to the maximum delay ($\omega + A_T \simeq 90^\circ$) is close to the periastron, i.e., at $A_T \simeq 16.2^\circ$. For a different choice of ω giving the maximum delay closer to the apastron, one would see that the maximum value of the bending delays are larger for lower eccentricities. This would happen because for the orbital phases closer to the apastron, a more eccentric orbit has a larger distance between the pulsar and the companion than that of a less eccentric orbit of the same size.

Figs. 3.8d and 3.9d show the values of the longitudinal and the latitudinal bending delays respectively for the values of i as 85° , 86° , and 87° . As expected, the bending effect becomes stronger when the value of i increases. In Figs. 3.8e, 3.8f, 3.9e, and 3.9f we study the effect of the value of η_p on the value of the bending delays. For both of the longitudinal and the latitudinal delays, we use both positive and negative values of η_p . We see that the magnitude of the maximum value of the longitudinal bending delay increases when $|\eta_p|$ decreases while that of the latitudinal bending delay decreases when $|\eta_p|$ decreases. The sign reversal of η_p flips the shape of the longitudinal delay curve horizontally and the latitudinal delay curve vertically.

From different sub-panels of Figs. 3.8 and 3.9, we also see that the value of the orbital phase corresponding to the strongest bending depends only on the values of η_p and i . However, this value is always close to 90° . In these figures, we do not show the dependence of the bending delays on the value of λ_p , as already discussed, the pulsar will not be visible at all for a significantly different value of λ_p .

3.3.3 Distortion of the beam and the change in the pulse profile due to the bending

We employ the beam model, which represents a distribution of light rays, as described in Sec. 1.3.2. After getting the distribution of the starting points of the light rays, we study the effect of bending on those. For this purpose, we use the parameters for the pulsar–black hole binary as given in Table 3.1 and apply the formalism described in Sec. 3.1 to each of the light rays. To visualise the effect of the bending on the surface density of the light rays over the cross section of the beam well, we first choose the orbital phase as 90° . Figs. 3.10a and 3.10b show the surface probability density of the apparent starting points of the light rays after light bending for $i = 90^\circ$ and $i = 89.8^\circ$, respectively. In Figs. 3.11a and 3.11b, we show the surface probability density of the apparent starting point of the light rays for $i = 90^\circ$ but at two different values of the orbital phase, e.g., 89° and 91° .

In each of the cases, different panels are for different times within one rotational period of the pulsar. In one rotational period of the pulsar, the orbital phase does not change significantly, but the direction of emission changes over time due to the rotation of the beam. This results in a change in the surface density of the light rays on the cross section of the beam after bending, as the bending of light depends on the direction of emission. In each of the plots, in addition to the surface probability density of the light rays, the location where the LoS cuts the cross section has been shown with a magenta diamond and the theoretical boundary of the cross-section of the beam without bending, i.e., the circle of radius $\mathcal{R}_{1.4,10}^{\text{out}}$ is marked with a red circle.

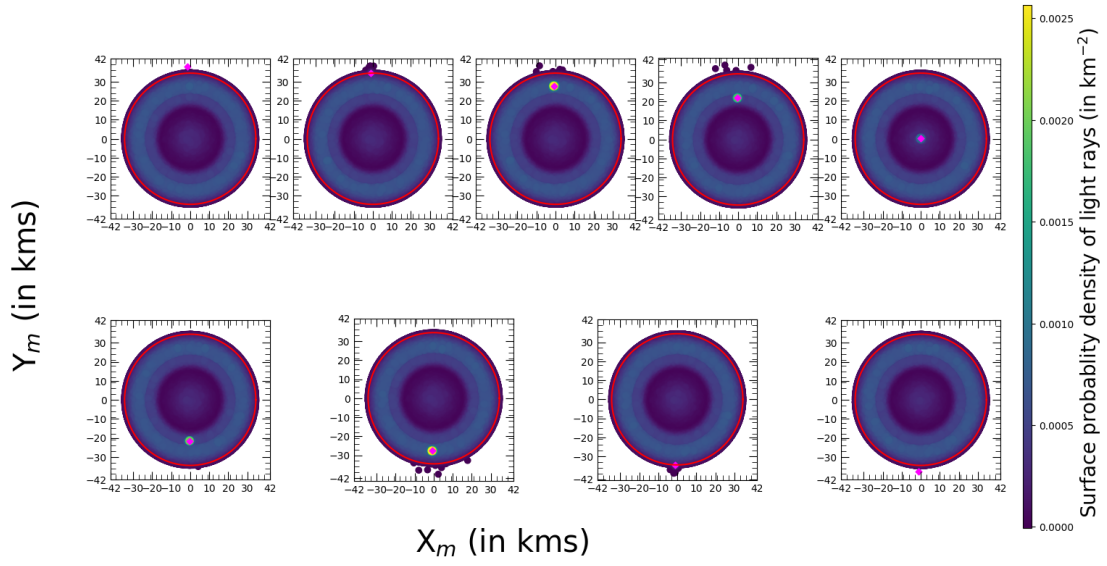
When the inclination angle and the orbital phase, are both equal to 90° (Fig. 3.10a), i.e., the pulsar is at the superior conjunction, the beam emitted by the pulsar reaches the observer by crossing the companion. Although the radius of the beam is about 30 km where it starts as seen in Fig. 1.2, it becomes much larger when it crosses the companion, as obtained by using the value of h as the distance between the pulsar and the companion in Eq. (1.14). Hence, the companion is like a point with respect to the cross-section of the beam that cuts it. Due to its strong gravity, this point-like companion acts like a convex lens and focuses all the light rays from a ring-shaped region around it. As a result, the density of the light rays increases around the point where it cuts the companion. When we trace back the bent beam at the emission height and plot the surface probability distribution of these apparent starting points of the light rays, we see a small circular region with enhanced intensity. This feature is noticeable in various panels of Figs. 3.10 and 3.11 as small circular patches with different colours that represent larger values of the surface probability density. We note that due to bending, sometimes there are cluster of light rays outside the circle of radius $\mathcal{R}_{1.4,10}^{\text{out}}$, and this can happen even when the LoS is outside this circle. If the LoS is surrounded by these rays, then the effective size of the beam increases, leading to large values of $\Delta\Phi_0$ when calculated numerically, and hence, large values of the latitudinal bending delay. This fact has been already discussed. In addition to those clusters, there are some light rays that seem to start far from the others. These are the rays with very low impact parameters. The exact number of such rays depends on the value of I_0^{out} . However, these isolated points do not affect any of our calculations.

We note that when the beam cuts the companion in the outer cone region, which is chosen to be the brightest of the components, the surface density enhancement is the largest. This is the case for the third and the seventh panels of Figs. 3.10a and 3.10b, the fourth panel of Fig. 3.11a, and the sixth panel of Fig. 3.11b. However, if the companion cuts the outer cone near its outer or inner edge, where the intensity is lower than the peak value, the the surface density enhancement is not that large. This is the case for the seventh panel of Fig.

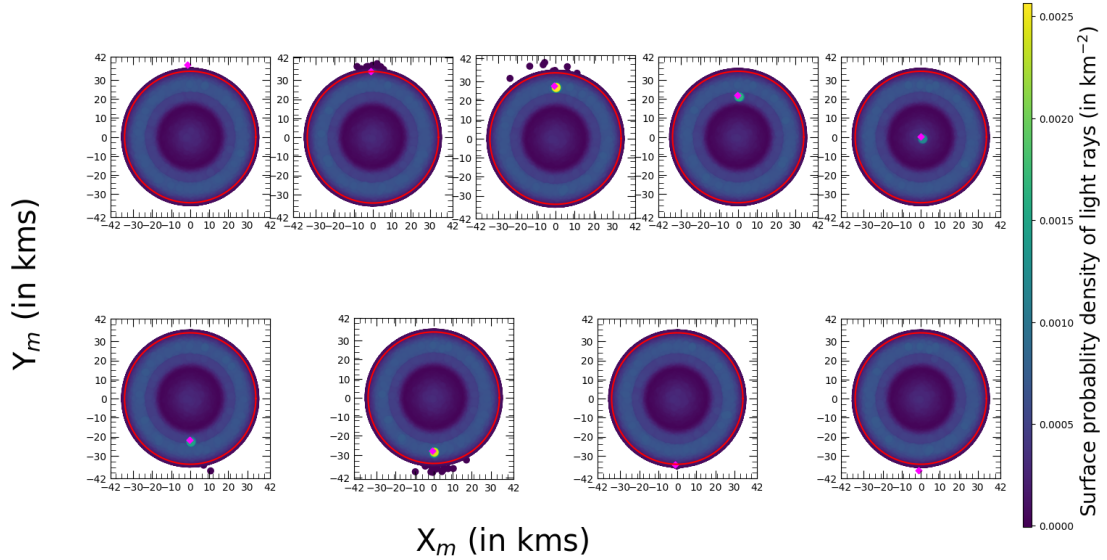
3.11a and the second panel of Fig. 3.11b. When the beam cuts the companion in the inner cone region, which is chosen to be fainter than the outer cone, the density enhancement is also smaller. This is the case for the fourth and the sixth panels of Figs. 3.10a and 3.10b, as well as for the sixth panel of Fig. 3.11a and for the third panel of Fig. 3.11b. When the beam cuts the companion in the core region, which is chosen to be the faintest of the components, the enhancement of the surface density is also very small. This is the case for the fifth panels of Figs. 3.10a, 3.10b, 3.11a, and 3.11b. In the fourth panel of 3.11b, the density enhancement is almost zero as the companion cuts the beam at near the outer edge of the core. In the second panels of Figs. 3.10a and 3.10b, the eighth panel of Figs. 3.10a, the second panel of Fig. 3.11a and the seventh panel of 3.11b, the beam cuts the companion at the edge of the outer cone where the surface probability distribution of the light rays is only 10% of its maximum value and hence the density enhancement is not noticeable. However, distortion of the beam, i.e., cluster of light rays outside the red circle is visible. In the first and the ninth panels of Figs. 3.10a and 3.10b, as well as in the first panel of Fig. 3.11a and eighth and ninth panels of Fig. 3.11b, the companion does not cut the beam at all and we don't see any enhanced density region. We do not even see any distortion of the beam as the companion is far enough from the beam.

Another phenomenon manifests in these figures. When the pulsar is at the superior conjunction, the LoS also cuts through the companion. The farther the pulsar is from the superior conjunction, the farther is the LoS from the companion. That is why in Fig. 3.10a, the LoS (magenta diamonds) is always at the centre of the enhanced intensity regions while in Fig. 3.10b, the LoS is shifted from the centre of the enhanced intensity regions, and in Figs. 3.11a and 3.11b, the LoS is even outside the enhanced intensity regions. That is why there is no difference between the pulse profiles without and with bending for $i = 90^\circ$ and $\omega + A_T$ either 89° or 91° , except the tail at the right side for the first case (Fig. 3.13a) and at the left side for the second case (Fig. 3.13a) caused by fact of the LoS being surrounded by the light rays outside the circle of radius $\mathcal{R}_{1.4,10}^{\text{out}}$ (or strong latitudinal bending), as seen in Figs. 3.11a, and 3.11b.

Next, we explore the change in the shape of the pulse profile due to the bending. To

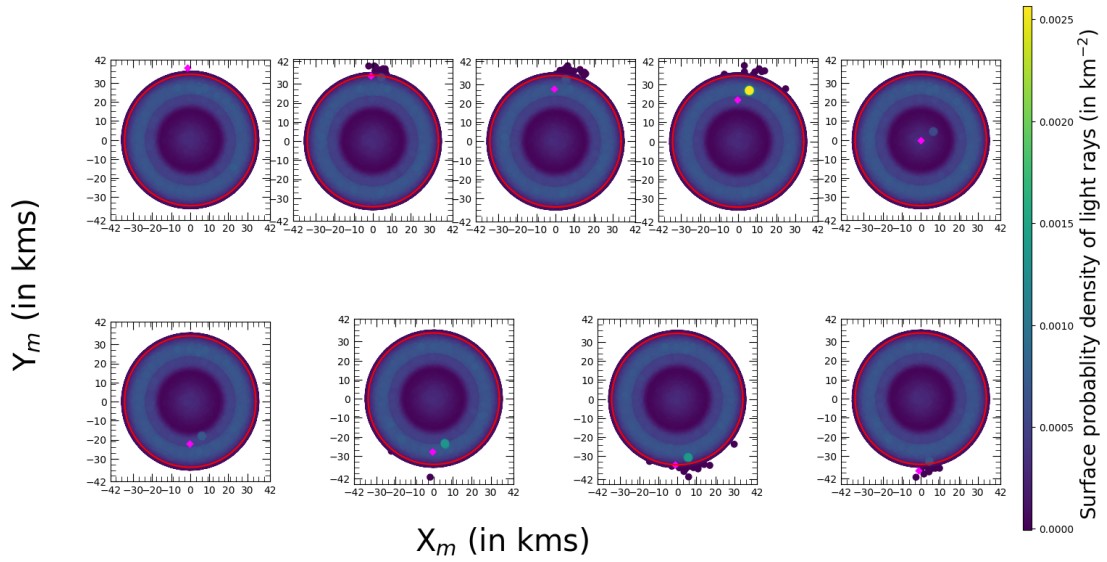


(a) The surface probability density of the light rays under bending for $\omega + A_T = 90^\circ$ and $i = 90^\circ$. The fifth plot is at t_{mid} , i.e., when the LoS is at the middle of the line along which it passes through the beam. The four plots before it are for $t_{\text{mid}} - 0.0298$ s (the first plot), $t_{\text{mid}} - 0.0274$ s (the second plot), $t_{\text{mid}} - 0.0218$ s (the third plot) and $t_{\text{mid}} - 0.0174$ s (the fourth plot), while the four plots after it are for $t_{\text{mid}} + 0.0174$ s (the sixth plot), $t_{\text{mid}} + 0.0218$ s (the seventh plot), $t_{\text{mid}} + 0.0274$ s (the eighth plot) and $t_{\text{mid}} + 0.0298$ s (the ninth plot).

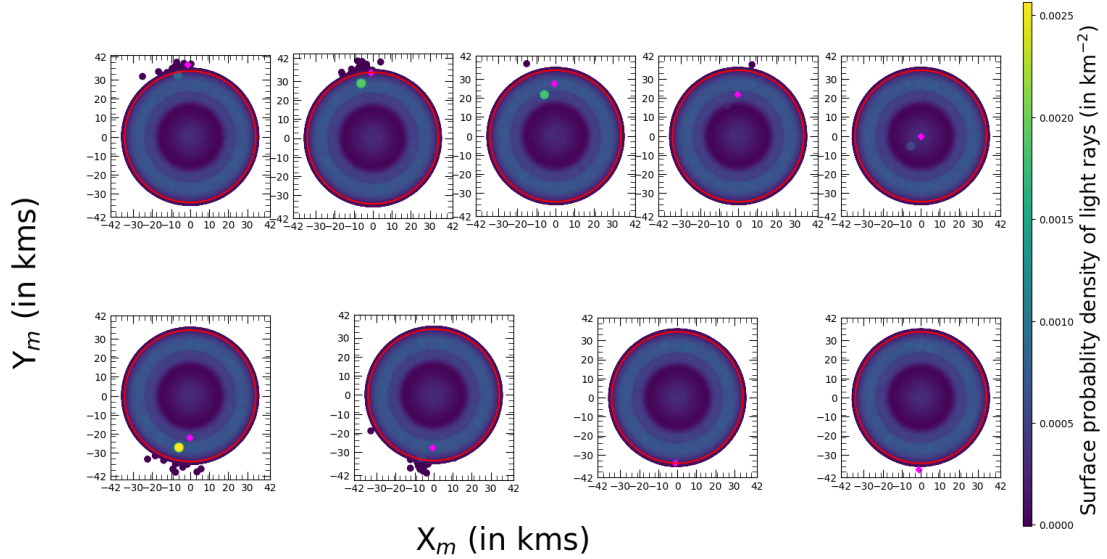


(b) The surface probability density of the light rays under bending for $\omega + A_T = 90^\circ$ and $i = 89.8^\circ$. The fifth plot is at t_{mid} , i.e., when the LoS is at the middle of the line along which it passes through the beam. The four plots before it are for $t_{\text{mid}} - 0.0298$ s (the first plot), $t_{\text{mid}} - 0.0274$ s (the second plot), $t_{\text{mid}} - 0.0218$ s (the third plot) and $t_{\text{mid}} - 0.0174$ s (the fourth plot), while the four plots after it are for $t_{\text{mid}} + 0.0174$ s (the sixth plot), $t_{\text{mid}} + 0.0218$ s (the seventh plot), $t_{\text{mid}} + 0.0274$ s (the eighth plot) and $t_{\text{mid}} + 0.0298$ s (the ninth plot).

Figure 3.10: Surface probability density of light rays on the cross-section of the pulsar beam in the $X_m Y_m$ plane, accounting for gravitational bending in a hypothetical pulsar–black hole binary. The top panel (a) shows results for $i = 90.0^\circ$, and the bottom panel (b) for $i = 89.8^\circ$. In both cases, we set $\omega + A_T = 90^\circ$, with all other parameters taken from Table 3.1. Each panel displays the intensity distribution at nine different times. For each instant, the location where the LoS intersects the beam cross-section is marked by a magenta diamond ($\diamond$). The red circle in each plot denotes the theoretical boundary of the beam cross-section without bending, i.e., a circle of radius $\mathcal{R}_{1.4,10}^{\text{out}}$.



(a) The surface probability density of the light rays under bending for $\omega + A_T = 89^\circ$ and $i = 90^\circ$. The fifth plot is at t_{mid} , i.e., when the LoS is at the middle of the line along which it passes through the beam. The four plots before it are for $t_{\text{mid}} - 0.0298$ s (the first plot), $t_{\text{mid}} - 0.0274$ s (the second plot), $t_{\text{mid}} - 0.0218$ s (the third plot) and $t_{\text{mid}} - 0.0174$ s (the fourth plot), while the four plots after it are for $t_{\text{mid}} + 0.0174$ s (the sixth plot), $t_{\text{mid}} + 0.0218$ s (the seventh plot), $t_{\text{mid}} + 0.0274$ s (the eighth plot) and $t_{\text{mid}} + 0.0298$ s (the ninth plot).

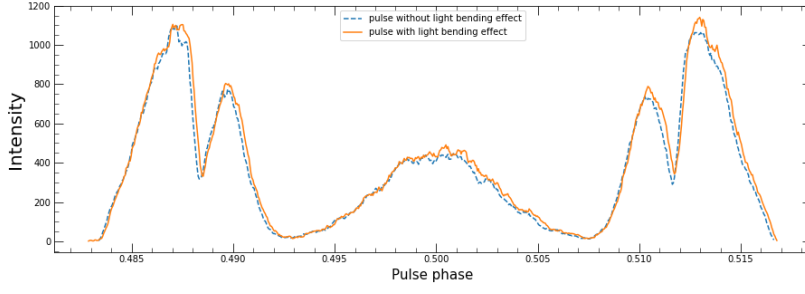


(b) The surface probability density of the light rays under bending for $\omega + A_T = 91^\circ$ and $i = 90^\circ$. The fifth plot is at t_{mid} , i.e., when the LoS is at the middle of the line along which it passes through the beam. The four plots before it are for $t_{\text{mid}} - 0.0298$ s (the first plot), $t_{\text{mid}} - 0.0274$ s (the second plot), $t_{\text{mid}} - 0.0218$ s (the third plot) and $t_{\text{mid}} - 0.0174$ s (the fourth plot), while the four plots after it are for $t_{\text{mid}} + 0.0174$ s (the sixth plot), $t_{\text{mid}} + 0.0218$ s (the seventh plot), $t_{\text{mid}} + 0.0274$ s (the eighth plot) and $t_{\text{mid}} + 0.0298$ s (the ninth plot).

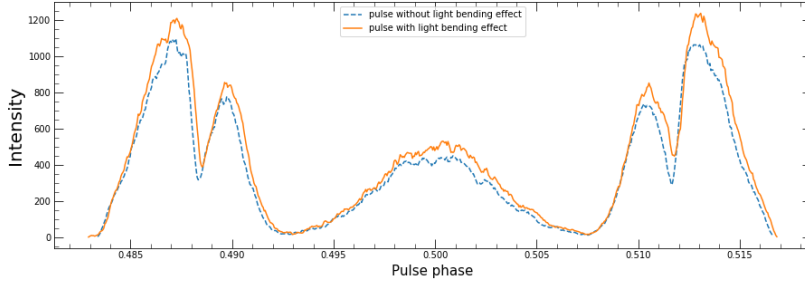
Figure 3.11: Surface probability density of light rays on the $X_m Y_m$ plane—i.e., the cross-section of the pulsar beam under gravitational bending in a hypothetical pulsar–black hole binary. The top panel (a) corresponds to $\omega + A_T = 89^\circ$, and the bottom panel (b) to $\omega + A_T = 91^\circ$. In both cases, we set $i = 90^\circ$, and all other parameters are adopted from Table 3.1. Each panel presents the intensity distribution at nine different times. For each time, in addition to the surface density of the light rays, the intersection point of the LoS with the beam cross-section is marked by a magenta diamond ($\diamond$). The red circle in each plot indicates the theoretical beam boundary without bending, represented by a circle of radius $\mathcal{R}_{1.4,10}^{\text{out}}$.

obtain a pulse profile from the distribution of the light rays on the cross section of the beam, we use a simple technique. We divide P_s into 20000 time-steps, and for each of these times, we first obtain the orientation of the beam in the I-frame (in which the LoS remain fixed) and then consider a very small circular region of radius $\mathcal{R}_{1.4,10}^{\text{out}}/50$ around the LoS on the $X_m Y_m$ plane and check whether there is any light rays in that region. When we get a non zero value, we can say that the LoS enters into the beam. The number of the light rays inside that circular region is taken as a proxy of the intensity. We note that this intensity is different than the surface probability density of the light rays mentioned earlier. This intensity is taken along the ordinate and the pulse phase is taken along the abscissa whenever a pulse profile is plotted. The pulse phase is defined to vary from $0 - 1$ which cover a full rotation of the magnetic axis. For the purpose of visualisation, we define the phase in such a way that the peak of the core component of the profile without the bending is at a phase 0.5. In other words, at the pulse phase of 0.5, the LoS is at the middle of the unbent beam. While plotting the profile, instead of plotting the full range of the pulse phase, we plot only the range over which the LoS is within the beam.

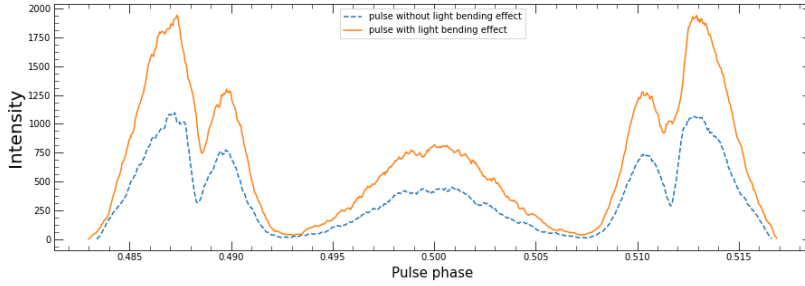
In Figs. 3.12a ,3.12b, 3.12c, and 3.12d, we show the pulse profiles for the value of the orbital phase as 90° and the values of the inclination angle as 89.7° , 89.8° , 89.9° , and 90° , respectively. In each of these cases, we compare the pulse profile under bending with the pulse profile without bending (using the distribution of the light rays of Fig. 1.2). Due to the bending, the profile strength increases as the LoS passes through the enhanced intensity region on the cross section of the beam as discussed earlier. This increase in the profile strength becomes larger as the inclination angle comes closer to 90° due to the increase in the effects of the longitudinal bending. From the above figures, we conclude that we can get a very strong pulse when both of the inclination angle and the orbital phase are close to 90° , as the LoS comes to the centre of the enhanced intensity region (Fig. 3.10a).



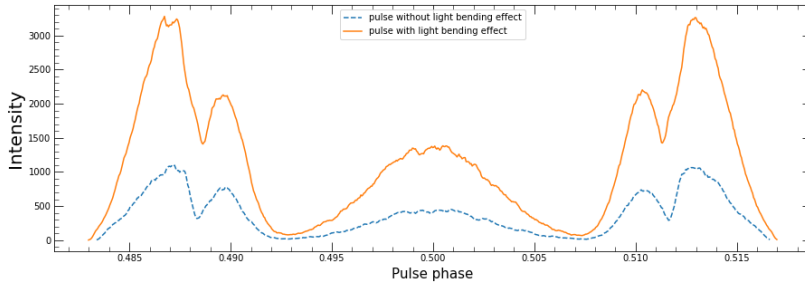
(a) Pulse profiles for $i = 89.7^\circ$.



(b) Pulse profiles for $i = 89.8^\circ$.

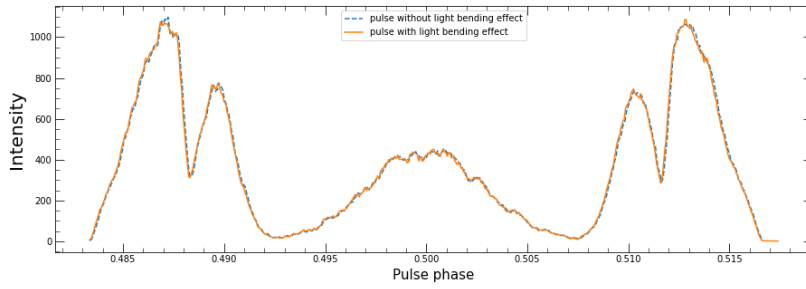


(c) Pulse profiles for $i = 89.9^\circ$.

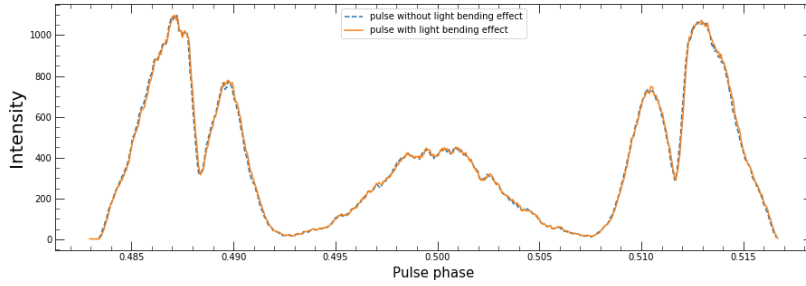


(d) Pulse profiles for $i = 90^\circ$.

Figure 3.12: Pulse profiles at four different orbital inclinations for a hypothetical pulsar–black hole binary, comparing cases with (solid orange lines) and without (blue dashed lines) gravitational bending. In each panel, the intensity refers to the number of light rays within a circular region of radius $\mathcal{R}_{1.4,10}^{\text{out}}/50$ centered on the line of sight (LoS). For all cases, the pulsar is positioned at the orbital phase $\omega + A_T = 90^\circ$, and the remaining parameters are taken from Table 3.1.



(a) Pulse profiles at orbital phase $\omega + A_T = 89^\circ$.



(b) Pulse profiles at orbital phase $\omega + A_T = 91^\circ$.

Figure 3.13: Pulse profiles for a hypothetical pulsar–black hole binary, comparing cases with (solid orange) and without (dashed blue) the influence of bending. Panel (a) shows results for $\omega + A_T = 89^\circ$ and panel (b) for $\omega + A_T = 91^\circ$, both with an inclination of $i = 90^\circ$. All remaining parameters follow Table 3.1. Here, intensity refers to the number of rays passing through a circular area of radius $\mathcal{R}_{1.4,10}^{\text{out}}/50$ centered at the LoS.

3.4 Conclusions

In this chapter, we have shown that for a non-rotating stellar-mass black hole companion, the latitudinal and longitudinal bending delays in pulsar timing are of the order of microseconds. These delays agree well with previously known approximate expressions, except when the pulsar is near superior conjunction. Similarly, distortions in the shape of the pulse profile become noticeable only in the vicinity of superior conjunction. For our choice of parameters (Table 3.1), we find that the pulsar moves from an orbital phase of 90° to 90.5° in approximately 304 s, whereas template profiles are typically generated by integrating pulses over a few minutes. Therefore, the distortion of the pulse profile can be visualized within this short interval. A natural question that arises is: what happens if the companion black hole possesses significant spin? We aim to address this question in the next chapter.

Chapter 4

Exploring the Light Bending Phenomenon for a Pulsar in a Binary With a Rotating Black Hole

In this chapter, we explore how gravitational light-bending influences the signal received from a pulsar orbiting a rotating black hole companion, with particular emphasis on systems containing stellar-mass black holes. To achieve this, we solve the null geodesic equations in the Kerr spacetime by employing the formalism devised by Kapec and Lupsasca [68] and Gralla and Lupsasca [69].

4.1 The effect of the spin of the companion on the bending delay: A full general relativistic study

In the Kerr spacetime, the paths of the light rays are described by a set of null geodesic equations that are commonly studied in Boyer-Lindquist coordinates where the coordinates x^μ are $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. For each of these coordinates as well as

other parameters, we use appropriate subscripts to represent the relevant reference frame, for example, the subscript ‘T’ is used for the T-frame, the subscript ‘bh’ is used for the bh-frame, etc. Additionally, the subscript ‘s’ is used to mean the values at the source (the pulsar) and the subscript ‘o’ is used to mean the values at the observation point. The integral forms of the null geodesic in Boyer-Lindquist coordinates are given by Gralla and Lupsasca [69]:

$$I_r = G_\theta , \quad (4.1)$$

and

$$\phi_o - \phi_s = I_\phi + D_\lambda G_\phi , \quad (4.2)$$

where we have defined a set of parameters as follows:

$$\begin{aligned} \mathcal{R}(r) = & r^4 + \left[\left(\tilde{a} \frac{GM_c}{c^2} \right)^2 - D_\lambda^2 - D_q \right] r^2 \\ & + 2 \left(\frac{GM_c}{c^2} \right) \left[D_q + \left\{ D_\lambda - \left(\tilde{a} \frac{GM_c}{c^2} \right) \right\}^2 \right] r \\ & - \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 D_q - \left(\frac{M_c^2 c^4}{E^2} \right) r^2 \Delta, \end{aligned} \quad (4.3a)$$

$$\begin{aligned} \Theta(\theta) = & D_q + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \cos^2 \theta - D_\lambda^2 \cot^2 \theta \\ & - \left(\frac{M_c^2 c^4}{E^2} \right) \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \cos^2 \theta, \end{aligned} \quad (4.3b)$$

$$I_r = \int_r \frac{dr}{\text{sign}(p_r) \sqrt{\mathcal{R}(r)}}, \quad (4.3c)$$

$$I_\phi = \oint_r \frac{\left(\tilde{a} \frac{GM_c}{c^2}\right) \left((2GM_c/c^2)r - D_\lambda \left(\tilde{a} \frac{GM_c}{c^2}\right)\right)}{\text{sign}(p_r) \sqrt{\mathcal{R}(r)} \Delta} dr, \quad (4.3d)$$

$$G_\theta = \oint_\theta \frac{d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}}, \quad (4.3e)$$

$$G_\phi = \oint_\theta \frac{\csc^2 \theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} d\theta, \quad (4.3f)$$

$$D_\lambda = cL_z/E, \quad (4.3g)$$

$$D_q = \mathcal{L}/E^2. \quad (4.3h)$$

In these equations, the symbol $\oint$ denotes the integrals to be the path integrals, evaluated along the trajectory connecting the coordinates of the source to the coordinates of the observer. The quantities E (the energy), L_z (the component of the angular momentum of the light ray along the relevant Z-axis), and $\mathcal{L}$ are the constants separating the polar and the azimuthal part of the geodesic equation [70] making D_q and D_λ constants of the motion. M_c represents the mass of the gravitating object. The dimensionless spin parameter is denoted by $\tilde{a}$, which relates to the spin angular momentum $\mathcal{S}_{\text{bh}} = \tilde{a} GM_c^2/c$ of the gravitating body. The four components of the momentum of the light rays in the temporal, radial, polar, and the azimuthal directions are denoted by $p_t, p_r, p_\theta, p_\phi$, respectively.

We also adopt the standard functions and parameters commonly used in the Kerr metric:

$$\Delta = r^2 - 2 \frac{GM_c}{c^2} r + \left(\tilde{a} \frac{GM_c}{c^2}\right)^2, \quad (4.4)$$

$$\rho^2 = r^2 + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \cos^2 \theta, \quad (4.5)$$

$$\Sigma^2 = \left[r^2 + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right]^2 - \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \Delta \sin^2 \theta, \quad (4.6)$$

$$\Omega = \frac{2 \left(\frac{GM_c}{c^2} \right) r \left(\tilde{a} \frac{GM_c}{c^2} \right)}{\Sigma^2}. \quad (4.7)$$

To solve the geodesic equations, we need to express D_q and D_λ in terms of quantifiable parameters. The method for determining these constants is discussed in the Appendix C.5. Once the constants of the motion are obtained, we proceed to solve the geodesic Eqs. (4.1) and (4.2) with the help of Eqs. (4.3) as discussed in details in Appendix C. Through this procedure, we obtain the final polar and azimuthal angles ($\theta_{\text{bh, f}}$ and $\phi_{\text{bh, f}}$) of the light rays in the bh-frame (bh-frame is defined in Sec. 2.2). The unit vector along the final direction of the light ray in the bh-frame can be written as:

$$\hat{n}_{\text{bh, f}} = [\cos \phi_{\text{bh, f}} \sin \theta_{\text{bh, f}}, \sin \phi_{\text{bh, f}} \sin \theta_{\text{bh, f}}, \cos \theta_{\text{bh, f}}]. \quad (4.8)$$

We transform this to the I-frame and get $\hat{n}_{\text{I, f}}$ as:

$$\hat{n}_{\text{I, f}} = R_y(-\lambda_p) R_z(-\eta_p) R_z(\eta_{\text{bh}}) R_y(\lambda_{\text{bh}}) \hat{n}_{\text{bh, f}}. \quad (4.9)$$

To estimate the values of the bending delays, we follow the same method as described in Sec. 3.1. First, we note a particular time, at which, in the absence of the bending, a light ray aligns with $\hat{N}_{\text{I}}$, i.e., with the line of sight (LoS). At this time, the phase of this light ray is zero and the co-latitude is ζ_L . Now, we identify the light ray which would have the phase zero and the co-latitude ζ_L at that particular time in the presence of bending. We do this with the help of the final directions of the light ray given by Eq. (4.9). After this we

find out what was the phase ($\Phi_{p,\text{old}}$) and colatitude (ζ_{old}) of that particular light ray. The change in the phase due to bending is $\Delta\Phi_p = \Phi_{p,\text{old}}$ and the change in the colatitude due to bending as $\Delta\zeta = \zeta_{\text{old}} - \zeta_L$.

4.1.1 Impact of Companion Spin on Bending Delay

The spin of the companion influences both the propagation and the direction of the light rays [11]. The light rays that travel in directions co-rotating with the spin of the companion bend more than the light rays that travel in the direction counter-rotating with the spin of the companion. This phenomenon affects both the longitudinal and latitudinal bending delays resulting in frame dragging bending delays. The longitudinal frame dragging bending delay is defined as:

$$\text{FD}_{\text{long}} = (\tau_{\text{long},\tilde{a}}) - (\tau_{\text{long},0}), \quad (4.10)$$

where $\tau_{\text{long},\tilde{a}}$ and $\tau_{\text{long},0}$ are the longitudinal bending delays considering the spin and ignoring the spin of the companion, respectively. We use Eq. (2.53) to estimate the values of $\tau_{\text{long},\tilde{a}}$ and $\tau_{\text{long},0}$.

Similarly, the latitudinal frame dragging bending delay is defined as:

$$\text{FD}_{\text{lat}} = (\tau_{\text{lat},\tilde{a}}) - (\tau_{\text{lat},0}), \quad (4.11)$$

where $\tau_{\text{lat},\tilde{a}}$ and $\tau_{\text{lat},0}$ are the latitudinal bending delays considering the spin and ignoring the spin of the companion, respectively. Eq. (2.54) is used to estimate the values of $\tau_{\text{lat},\tilde{a}}$ and $\tau_{\text{lat},0}$.

4.2 Numerical study

As previously discussed, we adopt a core–double cone beam model. Using the formalism outlined in Sec. 3.2, we model the emission beam at 1.4 GHz and simulate multiple initial positions of light rays on the $X_m Y_m$ plane. The trajectories of these rays are then traced through curved spacetime, incorporating the effects of gravitational bending as described in Sec. 4.1. From the rays that remain visible after bending, we compute the bending delays and construct both the distorted beam pattern and the corresponding pulse profile.

Due to the absence of any confirmed pulsar-black hole binary, we once again consider hypothetical pulsar-black hole binaries with standard values for relevant parameters that are not too different from the theoretical predictions of one of the models by Chattopadhyay et al. [4], namely ZM–001. The relevant parameters are given in Table 4.1. In this section we need three additional parameters $\tilde{a}$, λ_{bh} and η_{bh} . In most of our investigation we choose the orbital plane of the pulsar as the equatorial plane of the black hole, i.e., $\eta_{\text{bh}} = -90^\circ$ and $\lambda_{\text{bh}} = i$. The spin parameter is varied as $\tilde{a} = 0, 0.5, 0.9$. As discussed in Sec. 3.3.2, temporal variations of different parameters are negligible over a few orbital periods.

To determine the light ray that aligns with the line of sight (LoS) after bending at each orbital phase of the pulsar, we employ an iterative search method. First, we create an $N \times N$ square grid around the LoS, which has the phase (or the longitude) $\Phi = 0$ and the co-latitude ζ_L (see Fig. 1.1). Hence, the edge of the grid are at longitudes $\pm\Phi_{\text{div}}$ and co-latitudes $\zeta_L \pm \zeta_{L\text{div}}$, where Φ_{div} and $\zeta_{L\text{div}}$ are some pre-defined numbers determining the size of the grid. We create initial positions of the light rays at each of these grid points and trace their paths using the formalism described in Sec. 4.1. We then identify the light ray from this grid that is closest to the LoS after bending. After this, we create another $N \times N$ square grid (smaller in size) around this closest point, generate initial positions of the light rays at each of these grid points, trace their paths and find the light ray (from this new grid) that comes closest to the LoS after bending. We continue this process for 20 iterations, which is more than sufficient, as after 15 iterations, the closest light ray

Table 4.1: This table summarizes the parameters used for a representative pulsar–black hole binary system. Listed from top to bottom are: pulsar spin period (P_s), spin-down rate ($\dot{P}_s$), orbital period (P_b), orbital eccentricity (e), inclination of the orbit i , the pulsar mass (M_p), black hole mass (M_c), longitude of periastron (ω), orbital inclination relative to the sky plane (i), the angle between the pulsar’s spin and magnetic axes (α), the angle between the X_s -axis and the projected pulsar spin axis on the sky plane (η_p), the angle between the pulsar spin axis and the Z_s -axis (λ_p), and similarly, η_{bh} is the angle between the X_s -axis and the projected spin axis of the black hole on the sky plane and λ_{bh} is the angle between the black hole spin axis and the Z_s -axis. $\tilde{a}$ is the dimensionless spin parameter of the black hole. Values for P_s , $\dot{P}_s$, and P_b , e , M_c , M_p are taken from Chattopadhyay et al. [4], while the remaining parameters are selected to satisfy the visibility condition $|180^\circ + \mathcal{W}_{1.4,10}^{\text{out}} - \alpha| \leq |\lambda_p| \leq |180^\circ - \mathcal{W}_{1.4,10}^{\text{out}} - \alpha|$, where $\mathcal{W}_{1.4,10}^{\text{out}}$ is the outer cone half-opening angle at 1.4 GHz, defined by the angular distance at which intensity falls to 10% of its peak, following Chapter 1. Time-dependent parameters are quoted at $t = 0$ (i.e., when $A_T = 0$), and in some cases, parameter values are varied to explore their effects, with two values listed where applicable—the first being used more frequently. Unless stated otherwise, we typically assume $\lambda_{bh} = i$. Exact parameter values for each figure are specified in the corresponding captions.

Parameters	Value
P_s [s]	1.67
$\dot{P}_s$ [s s ⁻¹]	1.38×10^{-15}
P_b [days]	5.5
e	0.35
ω [deg]	73.804
i [deg]	87.5 (or 90.0)
M_p [$M_\odot$]	1.8
M_c [$M_\odot$]	14.5
α [deg]	50.0
η_p [deg]	45.0
λ_p [deg]	130.0
η_{bh} [deg]	-90.0
λ_{bh} [deg]	87.5 (or 90.0)
$\tilde{a}$	0.9 (or 0.5)

remains the same in two successive iterations. In other words, after 15 iterations, the change in the values of the bending delays found in two successive iterations is less than 0.1 nanoseconds. At the first step, we chose both Φ_{div} and ζ_{Ldiv} as 0.001 radian and in each step reduce both by a factor of 5. The value of N is kept fixed as 11 throughout. We have also verified that increasing the values of Φ_{div} , ζ_{Ldiv} , and N does not affect the values of the bending delays.

First, we verify whether we get exactly the same results if we set the black hole spin parameter $\tilde{a} = 0$ in the formalism followed in this work, i.e., tracing the path of the light rays in the Kerr spacetime around the black hole with those obtained using the formalism adopted in Chapter 3, i.e., tracing the path of the light rays in the Schwarzschild spacetime around the black hole. In Fig. 4.1, we plot the orbital phase ($A_T + \omega$) in degrees along the ordinate and the difference in delays (calculated using the two formalism as mentioned above) in nanoseconds along the abscissa, the top panel showing the differences in the longitudinal delays while the bottom panel showing the differences in the latitudinal delays. We see that exactly at $A_T + \omega = 90^\circ$, the difference in the longitudinal bending delay is 0.9 nanosecond and latitudinal bending delay is 0.02 nanosecond. At all other orbital phases, the difference in the longitudinal bending delay remains below 0.004 nanosecond and the difference in the latitudinal bending delay remains below 0.0001 nanosecond. Hence, one can conclude that the agreement between the two formalism is satisfactory as it should be from a theoretical point of view. For the rest of the discussion, we solve the equations for the null geodesic in the Kerr spacetime even when we set $\tilde{a}$ as zero.

Next, we aim to quantify the impact of the spin parameter on the values of the bending delays. In Figs. 4.2a and 4.2b, we plot the latitudinal and the longitudinal bending delays respectively as functions of the orbital phase for $\tilde{a} = 0$ (black dashed line) and $\tilde{a} = 0.9$ (blue triangles). We note that, the vertical orange line in each of these figures correspond to $A_T = 0^\circ$ at which we set the time $t = 0$. Hence, the left side of this line actually represents the orbital phase post 360° . For the sake of simplicity $t = 0$ line is not shown

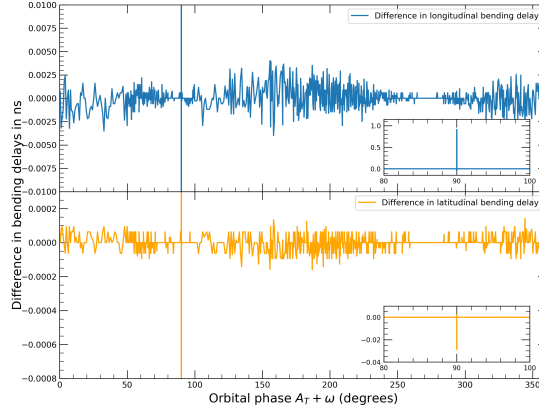


Figure 4.1: Differences in bending delays computed using two distinct formulations of null geodesics around a non-rotating black hole: one using the Schwarzschild solution and the other using the Kerr solution with the spin parameter set to $\tilde{a} = 0$. For both calculations, we adopt $\lambda_{\text{bh}} = i = 87.5^\circ$, and all other relevant parameter values are taken from Table 4.1.

in any other figure in this section. From Figs. 4.2a and 4.2b, we see that the difference is too small to be visualised in these plots. Hence, we decide to rather investigate the differences of the bending delays for zero and non-zero values of the spin parameters. These differences are already defined in Sec. 4.1.1 as the frame-dragging bending delays (or FD bending delays).

Figs. 4.4a and 4.4b present the latitudinal and longitudinal frame-dragging (FD) bending delays, respectively, both computed for a black hole with spin parameter $\tilde{a} = 0.9$. In both plots, the FD delays are shown in nanoseconds (ns) along the ordinate, and the orbital phase, $A_T + \omega$ is plotted in degrees along the abscissa. A discontinuity appears at the orbital phase $A_T + \omega = 90^\circ$ in both Figs 4.4a and 4.4b. The locations of the discontinuities in both figures have been highlighted by vertical grey dashed lines in the zoomed subplots. The discontinuity happens at two orbital phases where $\hat{r}_s$ comes in the plane containing $\hat{S}_{\text{bh}}$ and $\hat{N}_s$. As at this two points, the path of the light ray reverses its direction with respect to the spin of the black hole as demonstrated in Fig. 4.3. This condition can be mathematically expressed as:

$$\hat{N}_s \cdot (\hat{r}_s \times \hat{S}_{\text{bh}}) = 0. \quad (4.12)$$

The vectors are as defined in Secs. 2.2, 2.3. Using Eqs. (2.14), (2.4), and (2.3), we find

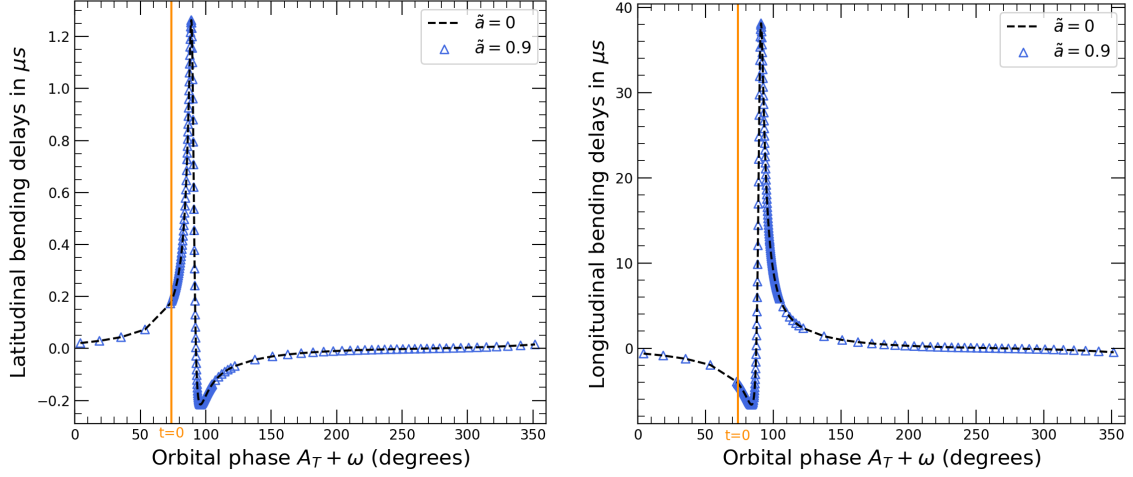


Figure 4.2: Bending delay variations across a full orbit for a hypothetical pulsar–black hole system, comparing the cases with spin parameters $\tilde{a} = 0$ (black dashed curve) and $\tilde{a} = 0.9$ (blue triangles). The system assumes $\lambda_{\text{bh}} = i = 87.5^\circ$, with all other parameters taken from Table 4.1. The left panel illustrates the delay due to latitudinal bending, while the right panel depicts the longitudinal component. A vertical orange line marks $A_T = 0^\circ$, which is taken as the zero point in time ($t = 0$).

that the condition specified in Eq. (4.12) is satisfied at orbital phases $A_T + \omega = 90^\circ$ and $A_T + \omega = 270^\circ$ when $\eta_{\text{bh}} = -90^\circ$, i.e., the spin axis of the black hole is in the $Y_s Z_s$ plane. The discontinuity can be seen in both the latitudinal and the longitudinal FD bending delays for $A_T + \omega = 90^\circ$ in Figs 4.4a and 4.4b. However, the discontinuity at $A_T + \omega = 270^\circ$ is very small because the bending itself is negligible at this orbital phase, thus the effect of the spin on the light bending is not perceptible.

Similarly, when $\eta_{\text{bh}} = -40^\circ$, the condition given in Eq. (4.12) is satisfied at $A_T + \omega = 92.98^\circ$ and $A_T + \omega = 272.98^\circ$. Figs. 4.5a and 4.5b present the latitudinal and the longitudinal FD bending delays for $\eta_{\text{bh}} = -40^\circ$ zooming near the larger discontinuity, i.e., in the range of $A_T + \omega$ as 80° – 100° . As expected, we see the discontinuities at $A_T + \omega = 92.975^\circ$, although the discontinuity is less prominent for the longitudinal FD delay as the delay itself is small at this orbital phase. The discontinuity at $A_T + \omega = 272.98^\circ$ cannot be seen as the FD delays themselves are very small and not plotted in this orbital phase. From Fig. 2.1, it is understandable that a change of λ_{bh} for a fixed η_{bh} does not affect the solution of Eq. (4.12).

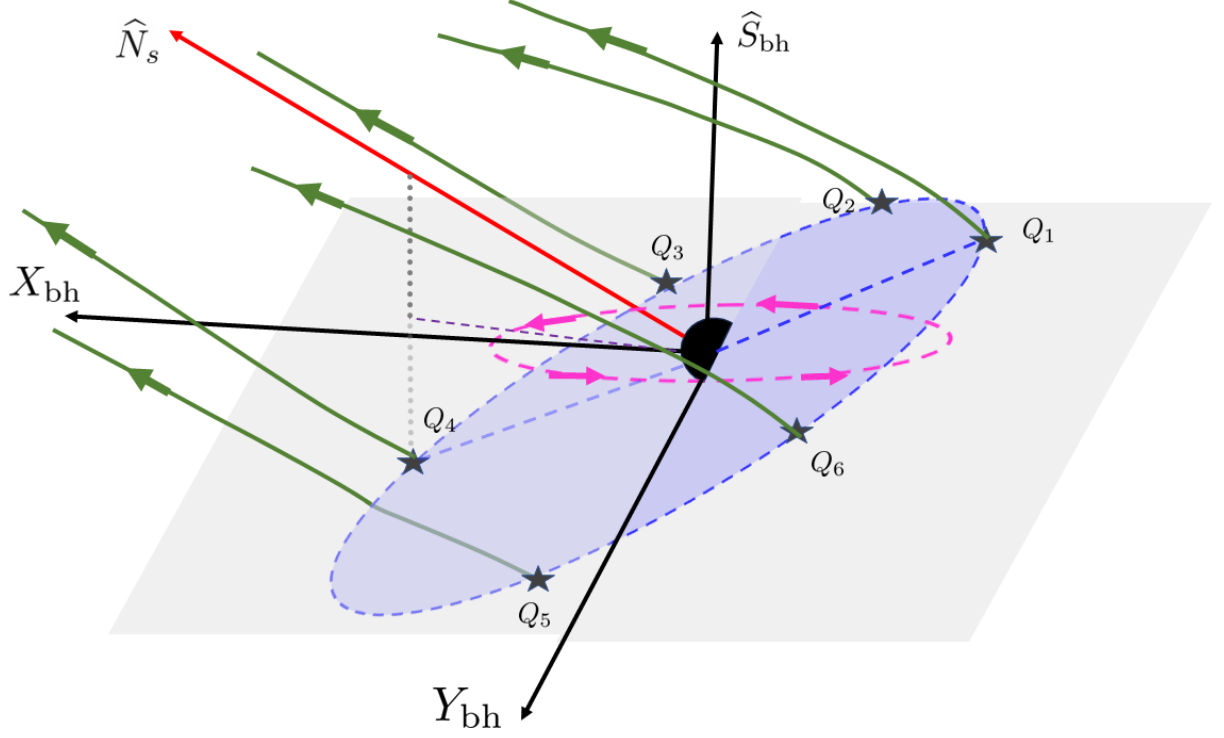


Figure 4.3: A schematic diagram depicting how the directions of light rays (shown in green) that align with the line of sight (LoS, red line) change depending on the spin of the black hole, indicated by a magenta dashed circle with arrows. The pulsar's orbital path (light-blue plane) includes several positions labeled as Q_1 through Q_6 , marked by black stars. The equatorial plane of the black hole is represented by the light-grey plane, and the black hole itself is partially illustrated as a filled black semicircle (not to scale). At positions Q_1 , Q_2 , and Q_3 , the light rays travel in the same direction as the black hole's spin, whereas at Q_4 , Q_5 , and Q_6 , their directions are opposite to the spin. The purple dashed line shows the projection of the LoS onto the equatorial plane of the black hole, while the green dashed line represents its projection onto the orbital plane. The points Q_1 , Q_2 , and Q_3 lie on one side of the green dashed line, and Q_4 , Q_5 , and Q_6 lie on the other.

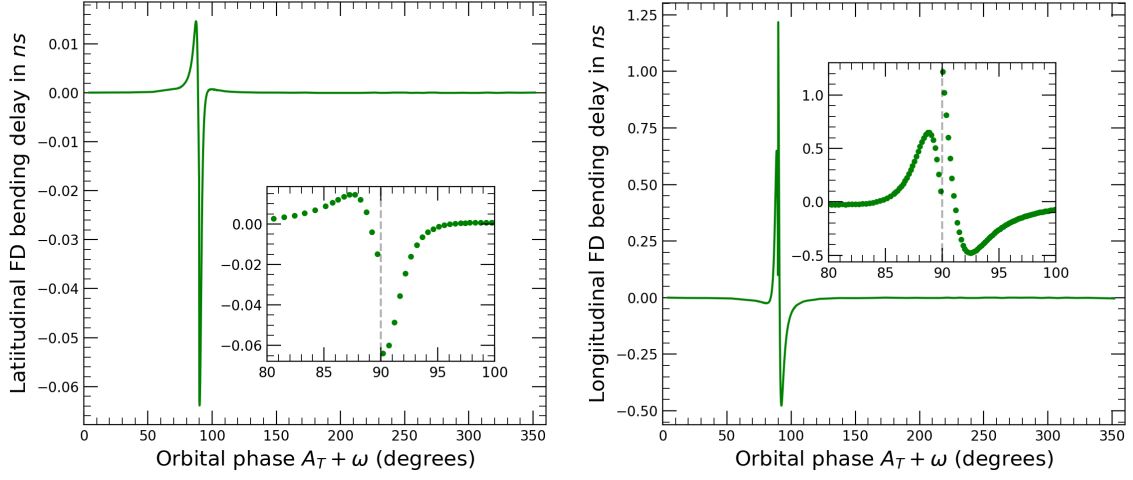


Figure 4.4: Frame-dragging (FD) bending delays for $\tilde{a} = 0.9$ calculated over a complete orbit for a hypothetical pulsar–black hole binary with $\lambda_{\text{bh}} = i = 87.5^\circ$. All other relevant parameters are taken from Table 4.1. The left panel illustrates the latitudinal component of the FD bending delay, while the right panel displays the longitudinal component. In both panels, the grey dashed vertical line within the zoomed inset highlights the position of the discontinuity.

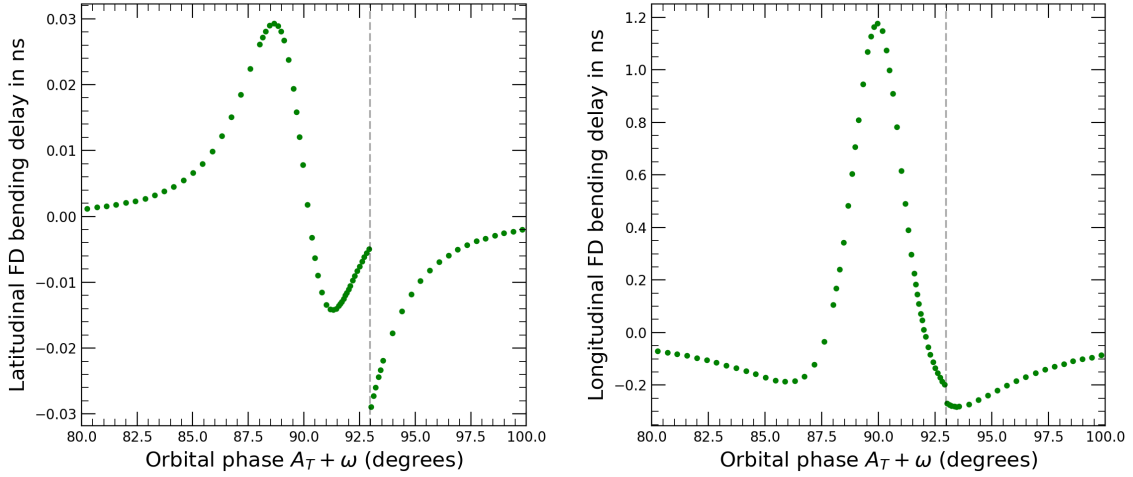
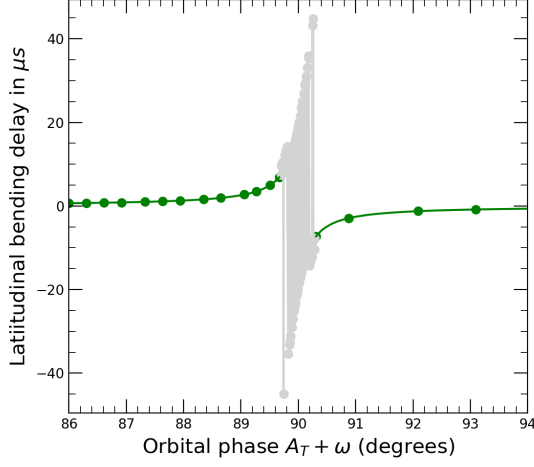


Figure 4.5: Frame-dragging (FD) bending delays computed for $\tilde{a} = 0.9$ across a complete orbital cycle in a hypothetical pulsar–black hole binary system. The parameters used here are $\eta_{\text{bh}} = -40^\circ$ and $\lambda_{\text{bh}} = i = 87.5^\circ$, with all other values taken from Table 4.1. The left panel displays the latitudinal FD bending delay, while the right panel shows the longitudinal FD bending delay. In both panels, the grey dashed vertical line marks the position of the discontinuity.

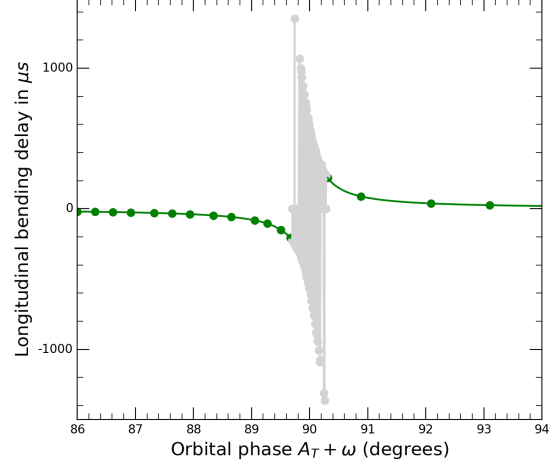
One intriguing aspect to investigate is the FD delays for an edge-on orbit ($i = 90^\circ$) with the pulsar rotating in the equatorial plane of the black hole, specified by $\eta_{\text{bh}} = -90^\circ$ and $\lambda_{\text{bh}} = 90^\circ$. As discussed in Chapter 3, for an edge-on binary, the bending delays exhibit irregularities near the superior conjunction ($A_T + \omega = 90^\circ$). At this point, the cross-section of the pulsar beam reaches the observer by crossing the black hole, which acts as a convex lens. This lensing effect converges light rays from a ring-shaped region towards the LoS, causing these irregularities. In Figs. 4.6a and (4.6b), we plot the latitudinal and longitudinal bending delays, respectively, for $\tilde{a} = 0$ (solid line) and $\tilde{a} = 0.5$ (green filled circles) for $i = 90^\circ$, $\eta_{\text{bh}} = -90^\circ$, and $\lambda_{\text{bh}} = 90^\circ$, zooming near the orbital phase where the irregularities are expected. These irregular regions are highlighted in grey. Both delays exhibit irregularities between the orbital phases $A_T + \omega = 89.7^\circ$ and $A_T + \omega = 90.3^\circ$. In Figs. 4.6c and 4.6d, we plot the the latitudinal and the longitudinal FD delays, respectively, and see similar irregular regions.

In Chapter 3, we showed that the values of both the longitudinal and the latitudinal bending delays increase with the increase in the values of the companion mass (M_c), the orbital inclination angle (i), and the orbital eccentricity (e), and decrease with the increase of the orbital period (P_b). Here we noticed similar trends when we solve the equations for the null geodesic (in each case, for a fixed value of $\tilde{a}$) for the exact same physical reasons. It seems more interesting to explore what happens to the FD bending delays, especially near the orbital phase of 90° where the magnitudes of the FD bending delays are large (as seen in Fig. 4.4). Hence, we investigate how the FD bending delays change if we change only one parameter at a time keeping the remaining ones fixed as those mentioned in Table. 4.1, with $\tilde{a} = 0.5$ and $i = 87.5^\circ$.

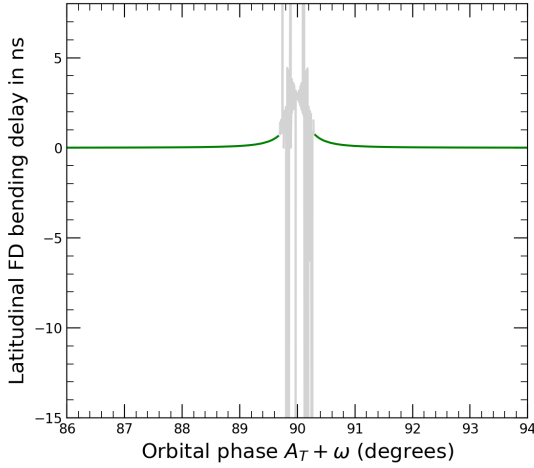
First, in Figs. 4.7a and 4.8a, we plot the latitudinal and the longitudinal FD bending delays against the orbital phase, respectively, for companion masses of 10, 15, 20, 25, and 30 solar masses. We see that as the mass of the companion increases, the values of the FD bending delays also increase. This happens because the spin angular momentum the



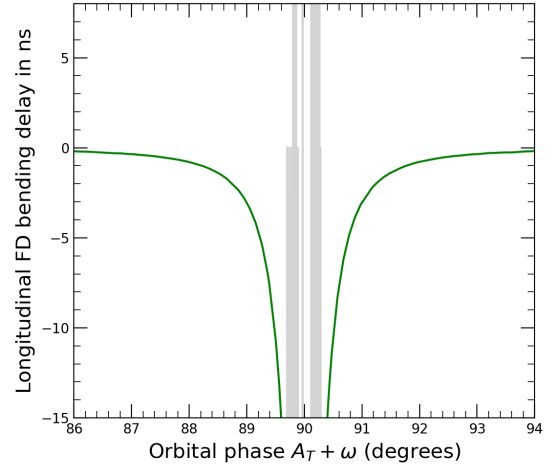
(a) Latitudinal FD bending delays for $\tilde{a} = 0.5$, $\lambda_{\text{bh}} = i = 90^\circ$.



(b) Longitudinal FD bending delays for $\tilde{a} = 0.5$, $\lambda_{\text{bh}} = i = 90^\circ$.



(c) Latitudinal FD bending delays for $\tilde{a} = 0.5$, $\lambda_{\text{bh}} = i = 90^\circ$.



(d) Longitudinal FD bending delays for $\tilde{a} = 0.5$, $\lambda_{\text{bh}} = i = 90^\circ$.

Figure 4.6: Bending delays and the frame-dragging (FD) bending delays for a hypothetical pulsar-black hole binary. The top left panel (panel a) shows the latitudinal bending delay and the top right panel (panel b) shows the longitudinal bending delay. The bottom left panel (panel c) shows the latitudinal FD bending delay and the bottom right panel (panel d) shows the longitudinal FD bending delay. We have chosen spin parameter $\tilde{a} = 0.5$, $\lambda_{\text{bh}} = i = 90^\circ$, with all other system parameters taken from Table 4.1. Irregularities appear in the delay curves between $A_T + \omega = 89.7^\circ$ and $A_T + \omega = 90.3^\circ$. The irregular regions are shown in gray.

black hole depends both on $\tilde{a}$ and mass of the black hole (M_c) as: $\mathcal{S}_{\text{bh}} = \tilde{a} GM_c^2/c$.

Next, in Figs. 4.7b and 4.8b, we plot the latitudinal and the longitudinal FD bending delays against the orbital phase, respectively, for values of the orbital period as 1 day, 2 days, 3 days, and 5 days. As expected, we see that the values of the FD bending delays increases as the value of the orbital period decreases. This happens because for fixed values of masses of the pulsar and the companion, a smaller value of the orbital period means a smaller size of the orbit, hence, the light rays travel closer to the companions in the region where the spacetime curvature is larger.

Figs. 4.7c and 4.8c show the latitudinal and the longitudinal FD bending delays for eccentricities 0.1, 0.3, 0.5, and 0.7. The maximum values of the FD bending delays are larger for higher eccentricities. This occurs because, for a higher eccentricity, the same size of the orbit (i.e., the same values of P_b , M_p , and M_c) results in a shorter distance between the pulsar and the companion near the periastron and nearby orbital phases, which increases the effect of the companion's spin on the geodesic path of the light rays.

Next, in Fig. 4.7d and 4.8d, we vary the inclination of the orbit in the range of 86.0° - 89.5° . As expected, the values of the FD bending delays increase as the inclination of the orbit increases. As the irregular region near the orbital phase of 90° appear for $i > 89.7^\circ$, in this figure we use only the values of i lower than that limit.

In all of the panels of Figs. 4.7 and 4.8, the vertical dashed lines at the orbital phase of 90° represent the locations of the discontinuities in the delay curves that has been discussed already.

One might intuitively think that the FD bending delays would not depend on the orientation of the spin axis of the pulsar. However, as the longitudinal and the latitudinal directions of the pulsar are determined by the direction of its spin axis, if the orientation of the spin axis changes, the latitude and the longitude of all the light rays coming from the pulsar change, leading to variations in the FD bending delays. This effect can be seen

in Figs. 4.7e, 4.7f, 4.8e and 4.8f. Fig. 4.7e shows the variation of the latitudinal FD delay with the variation of η_p , keeping it positive while Fig. 4.7f shows the variation of the latitudinal FD delay with the variation of η_p , keeping it negative. Similarly, Fig. 4.8e shows the variation of the longitudinal FD delay with the variation of η_p , keeping it positive, while Fig. 4.8f shows the variation of the longitudinal FD delay with the variation of η_p , keeping it negative. We see a horizontal flip in the shapes of the latitudinal FD delay curves and a vertical flip in the shapes of the longitudinal FD delay curves for a sign reversal of η_p . We did not attempt to investigate the variation of the FD bending delays with the variation of λ_p , as only a small range $134.51^\circ \geq \lambda_p \geq 125.49^\circ$ satisfy the visibility condition $|180^\circ + \mathcal{W}_{1,4,10}^{out} - \alpha| \leq |\lambda_p| \leq |180^\circ - \mathcal{W}_{1,4,10}^{out} - \alpha|$ for our choice of $\alpha = 50^\circ$, and our choice of $\lambda_p = 130^\circ$ lies almost middle of that range.

As the concept of $\tilde{a}$, η_{bh} , and λ_{bh} does not arise when the pulsar is in a binary system with a Schwarzschild black hole as studied in Chapter 3, the effect of these parameters on the latitudinal and the longitudinal bending delays is worth exploring. We have already seen in Fig. 4.2 that the change in the latitudinal and longitudinal bending delays with $\tilde{a}$ is imperceptible. Hence in Fig. 4.9, we demonstrate the effect of variation of $\tilde{a}$ on FD bending delays. We see that the values of both the latitudinal and the longitudinal FD delays decrease with the decrease of $\tilde{a}$ from 0.9 to 0.5 and to 0.1.

We find that a change in the value of η_{bh} changes the values of the latitudinal and the longitudinal bending delays in the nanosecond order and a change in the value of λ_{bh} changes the values of the latitudinal and the longitudinal bending delays in the sub-nanosecond order. The changes are maximum near the orbital phase of 90° . As these changes would not be visible if we plot the bending delays that are in the microsecond order, we rather plot the differences. For this purpose, we first define a set of parameters as:

$$\Delta\tau_{\text{long}\eta, \eta_1 - \eta_2} = \tau_{\text{long}\eta, \eta_1} - \tau_{\text{long}\eta, \eta_2}, \quad (4.13)$$

$$\Delta\tau_{\text{long}\lambda, \lambda_1 - \lambda_2} = \tau_{\text{long}\lambda, \lambda_1} - \tau_{\text{long}\lambda, \lambda_2}, \quad (4.14)$$

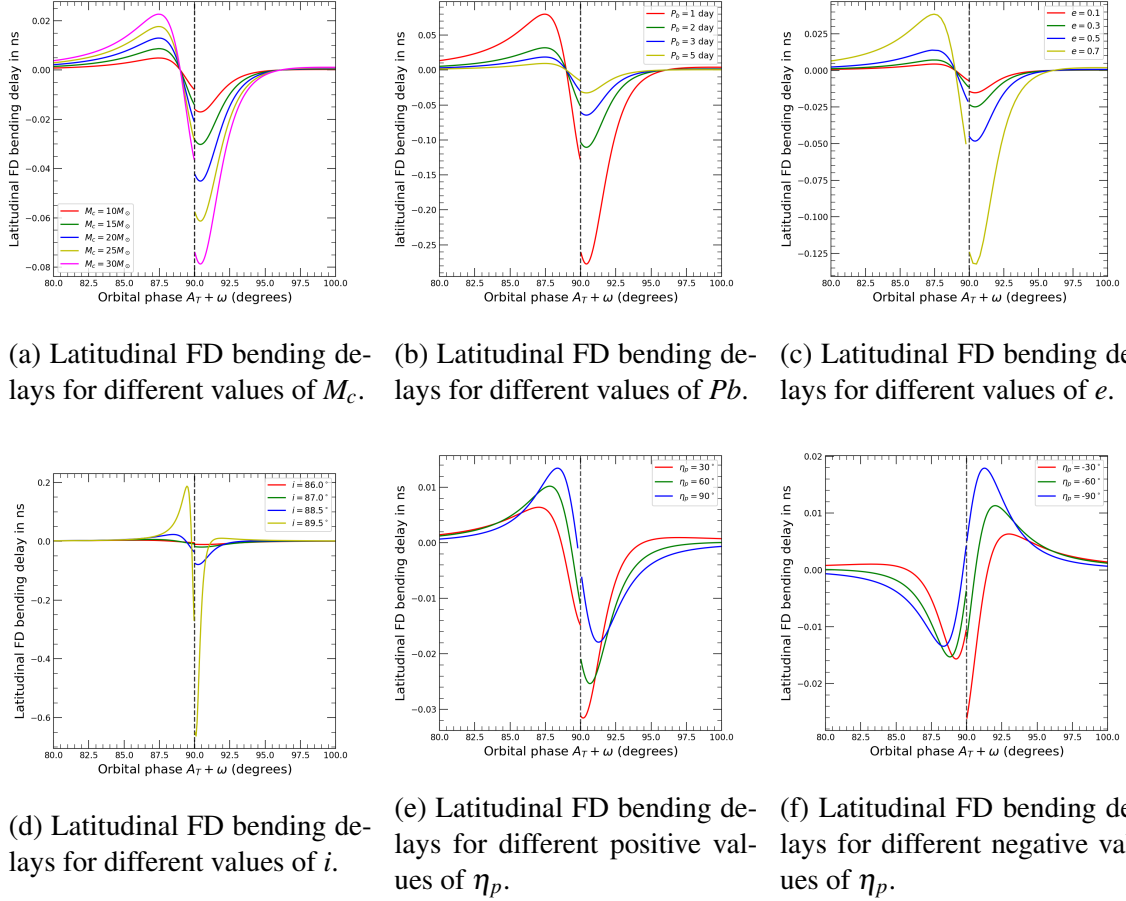
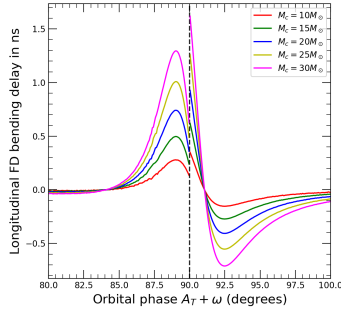
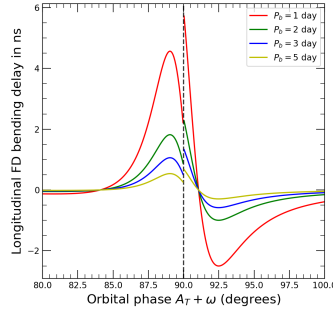


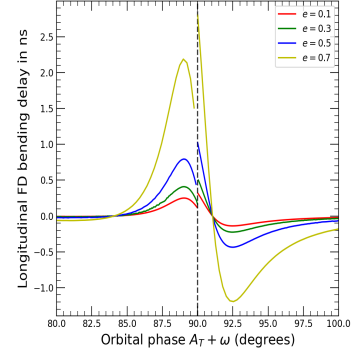
Figure 4.7: Latitudinal frame-dragging (FD) bending delay curves for hypothetical pulsar–black hole binaries. In each panel, one parameter is varied while all others are held constant as specified in Table 4.1. The spin parameter is fixed at $\tilde{a} = 0.5$, except in panel (f) where it is varied. The inclination $\lambda_{\text{bh}} = i = 87.5^\circ$ is used throughout, except in panel (c), where only i is varied while keeping λ_{bh} fixed at 87.5° . Vertical dashed lines mark the positions of the discontinuities in the delay curves.



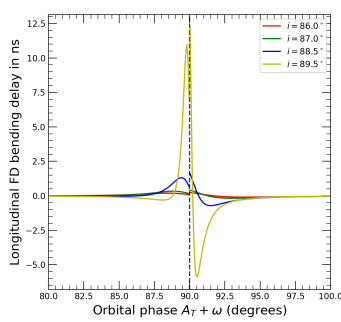
(a) Longitudinal FD bending delays for different values of M_c .



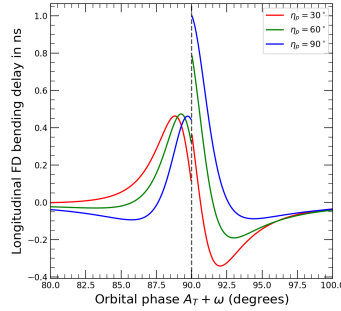
(b) Longitudinal FD bending delays for different values of P_b .



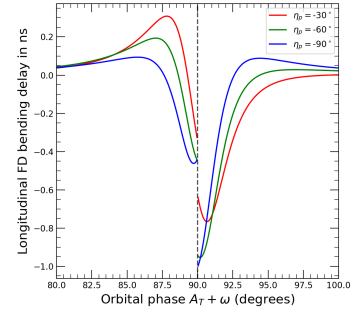
(c) Longitudinal FD bending delays for different values of e .



(d) Longitudinal FD bending delays for different values of i .

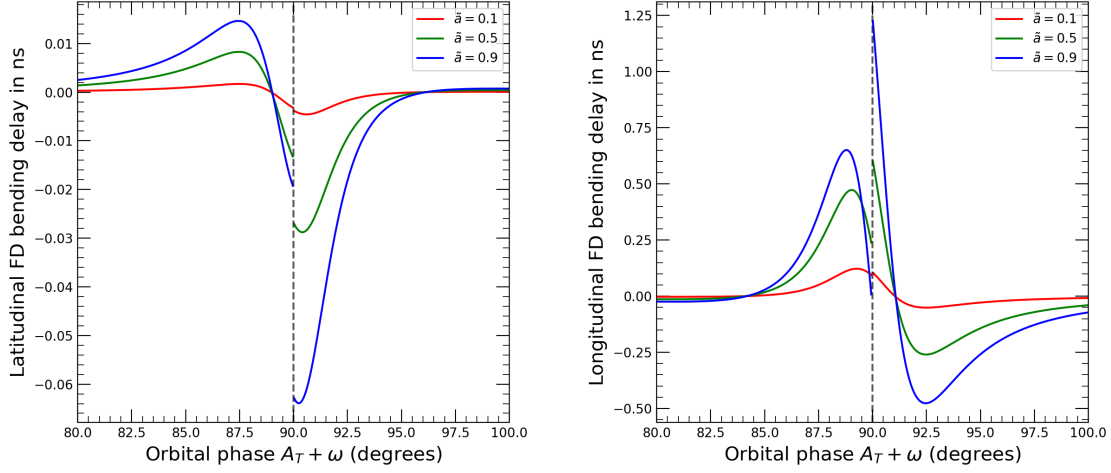


(e) Longitudinal FD bending delays for different positive values of η_p .



(f) Longitudinal FD bending delays for different negative values of η_p .

Figure 4.8: Latitudinal frame-dragging (FD) bending delay curves for hypothetical pulsar–black hole binaries. In each panel, a single parameter is varied while all others are held fixed as specified in Table 4.1. The spin parameter is set to $\tilde{a} = 0.5$ in all panels except (f), where it is varied. Likewise, the inclination angles are taken as $\lambda_{\text{bh}} = i = 87.5^\circ$, except in panel (c), where only i is varied while λ_{bh} remains fixed at 87.5° . The vertical dashed lines indicate the positions of discontinuities in the delay curves.



(a) Latitudinal FD bending delays for different values of $\tilde{a}$. (b) Longitudinal FD bending delays for different values of $\tilde{a}$.

Figure 4.9: Latitudinal (panel a) and longitudinal (panel b) frame-dragging (FD) bending delay curves for hypothetical pulsar–black hole binaries, shown for different values of the spin parameter $\tilde{a}$. The inclination angles are fixed at $\lambda_{\text{bh}} = i = 87.5^\circ$, and all other relevant parameters are adopted from Table 4.1. Vertical grey dashed lines indicate the positions of discontinuities in the delay curves.

$$\Delta\tau_{\text{lat}\eta, \eta_1 - \eta_2} = \tau_{\text{lat}\eta, \eta_1} - \tau_{\text{lat}\eta, \eta_2}, \quad (4.15)$$

and

$$\Delta\tau_{\text{lat}\lambda, \lambda_1 - \lambda_2} = \tau_{\text{lat}\lambda, \lambda_1} - \tau_{\text{lat}\lambda, \lambda_2}, \quad (4.16)$$

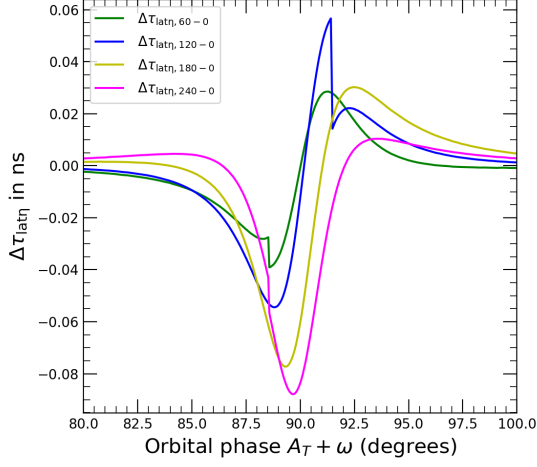
where $\tau_{\text{long}\eta, \eta_i}$ and $\tau_{\text{lat}\eta, \eta_i}$ mean the values of the longitudinal and the latitudinal bending delays, respectively, for $\eta_{\text{bh}} = \eta_i$ degrees with all other parameters having constant pre-assigned values (here as in Table 4.1). Similarly, $\tau_{\text{long}\lambda, \lambda_i}$ and $\tau_{\text{lat}\lambda, \lambda_i}$ mean the values of the longitudinal and the latitudinal bending delays, respectively, for $\lambda_{\text{bh}} = \lambda_i$ degrees with all other parameters having constant pre-assigned values (here as in Table 4.1). In Fig. 4.10, we plot the orbital phase in the range of 80° - 100° along the abscissa and $\Delta\tau_{\text{lat}\eta, \eta_i - 0}$, $\Delta\tau_{\text{long}\eta, \eta_i - 0}$, $\Delta\tau_{\text{lat}\lambda, \lambda_i - 0}$, and $\Delta\tau_{\text{long}\lambda, \lambda_i - 0}$ along the ordinate in different panels by varying $\eta_{\text{bh}} = \eta_i$ and $\lambda_{\text{bh}} = \lambda_i$ over their full allowed ranges of 0° - 360° and 0° - 180° , respectively. From Fig. 4.10, it is clear that $\Delta\tau_{\text{lat}\eta, \eta_i - 0}$, $\Delta\tau_{\text{long}\eta, \eta_i - 0}$, $\Delta\tau_{\text{lat}\lambda, \lambda_i - 0}$, and $\Delta\tau_{\text{long}\lambda, \lambda_i - 0}$ remain in the nanosecond order. However, all these parameters depend strongly on the

orbital phase in the range of 85° - 95° . In this range, they can even change the sign one or more times. The variation of η_{bh} has a stronger effect than the variation of λ_{bh} .

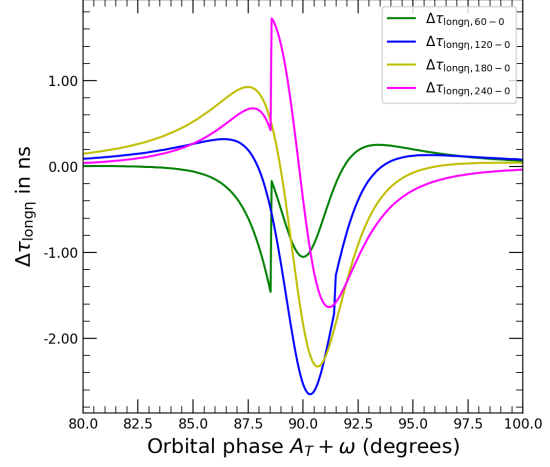
Next, we explore the effect of the orientation of the spin axis of the black hole, i.e., the values of η_{bh} and λ_{bh} on the FD bending delays. In Figs. 4.11a, 4.11b, 4.12a, and 4.12b, we study the effect of different choice of the value of η_{bh} on the FD bending delays. In these figures, we plot the values of the FD bending delays along the ordinates and the orbital phase in the range of 80° - 100° along the abscissa. Discontinuities occur at different orbital phases, as given by Eq. (4.12). For the value of η_{bh} being 40° , 90° , and 120° , the discontinuities occur at orbital phases of $(87.02^\circ, 267.02^\circ)$, $(90.0^\circ, 270.0^\circ)$, and $(91.44^\circ, 271.44^\circ)$, respectively, while for the value of η_{bh} being -40° , -90° , and -120° , the discontinuities occur at orbital phases of $(92.97^\circ, 272.97^\circ)$, $(90.0^\circ, 270.0^\circ)$, and $(88.55^\circ, 268.55^\circ)$, respectively. In Figs. 4.11a, 4.11b, 4.12a, and 4.12b, only the first discontinuity from each pair can be seen as the second one is out of the range plotted, and would be imperceptibly small even if those ranges were included as the values of the FD bending delays themselves are small at those orbital phases.

In Figs. 4.11c, 4.11d, 4.12c, and 4.12d, we study the effect of different choice of the value of λ_{bh} on the FD bending delays. We see that the values of the FD bending delays are larger when the value of λ_{bh} comes closer to 90° . In fact, the FD bending delays are maximum when $\lambda_{\text{bh}} = i$ ($i = 87.5^\circ$ in our case). We also found that the value of the FD bending delays are symmetric around $\lambda_{\text{bh}} = i$. As mentioned earlier, the location of the discontinuity does not depend on the value of λ_{bh} . In all panels of Figs. 4.11 and 4.12, we use the values of the relevant parameters as those given in Table 4.1 with $\tilde{a} = 0.5$ except the parameter that is varied.

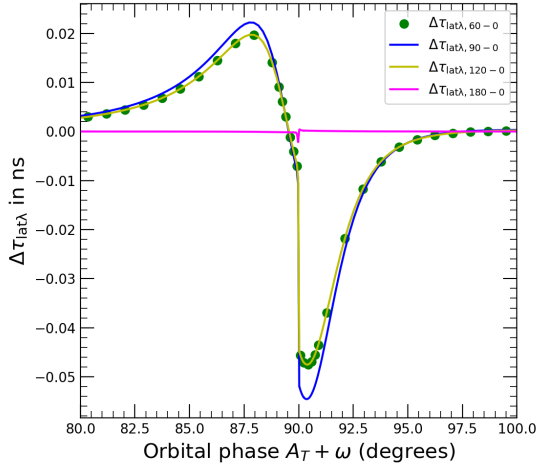
In the previous chapter (Chapter 3), we showed that the photon distribution on the cross-section of the beam changes due to bending, resulting in a change in the pulse profile. Although the overall shape of the profile remains unaltered, the heights of the peaks increase and sometimes small features appear. These effects were seen to be more prominent closer



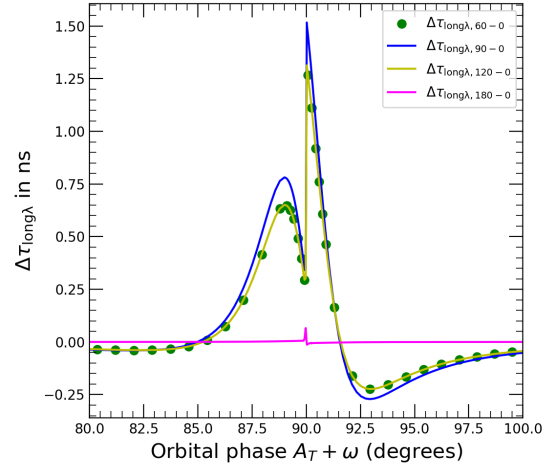
(a) Difference in the latitudinal bending delays for various values of η_{bh} .



(b) Difference in the longitudinal bending delays for various values of η_{bh} .

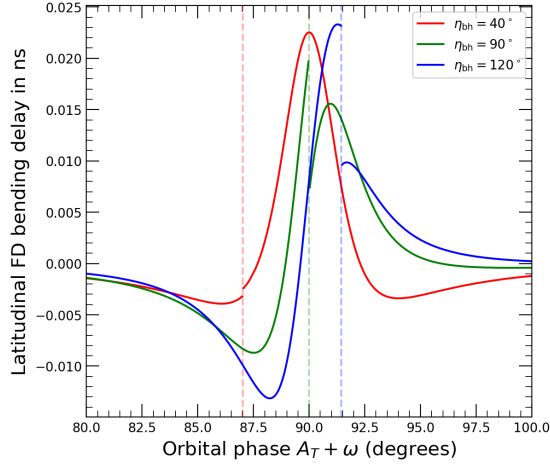


(c) Difference in the latitudinal bending delays for various values of λ_{bh} .

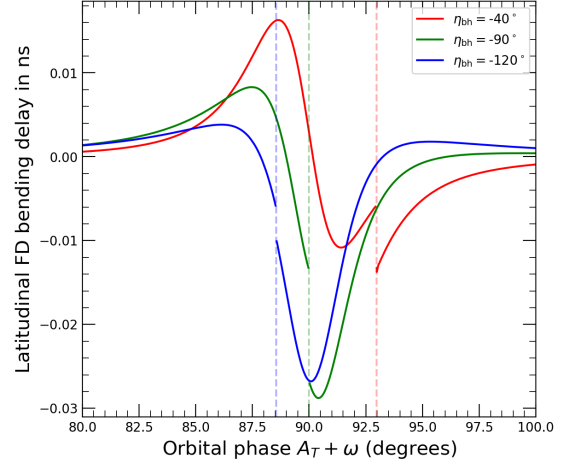


(d) Difference in the longitudinal bending delays for various values of λ_{bh} .

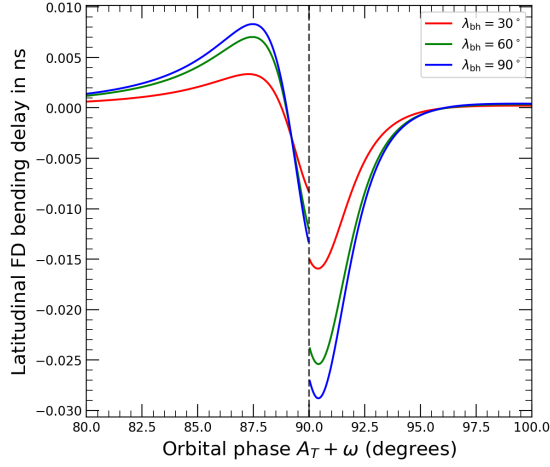
Figure 4.10: Effects of varying η_{bh} and λ_{bh} on the latitudinal and longitudinal bending delays for hypothetical pulsar–black hole binaries with $\tilde{a} = 0.9$ and $i = 90^\circ$. Each panel shows the delay differences over a restricted orbital phase range (in degrees) where the variations are most prominent. Panel (a) (top right) displays $\Delta\tau_{\text{lat}\eta, \eta_1-0}$, panel (b) (top left) shows $\Delta\tau_{\text{long}\eta, \eta_1-0}$, panel (c) (bottom right) presents $\Delta\tau_{\text{lat}\lambda, \lambda_1-0}$, and panel (d) (bottom left) shows $\Delta\tau_{\text{long}\lambda, \lambda_1-0}$. These quantities are defined in the main text. All relevant parameters are taken from Table 4.1, except for the ones being varied. For panels (a) and (b), λ_{bh} is set equal to i .



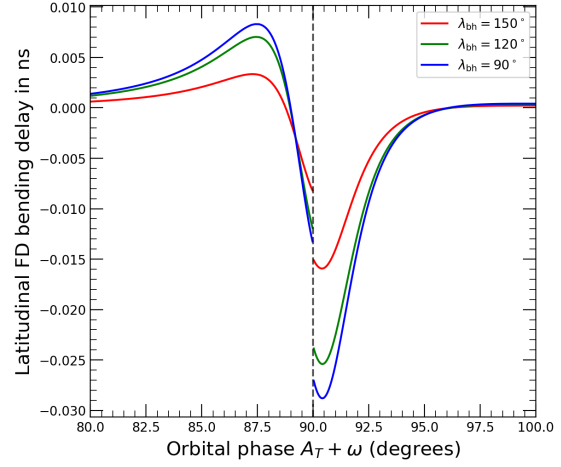
(a) Latitudinal FD bending delays for different positive values of η_{bh} .



(b) Latitudinal FD bending delays for different negative values of η_{bh} .

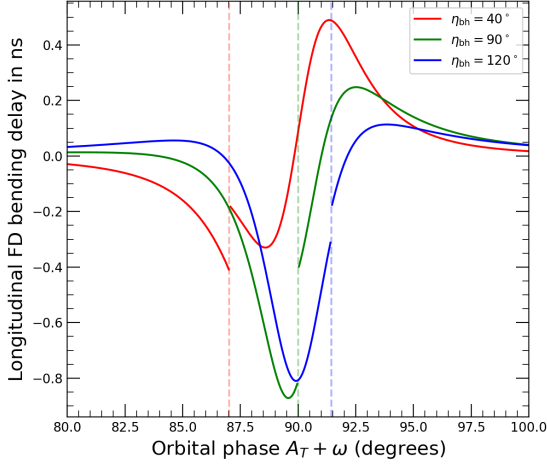


(c) Latitudinal FD bending delays for different values of $\lambda_{\text{bh}} \leq 90^\circ$.

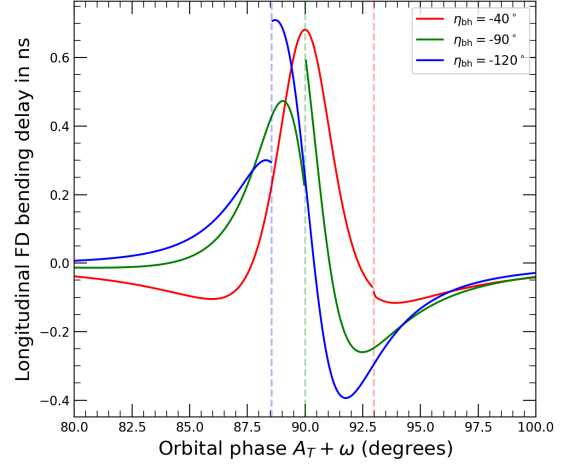


(d) Latitudinal FD bending delays for different values of $\lambda_{\text{bh}} \geq 90^\circ$.

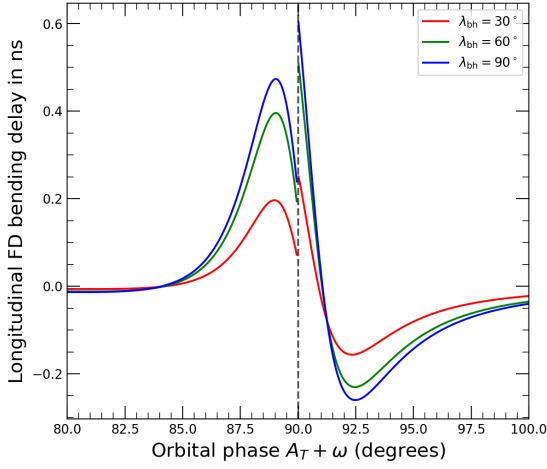
Figure 4.11: Latitudinal FD bending delays for hypothetical pulsar–black hole binaries with $\tilde{a} = 0.5$, shown for different orientations of the black hole’s spin axis. In each panel, only one parameter is varied while all others are kept fixed, with $i = 87.5^\circ$ and remaining values taken from Table 4.1. In panels (a) and (b), where η_{bh} is varied, we set $\lambda_{\text{bh}} = i$.



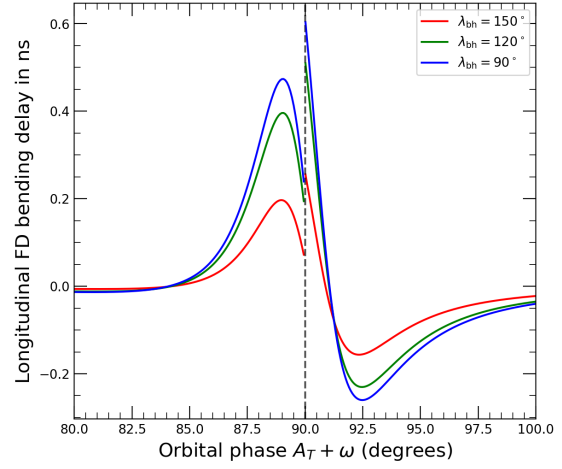
(a) Longitudinal FD bending delays for different positive values of η_{bh} .



(b) Longitudinal FD bending delays for different negative values of η_{bh} .



(c) Longitudinal FD bending delays for different values of $\lambda_{\text{bh}} \leq 90^\circ$.



(d) Longitudinal FD bending delays for different values of $\lambda_{\text{bh}} \geq 90^\circ$.

Figure 4.12: The longitudinal FD bending delay curves for hypothetical pulsar-black hole binaries, where each panel illustrates the effect of varying a single parameter. All other parameters are fixed to the values listed in Table 4.1, with $\tilde{a} = 0.5$ and $i = 87.5^\circ$, unless stated otherwise. In panels (a) and (b), η_{bh} is varied while keeping $\lambda_{\text{bh}} = i$.

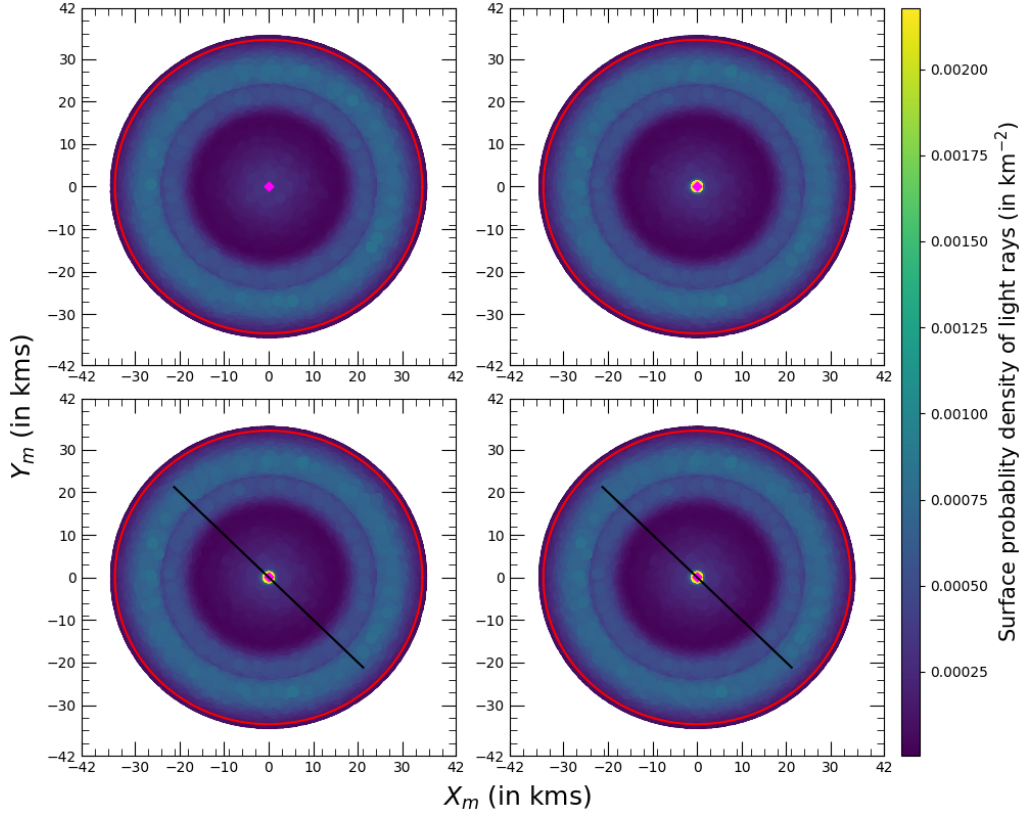


Figure 4.13: Photon distribution across the beam cross-section of a pulsar orbiting a black hole, assuming $\lambda_{\text{bh}} = i = 90^\circ$. The top-left panel shows the undistorted beam without gravitational bending. The top-right, bottom-left, and bottom-right panels display the distorted photon distributions due to bending for spin parameters $\tilde{a} = 0, 0.5$, and 0.9 , respectively. All other parameters are taken from Table 4.1. In each case, the intensity profile corresponds to the moment when the line of sight (LoS) passes through the center of the beam. In the panels with $\tilde{a} \neq 0$, the black solid line indicates the projection of the black hole's spin axis onto the beam cross-section.

to the superior conjunction, i.e., for $i = 90^\circ$ and $A_T + \omega = 90^\circ$. This happens because in this situation, the LoS contains the gravitating body (the companion black hole) and hence a large number of photons that would be away from the LoS in absence of the bending, get diverted towards the LoS. In this chapter, we investigate whether in this situation, the spin of the black hole plays any significant role in the photon distribution across the beam. For this purpose, we choose $\lambda_{\text{bh}} = i = 90^\circ$, $\eta_{\text{bh}} = -90^\circ$.

In Fig. 4.13, we show the photon distributions across the beam (on the $X_m Y_m$ plane). The top-left panel is the photon distribution of the original beam (without any bending), the top-right panel is the photon distribution of the distorted (due to bending) beam when

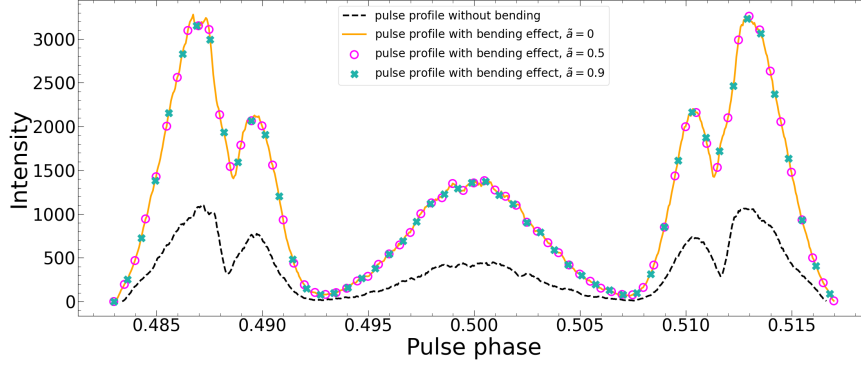


Figure 4.14: Pulse profiles of a pulsar in a binary system with a black hole companion, assuming $\lambda_{\text{bh}} = i = 90^\circ$ and orbital phase $A_T + \omega = 90^\circ$. The black dashed curve represents the profile without gravitational bending. The profiles including bending are shown for different spin parameters of the black hole, e.g., $\tilde{a} = 0$ (solid orange line), $\tilde{a} = 0.5$ (magenta unfilled circles), and $\tilde{a} = 0.9$ (sea-green crosses). All other parameters are consistent with those listed in Table 4.1.

$\tilde{a} = 0$, the bottom-left panel shows the photon distribution of the distorted beam (due to bending) when $\tilde{a} = 0.5$, and the bottom-right panel shows the photon distribution of the distorted (due to bending) beam when $\tilde{a} = 0.9$. In these panels, in addition to the colour coded photon distributions, we see a solid black line when $\tilde{a} \neq 0$. This line represents the projection of the spin axis on the cross-section of the beam. For each case, we show the intensity distribution when the LoS is at the middle of the beam¹. We see that the bending changes the intensity distribution in comparison to the original beam, however the spin of the black hole does not show any visible effect.

To establish the above observation even strongly, in Fig. 4.14, we plot pulse profiles obtained from the same pulsar-black hole binary without and with the effect of the bending, and for various values of $\tilde{a}$ when the bending is considered. We use the same formalism as explained in Sec. 1.3.2 to get a dimensionless proxy for the intensity. This intensity is plotted along the ordinate and the pulse phase is plotted along the abscissa. The pulse phase is defined to vary from 0 to 1, covering a full rotation of the magnetic axis. For visualisation purposes, we define the phase in such a way that the peak of the core component

¹In Chapter 3, we showed that when the LoS sweep across the beam, at each position of the LoS, the photon distribution on the $X_m Y_m$ plane is slightly different (see Figs. 3.10a and 3.10a).

of the profile without the bending is at the phase of 0.5. In other words, at a pulse phase of 0.5, the LoS is at the centre of the unbent beam. When plotting the profiles, we show only the range over which the LoS is within the beam, instead of the full pulse phase range. From Fig. 4.14, it is evident that the pulse becomes stronger, i.e., the intensity increases (due to the enhanced number of light rays along the LoS) due to bending, but the value of $\tilde{a}$ does not play any role in this enhancement.

While obtaining Figs. 4.13 and 4.14, we used the procedure as described in Sec. 3.3.3.

4.3 Exploring the bending delay when the companion is a supermassive black hole

At this point, a logical curiosity might arise, what happens when a pulsar orbits a supermassive black hole (SMBH)? How does the light bending phenomenon affect the signal of the pulsar? As we do not have any such system discovered yet, we again choose a hypothetical system with parameters the same as the Table 4.1, except the mass of the black hole M_c is replaced by $10^6 M_\odot$.

Note that, SMBHs are typically found in dense stellar environments, such as the centres of galaxies. Such an SMBH, Sgr A*, is known at the centre of our Galaxy. Many astrophysical bodies (various types of stars and perhaps stellar mass black holes) are orbiting Sgr A*, usually in wide orbits, such as S1 and S2 [71, 72, 73, 74] and in each of these cases, one can treat the SMBH and the star as a binary. We have chosen a hypothetical pulsar-SMBH binary in a comparatively short orbit and assume that the other astrophysical bodies around the SMBH do not affect this binary. There might be systems involving the SMBH and a pulsar, where many-body physics should be used to model the orbital dynamics, e.g., a neutron star-neutron star or a neutron star - white dwarf system orbiting around the SMBH. If the first orbit is much smaller than the second, one can consider it as

a hierarchical triple and model orbital dynamics perturbatively. The delays of the pulsar signal also can be modelled accordingly. If the objects in the many body systems is of nearly equal mass and/or orbital size, then it would be extremely difficult to model.

We first check the variation of different parameters over one orbit. With the choice of parameters as in Table 4.1 except $M_c = 10^6 M_\odot$ and with $i = \lambda_{\text{bh}} = 87.5^\circ$, we use the formalism presented in Sec. 2.3 to find that the rate of change of different parameters are very small and can be ignored over a few orbit. Specifically, over one orbit, ω changes by $2.01^\circ, 1.95^\circ$, and 1.91° for $\tilde{a} = 0, 0.5$, and 0.9 , respectively, η_p changes by $-0.54^\circ, -0.53^\circ$, and -0.52° for $\tilde{a} = 0, 0.5$, and 0.9 , respectively, λ_p changes by $-0.71^\circ, -0.70^\circ$, and -0.69° for $\tilde{a} = 0, 0.5$, and 0.9 , respectively. Similarly, P_b changes by -2.37×10^{-5} s, e changes by -1.23×10^{-11} and i changes by $(7.95 \times 10^{-12})^\circ$ regardless of the spin of the black hole.

We note that for our choice of parameters, the average separation between the pulsar and the black hole in the range of 401.5 to $833.8 GM_c c^{-2}$ (for stellar-mass black hole, it was 7.02×10^5 to $1.45 \times 10^6 GM_c c^{-2}$). That is why the evolution and precession of the orbit and the spin of the pulsar are not large over one orbit and the formalism presented in Sec. 2.3 is valid. However, this orbital precession demands special care in the timing analysis as usually, pulsar timing algorithms fits for a particular value of ω when ToAs for only a few orbits are available and then adds $\dot{\omega}$ in the fitting algorithm when ToAs of a large number of orbits are available.

Moreover, we choose $P_s = 1.67$ s, which was motivated by the detailed study of binary evolution by Chattopadhyay et al. [4] for pulsar - stellar-mass black hole binaries. Inspired by this, we use the same value of P_s even for a pulsar - supermassive black hole. For our choice of orbital parameters and masses, if we use $P_s = 1$ ms, the only significant change would be in i which would change by $(1.33 \times 10^{-8})^\circ$ regardless of the spin of the black hole. This is still very small.

More rigorous calculation for tighter (hence more relativistic) binaries are available in the

literature. In particular, Singh et al. [75], Li et al. [76], and Li et al. [77] explored properties of pulsars around supermassive and massive black holes. They modelled the spin-spin and spin-curvature using with the help of with the help of Mathisson-Papapetrou-Dixon (MPD) equations and the Kerr metric around the black hole. They used a fast pulsar with $P_s = 1$ ms and the average separation between the pulsar and the black hole in the range of 10 to $100 \, GM_c c^{-2}$, the lower the separation, the higher is the general relativistic effects. We note that Singh et al. [75] showed that the motion of the pulsar out of the initial orbital plane is negligible when the average separation between the pulsar and the supermassive black hole is much larger than $40 \, GM_c c^{-2}$. This agrees with our results of small change in i (although obtained by a different formalism).

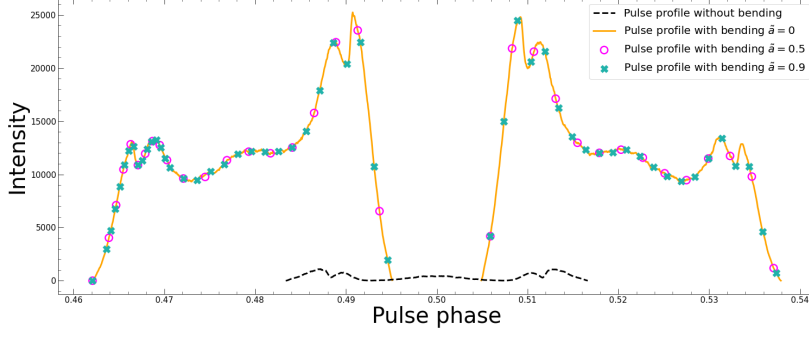
Our numerical results of spin precession following Eqs. (2.17), (2.18), (2.19), and (2.20) agree with recent results of Majár [78] and Majár and Mikóczi [79]. We note that these equations do not give the scale of the nutation. However, [76] (see their figure 5) showed that the scale of nutation is very small, in the order of 10^{-4} radian even when the separation is as small as $20 \, GM_c c^{-2}$. Hence, we do not worry about the nutation. We first investigate how the strong bending affects the pulsar beam resulting in a change in the shape and strength of the pulse profile.

In Fig. 4.15, we show the pulse profile for two different orbital phases, $A_T + \omega = 90^\circ$ (upper panel) and $A_T + \omega = 85^\circ$ (lower panel), both for $i = 90^\circ$. In each of the cases, we show the profiles if there was no bending, and with bending for $\tilde{a} = 0$, $\tilde{a} = 0.5$, and $\tilde{a} = 0.9$. For $i = 90^\circ$ and $A_T + \omega = 90^\circ$, we see a drastically different shape of the pulse profile in comparison to the unbent profile. There are two pulse components symmetrically separated from the center of the unbent pulse profile. This is quite different than what we see when the companion is a stellar-mass black hole in Fig. 4.14 where for the same values of i and $A_T + \omega$, the shape of the pulse profile is similar to the unbent profile, only the strength of each components increases. For the supermassive black hole companion, even when we choose $i = 90^\circ$ and $A_T + \omega = 85^\circ$, the profile shape is significantly different than the

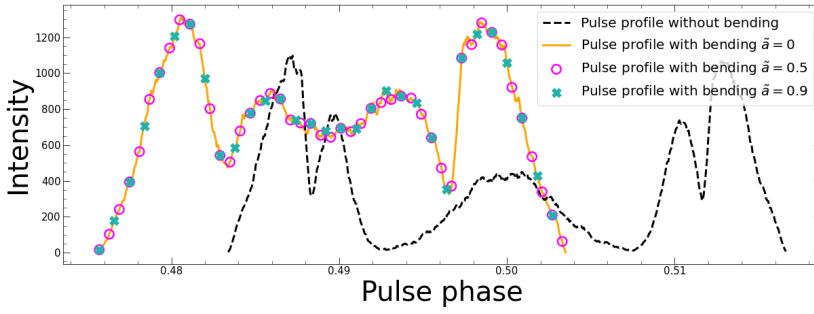
unbent profile, the pulse profile undergoes moderate distortion. The core component of the pulse vanishes, while the peaks corresponding to the conal components become wider. Additionally, there is an overall leftward shift between the bent and the unbent profiles. This intrigued us to check what happens if we keep the value of $A_T + \omega$ fixed at 90° but change the value of i to 85° , as shown in 4.16. We see that the shape of the profile is somewhat similar to the case of $i = 90^\circ$, $A_T + \omega = 85^\circ$ (Fig. 4.15b), only the shift of the bent profile is now leftward in comparison to the unbent profile. Interestingly, in none of the cases, any difference between the profiles with bending for different values of $\tilde{a}$ is observed. We explain these features at and near the superior conjunctions with the help of simplified geometry in Appendix D.

This motivates us to investigate whether the shape and strength of the pulse profile come closer to the unbent profile if the configuration deviates significantly from the superior conjunction. In Fig. 4.17a, we set $i = 80^\circ$ and see that for $A_T + \omega = 90^\circ$, although the enhancement of the conal components are very small, the core component is very weak and the pulse shape is still significantly different and shifted rightwards from the unbent profile. However, for $A_T + \omega = 200^\circ$ the profile is nearly similar to the unbent one (Fig. 4.17b). Interestingly, the bending is so weak at this orbital phase that the bent profile is nearly similar to the unbent one even if we set $i = 90^\circ$. The reason is the fact that as the distance of the pulsar from the observer is smaller than the distance of the black hole from the observer, the light rays that reach the observer never pass very close to the black hole. We have also checked what happens for $i = 60^\circ$ and see that for $A_T + \omega = 90^\circ$ (not shown), the enhancement of the conal components are very small and the core component is visible. There is still a shift between the bent and the unbent profiles, but much less than the case of $i = 80^\circ$ and see that for $A_T + \omega = 90^\circ$ (Fig. 4.17a).

Now, we investigate what happens to bending delay curves. We choose $i = 80^\circ$ where bending is strong near $A_T + \omega = 90^\circ$ but weak at other orbital phases like $A_T + \omega = 200^\circ$. In Fig. 4.18, we plot the orbital phase ($A_T + \omega$) in degrees along the ordinate and the



(a) Pulse profiles for $i = 90^\circ$, $A_T + \omega = 90^\circ$.



(b) Pulse profiles for $i = 90^\circ$, $A_T + \omega = 85^\circ$.

Figure 4.15: Comparison of pulse profiles for a pulsar orbiting a black hole with $\lambda_{\text{bh}} = i = 90^\circ$. The black dashed curve represents the profile without gravitational bending, while the profiles including bending are shown for different values of the black hole spin parameter: $\tilde{a} = 0$ (solid orange), $\tilde{a} = 0.5$ (magenta open circles), and $\tilde{a} = 0.9$ (sea-green crosses). All other relevant parameters are as listed in Table 4.1, except for the companion mass, which is set to $M_c = 10^6 M_\odot$.

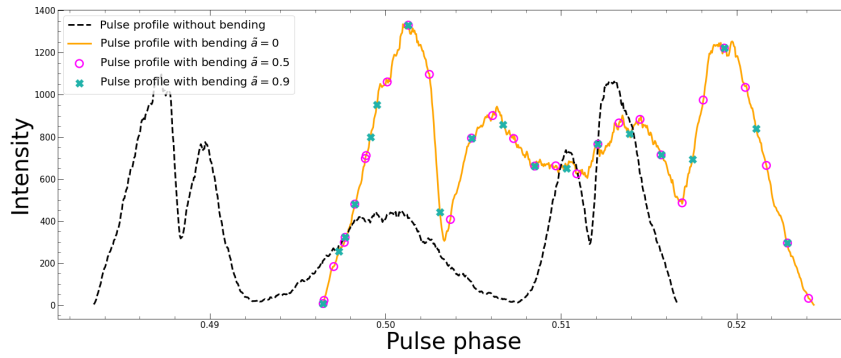
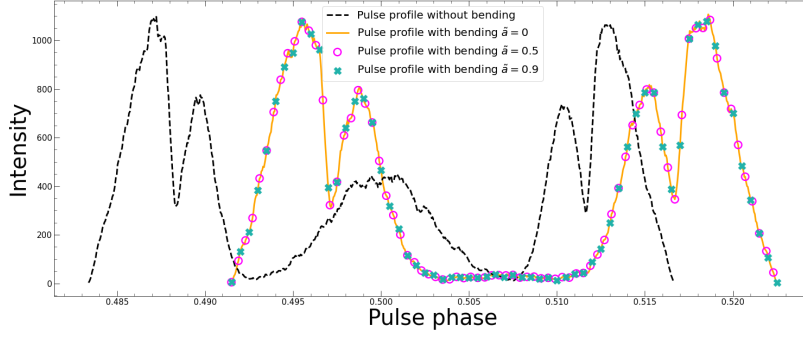
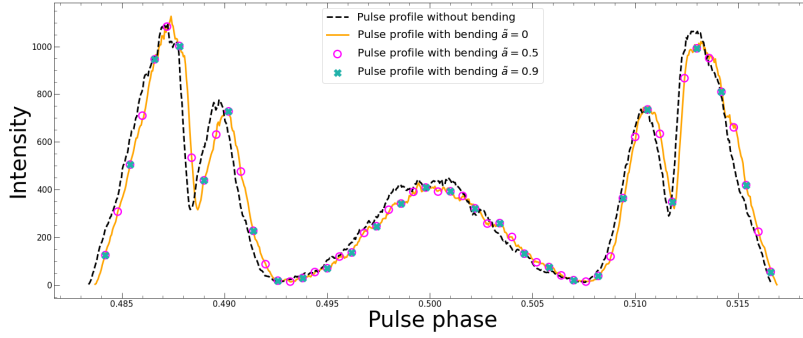


Figure 4.16: Pulse profile comparison for a pulsar orbiting a black hole with $\lambda_{\text{bh}} = i = 85^\circ$ and orbital phase $A_T + \omega = 90^\circ$. The black dashed line shows the profile without gravitational bending, while the profiles accounting for bending are displayed for different black hole spin parameters: $\tilde{a} = 0$ (solid orange), $\tilde{a} = 0.5$ (magenta open circles), and $\tilde{a} = 0.9$ (sea-green crosses). All other relevant parameters are the same as those in Table 4.1, except for the companion mass, which is set to $M_c = 10^6 M_\odot$.



(a) Pulse profiles for $i = 80^\circ$, $A_T + \omega = 90^\circ$.



(b) Pulse profiles for $i = 80^\circ$, $A_T + \omega = 200^\circ$.

Figure 4.17: Pulse profile comparison for a pulsar in orbit around a black hole companion, with $\lambda_{\text{bh}} = i = 80^\circ$. The black dashed curve represents the profile without gravitational bending, while the effects of bending are shown for different black hole spin parameters: $\tilde{a} = 0$ (solid orange), $\tilde{a} = 0.5$ (magenta open circles), and $\tilde{a} = 0.9$ (sea-green crosses). All other relevant parameters are as listed in Table 4.1, except for the companion mass, which is set to $M_c = 10^6 M_\odot$.

bending delays in millisecond along the abscissa. The left panel is for the latitudinal bending delay and the right panel is the longitudinal bending delay. In the case of the stellar-mass black hole companion, even for $i = 87.5^\circ$ (Fig. 4.2), the bending delays were in the order of microsecond. However, in the case of a supermassive black hole companion, we see that the bending delays are much larger, in the order of milliseconds. However, just like the case of stellar-mass black hole, we cannot distinguish bending delays for different values of $\tilde{a}$. Hence, we plot the value of the FD bending delays in Fig. 4.19, where the orbital phase in degrees is plotted along the ordinate and the values of the FD bending delays in microsecond are shown along the abscissa. The left panel is for the latitudinal FD bending delay and the right panel is the longitudinal FD bending delay. We see that the values of FD bending delays for supermassive black hole is in the order of microseconds while those were in the order of nanoseconds in the case of the stellar-mass black hole (Fig. 4.4).

We note that, in all calculations and figures of bending delays and FD bending delays, we set the time $t = 0$ when $A_T = 0^\circ$ or $A_T + \omega = 73.804^\circ$. We represent $t = 0$ by a vertical orange line in Figs. 4.2, 4.18, 4.19. In each figure, the left side of this orange line represent orbital phases post 360° . We see for the case of the supermassive black hole, at $t = P_b$, the value of $A_T + \omega$ is slightly larger than 73.804° due to precession of the orbit - visible from the spillage of the delay curves from left to right of the vertical orange lines. Moreover, at $t = P_b$, values of the bending delays and the FD bending delays are different than what it was at $t = 0$. This happens because spin precession affects the geometry determining the amount of bending. Interestingly, the discontinuities seen at $A_T + \omega = 90^\circ$ (when $\eta_{bh} = 90^\circ$) in FD bending delay curves for the stellar-mass black hole is not visible for the case of the supermassive black hole. We attribute this to comparatively larger changes in the configuration due to the comparatively larger spin and orbital precession for the second case.

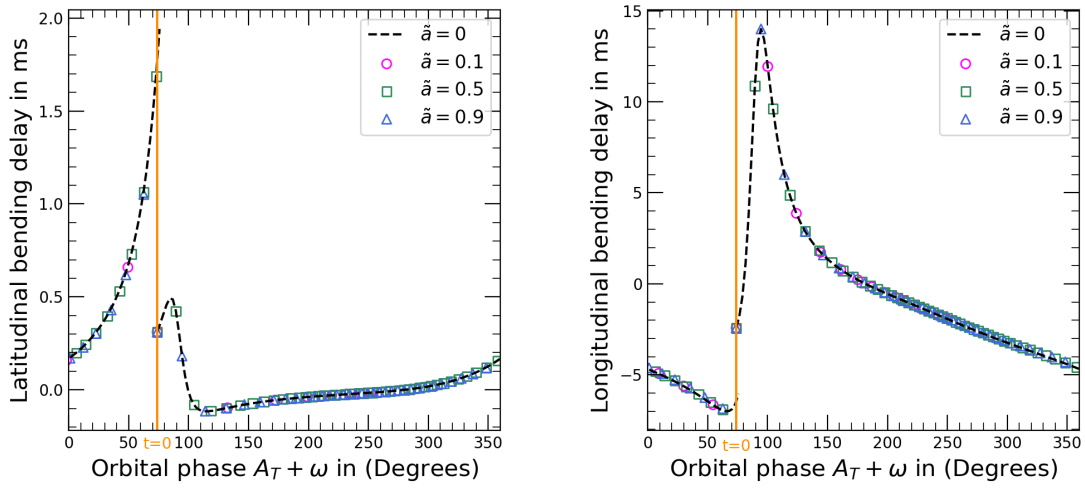


Figure 4.18: Bending delays across a full orbital cycle for a hypothetical pulsar–supermassive black hole binary system with $\lambda_{\text{bh}} = i = 80^\circ$. The left panel displays the latitudinal bending delay, while the right panel shows the longitudinal bending delay. Results are shown for different spin parameters of the black hole: $\tilde{a} = 0$ (solid black line), $\tilde{a} = 0.1$ (magenta circles), $\tilde{a} = 0.5$ (green squares), and $\tilde{a} = 0.9$ (blue triangles). All other relevant parameters follow Table 4.1, except for the black hole mass, which is set to $M_c = 10^6 M_\odot$. The vertical orange line in each panel marks $A_T = 0^\circ$, corresponding to time $t = 0$. Phases to the left of this line represent orbital positions beyond 360° .

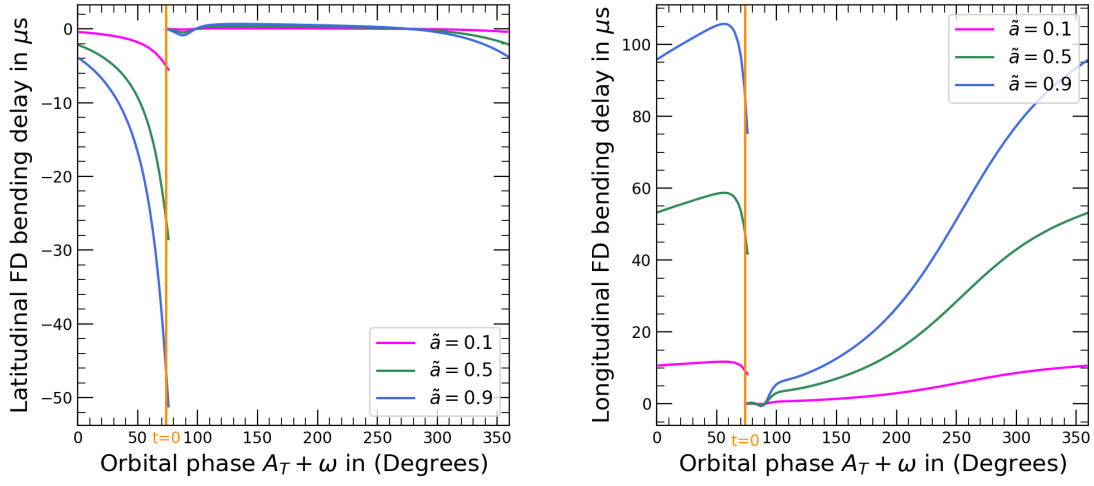


Figure 4.19: FD bending delays over a complete orbital cycle for a hypothetical pulsar–supermassive black hole binary with $\lambda_{\text{bh}} = i = 80^\circ$. The left panel presents the latitudinal FD bending delays, while the right panel shows the longitudinal ones. Each curve corresponds to a different black hole spin parameter: $\tilde{a} = 0$ (solid black), $\tilde{a} = 0.1$ (magenta circles), $\tilde{a} = 0.5$ (green squares), and $\tilde{a} = 0.9$ (blue triangles). All other relevant parameters are taken from Table 4.1, except the companion mass, which is set to $M_c = 10^6 M_\odot$. The vertical orange line in both panels marks $t = 0$ (i.e., $A_T = 0^\circ$); orbital phases to its left correspond to positions beyond 360° .

4.4 Conclusion

In this chapter, we examined how light bending affects the timing signals of pulsars in binary systems with spinning (Kerr) black hole companions, with a primary focus on stellar-mass black holes. Our analysis shows that the qualitative influence of various system parameters on the bending delays remains similar to the case of non-rotating (Schwarzschild) black holes. This is primarily because changes induced by the black hole's spin magnitude and orientation occur on nanosecond timescales, whereas other orbital and geometric parameters cause variations on the order of microseconds. As a result, the deformation of the pulsar beam and the associated changes in the observed pulse profile are only weakly sensitive to the black hole's spin.

We further analyzed the variation in bending delays due to spin by computing the difference between delay values with and without spin, and observed nanosecond-level discontinuities at orbital phases where the light ray's trajectory transitions across the spin axis of the black hole.

Additionally, we explored scenarios involving pulsars in orbit around supermassive black holes. In such cases, the significantly stronger gravitational field leads to noticeable enhancements and deformations in the pulse profile near superior conjunction—effects that are absent when the companion is a stellar-mass black hole. The bending delays in these systems are also approximately three orders of magnitude larger, under otherwise similar orbital and geometric conditions. This considerable increase in delay magnitude and profile distortion could pose significant challenges for precise pulsar timing in such extreme environments.

Chapter 5

Conclusion

In this thesis, we studied the effect of the light bending phenomenon on the signal of a pulsar in a binary system with either a non-rotating or rotating black hole companion, focusing mainly on the stellar-mass black holes. For this purpose, we solved the equations of the null geodesic in the Schwarzschild spacetime using the formalism devised by Chandrasekhar [65], and the equations of the null geodesic in the Kerr spacetime using the formalisms devised by Kapec and Lupsasca [68] and Gralla and Lupsasca [69]. We adopted the orbital parameters and masses of the pulsar-black hole binary from Chattopadhyay et al. [4], although that work favored non-rotating black holes only. As the bending takes place near the companion, the spin of the pulsar does not play any role in the estimation of the bending. However, it plays a significant role in the geometry, i.e., in determining the initial direction of the pulsar beam. To calculate the initial direction of the beam, in addition to creating the beam at different rotational phases, we also traced the pulsar in its orbit by solving Kepler's equations augmented by post-Keplerian parameters to account for the general relativistic effects. Additionally, we have taken care of the geodetic precession of the spin axis of the pulsar about the total angular momentum of the binary.

We have used the core double cone model of the pulsar beam and set the orientation of the

spin axis of the pulsar and its magnetic axis such that the line of sight traverses through the center of the cross section of the beam.

The light bending phenomenon results in two different delays, the longitudinal delay and the latitudinal delay. Both delays would be present, but usually one is much stronger than the other depending on the geometry of the system. The resultant bending delay curve, which is the sum of the two delay curves, would mostly follow the shape of the stronger one. Similar to the Shapiro delay, the total bending delay is also stronger for edge-on systems, especially at the orbital phases near the superior conjunction. It might therefore be difficult to decouple these delays [80]. However, the value of the Shapiro delay depends only on the masses of the pulsar and the companion as well as the orbital parameters, e.g., eccentricity (e), orbital period (P_b), orbital inclination (i), longitude of the periastron (ω), and true anomaly of the orbit (A_T). On the other hand, in addition to these parameters, the values of the angles α , η_p , and λ_p affect the value of the bending delay, where α is the angle between the pulsar's spin axis and its magnetic axis. The angles λ_p and η_p specify the orientation of the pulsar's spin axis in the sky frame, while λ_{bh} and η_{bh} define the orientation of the companion's spin axis in the same frame (see Sec. 2.1). The last two parameters (λ_{bh} , η_{bh}) are relevant only if the companion has spin.

We found that for a stellar-mass black hole companion, the magnitude of the spin as well as the orientation of the spin axis of the black hole introduce changes at the nanosecond order while bending delays are at the microsecond order. Hence, the impact of the companion's spin is illustrated by plotting the FD bending delays, which represent the difference in bending delays between a rotating and a non-rotating black hole, while keeping all other system parameters fixed. We noticed nanosecond scale discontinuities in the FD bending delay curves when $\hat{r}_s$, $\hat{S}_{bh}$, and $\hat{N}_s$ align in the same plane. However the bending delays become irregular (in the microsecond order) near the superior conjunction, where both the orbital phase and the orbital inclination approach ninety degrees for both non-rotating and rotating black hole companion.

Although we have studied the manifestation of the light bending effect within the framework of full general relativity, it is difficult to incorporate this in pulsar timing analysis. However, the simpler analytical expressions given by Rafikov et al. [2] or Doroshenko et al. [1] match the full general relativistic methods except in a few specific situations. The first is the case when the values of both the orbital phase and the inclination angle are close to 90° . In such a case, the bending delay suddenly becomes very strong. The second is the case when the orbital phase is near 270° , where the expressions of Rafikov et al. [2] gives some artificial features as seen in Fig. 3.6. Although the expression by Doroshenko et al. [1] is free from any such artificial feature, they provided the expression for the longitudinal bending delay only, not for the latitudinal bending delay.

In addition to introducing a delay, the bending of light also distorts the pulsar beam, resulting in changes to the observed pulse profile. For a stellar-mass black hole companion, this distortion becomes significant only when the pulsar is very close to the superior conjunction, i.e., when both the orbital phase and the inclination angle are close to 90° . Away from superior conjunction, the bending effect weakens, and no noticeable distortion occurs. As a result, the pulse shape closely resembles that of an isolated pulsar, as shown in Fig. 1.3.

However, near superior conjunction, the distortion of the beam becomes substantial, often leading to an enhancement in the pulsar signal. In some cases, the distorted beam results in a low-intensity tail appearing either at the beginning or end of the pulse profile, as seen in Fig. 3.13. In Fig. 3.12a, we observe that the profile variation due to bending is minimal for a pulsar–black hole binary with an inclination angle of $i = 89.7^\circ$. Moreover, the beam distortion and the corresponding changes in pulse shape are only weakly affected by the spin parameters of the black hole, including both its magnitude and orientation. Kopeikin and Mashhoon [81] reported similarly negligible differences in the propagation delays when comparing rotating and non-rotating black hole companions.

Therefore, when performing pulsar timing in such systems, it is essential to account for

pulse profile variations with orbital phase due to gravitational light bending. Since the bending is negligible near inferior conjunction and away from superior conjunction, the profile variations in these regions are also insignificant. This is consistent with the null detection of bending-induced profile variations reported by Hu et al. [80] for the Double Pulsar system, which has an inclination of $i = 89.35^\circ$ or 90.65° . Thus, pulse profiles near inferior conjunction can be reliably used as templates for extracting pulse times of arrival (ToAs) and for studying pulse shape variations. In contrast, using profiles near superior conjunction as templates would lead to erroneous timing results at other orbital phases.

Hence, we conclude that if a pulsar-black hole (stellar-mass) binary is discovered in the future, the analytical expressions of Rafikov and Lai [62] or Doroshenko et al. [1] would suffice to model the light bending delays, regardless of the spin of the black hole, provided the timing accuracy does not reach the nanosecond scale. However, care should be taken during the superior conjunction, where changes in the shape of the profiles could lead to discrepancies with the stable template profile used in timing analysis.

We have also explored some cases of bending phenomena for pulsars in a binary with a supermassive black hole, and found some interesting features in the pulse profile due to strong bending that are absent in the case of a stellar-mass black hole companion. We find significant enhancement and change in the shape of the profiles at and near the superior conjunction in comparison to the rotating stellar-mass black hole. Moreover, bending delays are also about three orders of magnitude higher than those in case of the stellar-mass black holes, when all other parameters are the same. These facts would make timing of such binaries difficult.

The distortion of the pulse profile when the pulsar is near superior conjunction (inclination of the orbit and orbital phase close to 90°) depends primarily on the mass of the companion black hole and the orbital separation. In our study of a supermassive black hole companion, we have shown that the resulting pulse-profile distortion differs significantly from the case of a stellar-mass black hole companion.

Other orbital parameters, such as the eccentricity, η_p , λ_p , and related angles, do not produce a qualitatively different change in the distorted profile compared to what is presented in this thesis. However, variations in the orbital period result in changes in the orbital separation, and these changes can influence the pulse-profile distortion, affecting both the peak intensity and the overall shape of the distorted profile.

Future Directions:

- As pulsar-black hole binaries are considered a natural tool to test theories of gravity with the help of accurate pulsar timing, it is essential to understand all general relativistic effects that the pulsar undergoes such systems. In this thesis, we only focus on the effect of the light bending phenomenon only. The propagation of the light rays in the gravitational field of the companion also introduces delays in the signal. The actual pulse time arrival and the variation of the pulse profile can be understood properly only when all the effects are included.
- For pulsar - black hole binaries, the values of bending delays obtained in our full general relativistic calculations deviate from the values obtained using existing analytical expressions at some orbital phases. Moreover, the shape of the pulse also changes. These phenomena might result in difficulties in obtaining correct timing solutions in the future. Hence, in future works, we would like to quantify these issues with the help of simulated profiles and injected values of bending delays in Time of Arrivals.
- Presently we studied the change in the pulse profiles due to the bending using only the core-double cone model. It will be interesting to perform similar studies in more intricate models of the beam. Such studies might offer new insights into the detection and characterization of such systems.

Appendix A

Rotational Transformations

We have defined various frames in Secs. 2.2, 1.3.1 and we need to convert vectors from one frame to another. We will discuss all those transformations in this section. First, let us discuss some general properties of rotational transformations. As we know the angle between various axes of the relevant frames, it is generally easier to first understand the conversion between frames using passive rotations. Afterward, we can transform a vector from one frame to another using active rotations and remembering the relations between the active and the passive rotations as:

$$R_X^A(\theta) = R_X^P(-\theta) , \quad (\text{A.1a})$$

$$R_Y^A(\theta) = R_X^P(-\theta) , \quad (\text{A.1b})$$

$$R_Z^A(\theta) = R_X^P(-\theta) . \quad (\text{A.1c})$$

$$R_{\zeta_1}^A(\theta)R_{\zeta_2}^A(\phi) = R_{\zeta_2}^P(-\phi)R_{\zeta_1}^P(-\theta) . \quad (\text{A.1d})$$

In Eq. (A.1), the superscript A stands for active rotation, i.e., rotation of a vector while the superscript P stands for the passive rotation, i.e., the rotation of the axis. The subscript denotes the axis of rotation and the amount of the rotation is given in the parenthesis. The

standard rotation matrices $R_X^P(\theta)$, $R_Y^P(\theta)$, $R_Z^P(\theta)$, etc. are known.

A.1 Conversion from the I-frame to b-frame

The I-frame is defined with its origin at the center of the pulsar. We shift this origin to the barycenter of the binary system, defining a new frame called the I'-frame, as introduced in Section 2.2.

Let a vector in the I-frame be denoted by $\vec{V}_I$ and the same vector in the I'-frame by $\vec{V}_{I'}$. Since both frames are related by a simple shift in origin (with no rotation), the vector remains unchanged:

$$\vec{V}_I = \vec{V}_{I'}. \quad (\text{A.2})$$

To express vectors in the b-frame, we proceed in two steps:

1. Transform the vector from the I'-frame to the s-frame.
2. Transform the result from the s-frame to the b-frame.

A.1.1 Conversion from the I'-frame to the s-frame

We know that the angle between the Z_s and $Z_{I'}$ axes is λ , and the angle between the X_s -axis and the projection of the $Z_{I'}$ -axis on the $X_s Y_s$ plane is η . Using this information, it is easier to visualise the conversion of the s-frame to the I'-frame. As shown in Fig. A.7, by rotating the s-frame about Y_s -axis by an angle λ , we get a new frame $X'_s Y'_s Z'_s$ where Y'_s is along Y_s , i.e., Y'_s is in $X_s Y_s$ plane. Z'_s and X'_s are in the $Z_s X_s$ plane. The angle between the Z_s and Z'_s -axis and the X_s and X'_s axes is λ .

Now we rotate $X'_s Y'_s Z'_s$ frame about Z_s by an angle η to get $X_{I'} Y_{I'} Z_{I'}$ frame. As the rotation is about Z_s -axis, $Y_{I'}$ remains in the $X_s Y_s$ plane, i.e., in the sky-plane.

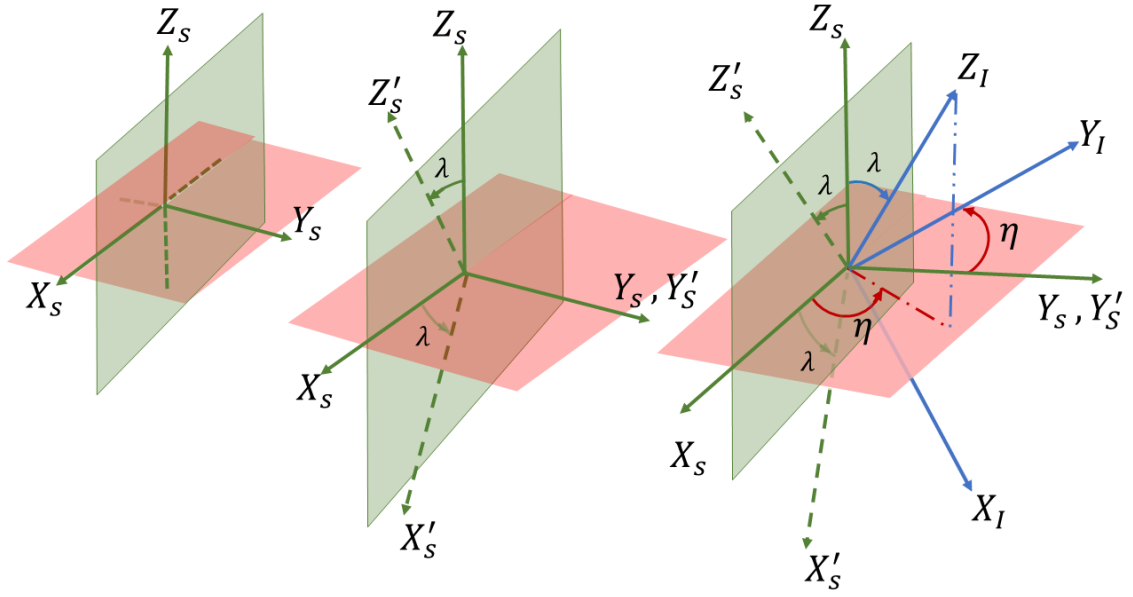


Figure A.1: The transformation from the s-frame to the I-frame. The green plane is the $Z_s X_s$ plane and the red plane is the $X_s Y_s$ plane.

The above transformation is shown in Fig. A.7 and can be represented as:

$$X_s Y_s Z_s \xrightarrow{R_{Y_s}^P(\lambda)} X'_s Y'_s Z'_s \xrightarrow{R_{Z'_s}^P(\eta)} X_I Y_I Z_I . \quad (\text{A.3})$$

Hence, the reverse transformation can be written as:

$$X_I Y_I Z_I \xrightarrow{R_{Z_s}^P(-\eta)} X'_s Y'_s Z'_s \xrightarrow{R_{Y'_s}^P(-\lambda)} X_s Y_s Z_s . \quad (\text{A.4})$$

Thus, if a vector is denoted by $\vec{V}_I$ in the I-frame and by $\vec{V}_s$ in the s-frame, these two would be related by

$$\vec{V}_s = R_Z^A(\eta) R_Y^A(\lambda) \vec{V}_I . \quad (\text{A.5})$$

Note that, as Y_I lies in the sky-plane, if the $X_I Y_I Z_I$ frame is parallelly transported to any other point as the origin to get a new frame $X'_I Y'_I Z'_I$, Y'_I will remain in the sky-plane.

A.1.2 Conversion from the s-frame to b-frame

Now we perform the conversion from the s-frame to the b-frame. We know that the $X_s Y_s$ plane is the sky plane and the $X_b Y_b$ plane is the orbital plane. The X_s -axis is defined along the line of the ascending node, i.e., the intersection of the sky-plane and the orbital plane. We also know that the angle between Z_s -axis and Z_b is i , and the angle between the X_s -axis and the X_b -axis is ω .

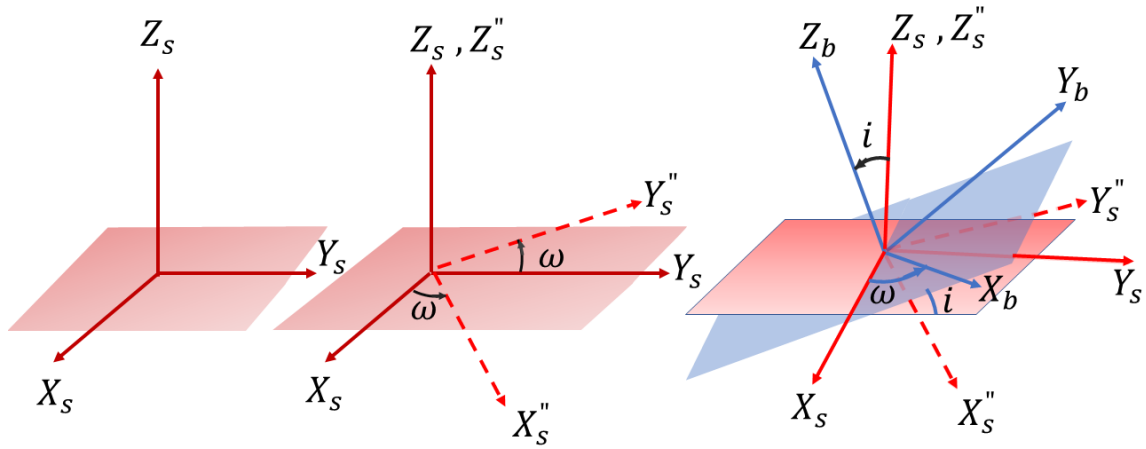


Figure A.2: The transformation from the s-frame to the b-frame. The red plane is the $X_s Y_s$ plane (the sky plane), and the blue plane is the $X_b Y_b$ plane (the orbital plane). The X_s -axis is the line of the node, that is along the intersection of the sky-plane and the orbital plane.

As shown in Fig. A.2, by rotating the s-frame about Z_s -axis by an angle ω , we get a new frame $X_s'' Y_s'' Z_s''$ where Z_s'' is along Z_s and X_s'' and Y_s'' are in the $X_s Y_s$ plane. The angle between X_s and X_s'' axes and Y_s and Y_s'' axes is ω .

Next, we rotate the $X_s'' Y_s'' Z_s''$ frame about the X_s -axis by an angle i to get the $X_b Y_b Z_b$ frame. The angle between the X_s'' and X_b axes, Y_s'' and the Y_b axes, and the Z_s'' and Z_b axes is i . This means the angle between the Z_s -axis and the Z_b -axis, as well as the sky-plane (the $X_s Y_s$ plane, shown in red in Fig. A.2) and the orbital plane (the $X_b Y_b$ plane,

shown in blue in Fig. A.2) is also i . This is the definition of the orbital inclination angle. Moreover, the angle between the the angle between the X_s -axis and the X_b -axis is ω as seen in Fig. A.2. The X_s -axis is along the intersection of the sky-plane (the $X_s Y_s$ plane) and the orbital plane (the $X_s Y_s$ plane of the new frame), i.e., along the line of the node as we have defined its direction.

The above transformation is shown in Fig. A.2 and can be represented as:

$$X_s Y_s Z_s \xrightarrow{R_{Z_s}^P(\omega)} X_s'' Y_s'' Z_s'' \xrightarrow{R_{X_s}^P(i)} X_b Y_b Z_b . \quad (\text{A.6})$$

Thus, if a vector is denoted by $\vec{V}_s$ in the s-frame and by $\vec{V}_b$ in the b-frame, these two would be related by

$$\vec{V}_b = R_Z^A(-\omega) R_X^A(-i) \vec{V}_s . \quad (\text{A.7})$$

or,

$$\vec{V}_s = R_X^A(i) R_Z^A(\omega) \vec{V}_b . \quad (\text{A.8})$$

Note that, instead of Eq. (A.6), if the second transformation is a rotation of the the $X_s'' Y_s'' Z_s''$ frame about the Y_s -axis by an angle i , we will still get a new frame whose Z -axis would make an angle i with the Z_s -axis, but the X_s -axis will not remain along the intersection of the sky-plane (the $X_s Y_s$ plane) and the orbital plane (the $X_s Y_s$ plane of the new frame), i.e., the X_s -axis will not be the line of node, which was our assumption. This wrong transformation is shown in Fig. A.3.

A.1.3 Conversion from I-frame to b-frame

Combining Eqs. (A.5) and (A.7), we get,

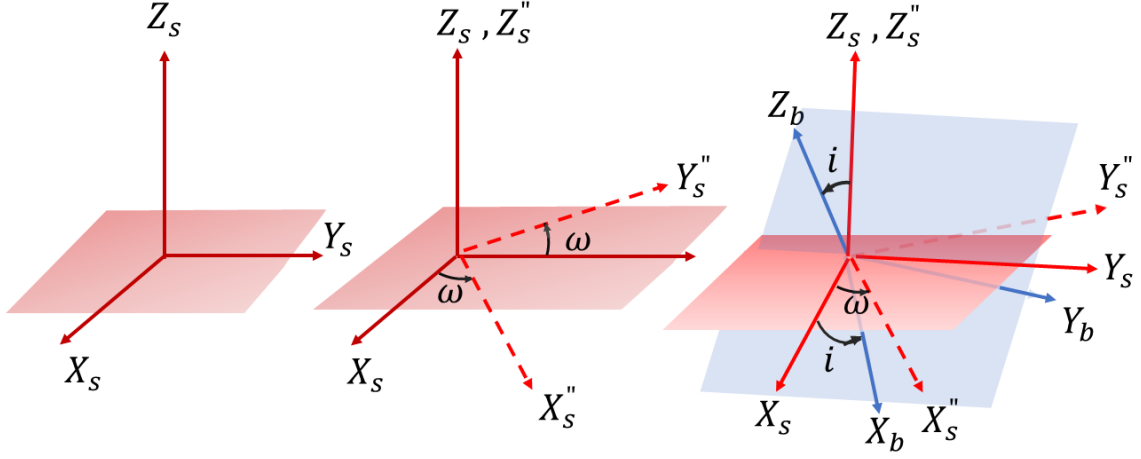


Figure A.3: The wrong transformation from s to b frame. The red plane is the $X_s Y_s$ plane (the sky plane), and the blue plane is the $X_b Y_b$ plane (the orbital plane). The Y_s -axis is the line of the node, that is along the intersection of the sky-plane and the orbital plane, which contradicts our choice.

$$\vec{V}_b = R_Z^A(-\omega) R_X^A(-i) \vec{V}_s = R_Z^A(-\omega) R_X^A(-i) R_Z^A(\eta) R_Y^A(\lambda) \vec{V}_I. \quad (\text{A.9})$$

A.2 Conversion from the L-frame to T-frame

We consider the L-frame is centered at the gravitating body such that the X_L, Y_L -axis is in the plane of the light ray (defined by the cross product of $\hat{r}_b$ and $\hat{n}_b$). ψ_L is the azimuthal position of the light ray in L-frame, i.e., measured from the X_L -axis. We know that $\psi_L = 0$ corresponds to minimum distance between the gravitating body and the light ray. Hence, we can say that the X_L -axis is along the minimum distance between gravitating body and the light ray in its trajectory. To specify the direction of the Y_L -axis, we adopt a convention that when the light ray initially moves towards the gravitating body, ψ_L is negative and when light ray initially moves away from the companion, ψ_L is positive.

The T-frame is defined in such a way that the Y_T -axis is chosen along the radial position

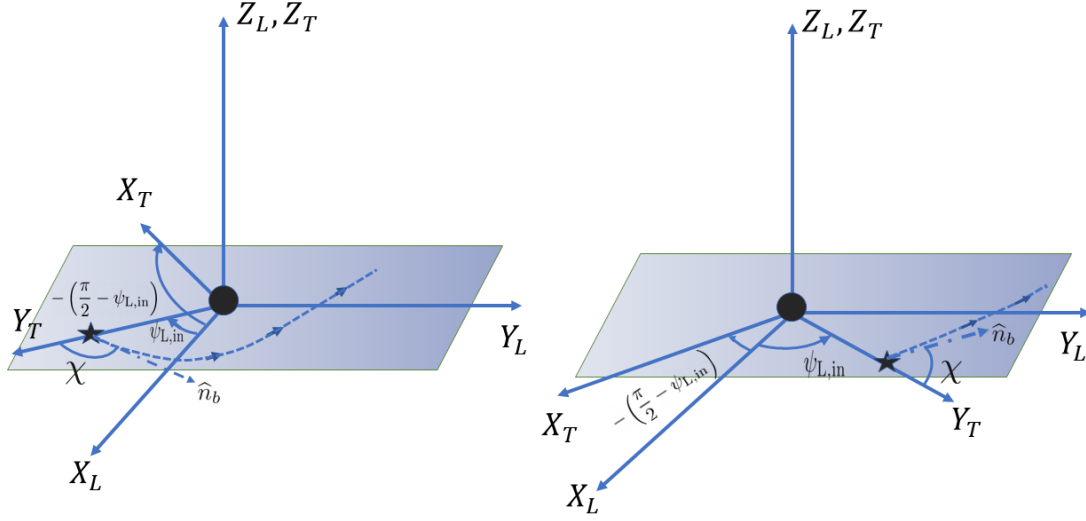


Figure A.4: The transformation from the L-frame to the T-frame. The blue plane is the $X_L Y_L$ plane. The left figure is for the case when $\psi_{L,in}$ is negative and the right figure is for the case when $\psi_{L,in}$ is positive.

of the pulsar, i.e, along $\hat{r}_b$ and we chose the Z_T -axis to be along the Z_L -axis, i.e,

$$\hat{y}_T = \hat{r}_b, \quad (\text{A.10})$$

and

$$\hat{z}_T = (\hat{r}_b \times \hat{n}_b) / |\hat{r}_b \times \hat{n}_b|. \quad (\text{A.11})$$

Hence, we can say that the $X_L Y_L$ plane and the $X_T Y_T$ plane are the same plane, i.e, the plane of the light ray. Now, if $\psi_{L,in}$ is the azimuthal position of the light ray in the L-frame at the time of emission, then, the angle between the X_L -axis and the Y_T -axis is $\psi_{L,in}$. Moreover, the angle between the X_L -axis and X_T -axis is $-(\pi/2 - \psi_{L,in})$ as shown in Fig. A.4. Hence, the T-frame can be obtained from the L-frame by simply rotating the L-frame by an angle $-(\pi/2 - \psi_{L,in})$ about the Z_L -axis:

$$X_L Y_L Z_L \xrightarrow{R_{Z_L}^P(-(\pi/2 - \psi_{L,in}))} X_T Y_T Z_T. \quad (\text{A.12})$$

Thus, if a vector is denoted by $\vec{V}_L$ in the L-frame and by $\vec{V}_T$ in the T-frame, these two would be related by

$$\vec{V}_T = R_Z^A((\pi/2 - \psi_{L.in})) \vec{V}_L . \quad (A.13)$$

A.3 Conversion from the T-frame to b-frame

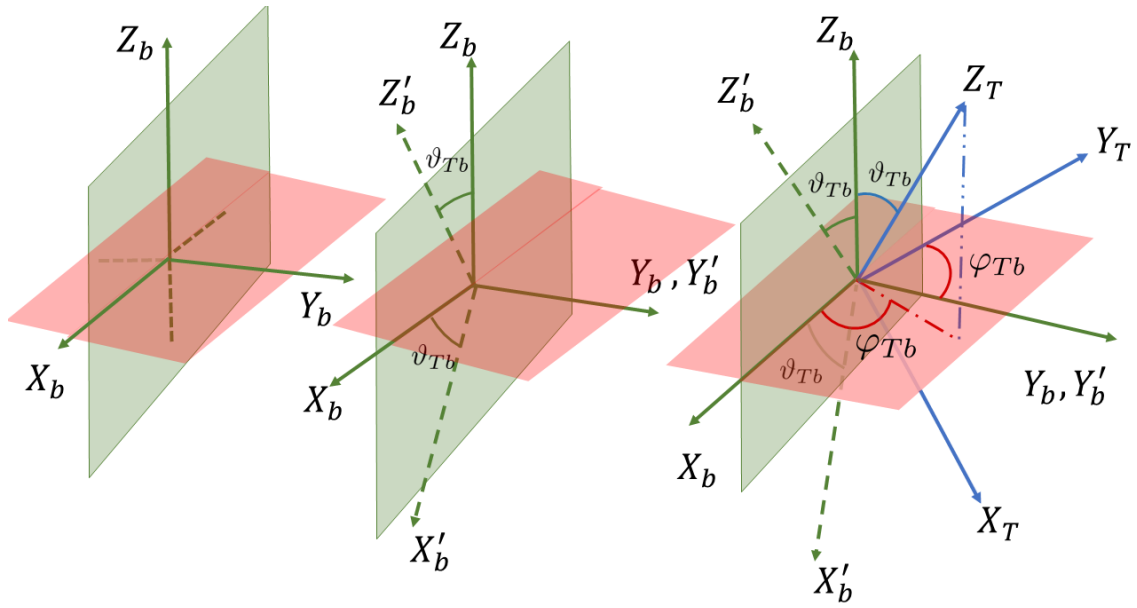


Figure A.5: The transformation from the b-frame to the T-frame. The green plane is the $Z_b X_b$ plane and the red plane is the $X_b Y_b$ plane or the orbital plane.

We know that the Y_T -axis is chosen along the $\hat{r}_b$ direction which means that the Y_T -axis is in the orbital plane. We also know that both of the X_b -axis and the Y_b -axis is also in the orbital plane.

If the angle between the Z_T -axis and the Z_b -axis is θ_{Tb} and the angle between the Y_T -axis and the Y_b -axis is ϕ_{Tb} , then we can write:

$$\theta_{Tb} = \cos^{-1}(\hat{z}_T \cdot \hat{z}_b) = \cos^{-1}(\hat{z}_T \cdot [0, 0, 1]) , \quad (A.14)$$

and

$$\phi_{Tb} = \cos^{-1}(\hat{y}_T \cdot \hat{y}_b) = \cos^{-1}(\hat{y}_T \cdot [0, 1, 0]) . \quad (\text{A.15})$$

where the unit vectors $\hat{z}_b$ and $\hat{y}_b$ are represented by $[0, 0, 1]$ and $[0, 1, 0]$, respectively. The values of θ_{Tb} and ϕ_{Tb} can be calculated using Eqs. (A.14) and (A.15) as we know $\hat{z}_T$ (Eq. (A.11)) and $\hat{y}_T$ (Eq. (A.10)).

From Fig. A.5, we can visualise the transformation from the b-frame to the T-frame. First, by rotating the b-frame about the Y_b -axis by an angle θ_{Tb} , we get a new frame $X'_b Y'_b Z'_b$. Here, the Y'_b -axis is along the Y_b -axis and both of the Z'_b -axis and the X'_b -axis are in the $Z_b X_b$ plane. The angle between the X_b -axis and the X'_b -axis is the same as the angle between the Z_b -axis and the Z'_b -axis, i.e., θ_{Tb} .

Next, we rotate the $X'_b Y'_b Z'_b$ frame about the Z_b -axis by an angle ϕ_{Tb} to get the $X_T Y_T Z_T$ frame. As this rotation is about the Z_b -axis, the Y'_b -axis remains in the $X_b Y_b$ plane, i.e., the Y_T -axis is also in the orbital plane.

The above two transformations are shown in Fig. A.5 and can be represented as:

$$X_b Y_b Z_b \xrightarrow{R_{Y_b}^P(\theta_{Tb})} X'_b Y'_b Z'_b \xrightarrow{R_{Z_b}^P(\phi_{Tb})} X_T Y_T Z_T . \quad (\text{A.16})$$

Hence, the reverse transformation can be written as:

$$X_T Y_T Z_T \xrightarrow{R_{Z_b}^P(-\phi_{Tb})} X'_b Y'_b Z'_b \xrightarrow{R_{Y_b}^P(-\theta_{Tb})} X_b Y_b Z_b . \quad (\text{A.17})$$

Thus, if a vector is denoted by $\vec{V}_T$ in the T-frame and by $\vec{V}_b$ in the b-frame, these two would be related by

$$\vec{V}_b = R_Z^A(\phi_{Tb}) R_Y^A(\theta_{Tb}) \vec{V}_T . \quad (\text{A.18})$$

Note that, if to the b-frame, one first applies a rotation about the X_b -axis by an angle θ_{Tb} followed by a rotation about the Z_b -axis by an angle ϕ_{Tb} , then also one would obtain a new frame whose Z axis would make an angle θ_{Tb} with the Z_b -axis and the Y -axis would make an angle ϕ_{Tb} with the Y_b -axis. However, in that case, the Y -axis of the new frame would not be in the orbital plane, which would contradict our choice of the Y_T -axis being along $\hat{r}_b$.

A.4 Conversion from the m-frame to I-frame

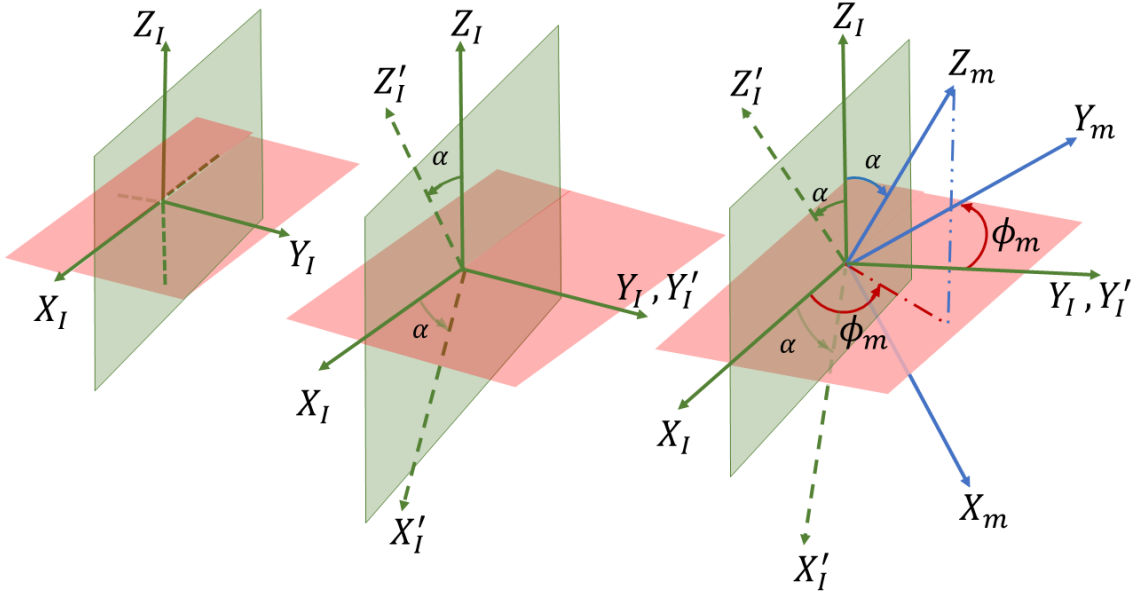


Figure A.6: The transformation from the I-frame to the m-frame. The green plane is the $Z_I X_I$ plane and the red plane is the $X_I Y_I$ plane.

The m-frame is chosen in such a way that the Z_m -axis is along the magnetic axis and the unit vector along the magnetic axis is denoted by $\hat{m}_I$ in I-frame. We know that the magnetic axis rotates around the spin axis (the Z_I -axis) making an angle α with it and the phase of the magnetic axis is denoted by Φ_m , which is the angle between the projection of $\hat{m}_I$ on the $X_I Y_I$ plane with the X_I -axis. Hence, the angle between the Z_I -axis and the Z_m -axis is α and the X_m -axis and the Y_m -axis is in a plane perpendicular to the Z_m -axis. There is no specific choice for the X_m and Y_m axes except the fact that the $X_m Y_m Z_m$ is a

right-handed Cartesian frame. Hence, the transformation from the I-frame to the m-frame can be done by in two steps as described below.

By rotating the I-frame about the Y_I -axis by an angle α , we get a new frame $X'_I Y'_I Z'_I$. Here Y'_I is along the Y_I -axis. The Z'_I -axis and the X'_I -axis is in the $Z_I X_I$ plane. The angle between the X_I -axis and the X'_I -axis is the same as the angle between the Z_I -axis and the Z'_I -axis, i.e., α .

Next, we rotate the $X'_I Y'_I Z'_I$ frame about the Z_I -axis by an angle Φ_m to get the $X_m Y_m Z_m$ frame. As the rotation is about the Z_I -axis, Y_m remains in the $X_I Y_I$ plane (the red plane in Fig. A.6). These transformations are shown in Fig. A.6 and can be represented as:

$$X_I Y_I Z_I \xrightarrow{R_{Y_I}^P(\alpha)} X'_I Y'_I Z'_I \xrightarrow{R_{Z_I}^P(\Phi_m)} X_m Y_m Z_m . \quad (\text{A.19})$$

Hence, the reverse transformation can be written as:

$$X_m Y_m Z_m \xrightarrow{R_{Z_I}^P(-\Phi_m)} X'_I Y'_I Z'_I \xrightarrow{R_{Y_I}^P(-\alpha)} X_I Y_I Z_I . \quad (\text{A.20})$$

Thus, if a vector is denoted by $\vec{V}_m$ in the m-frame and by $\vec{V}_I$ in the I-frame, these two would be related by

$$\vec{V}_I = R_Z^A(\Phi_m) R_Y^A(\alpha) \vec{V}_m . \quad (\text{A.21})$$

A.5 Conversion relation between the bh' -frame and the s-frame

We define a frame called the ‘bh-frame’ centered at the companion in such a way that the Z-axis of this frame, (Z_{bh}) is along the spin axis of the gravitating object (the black hole).

The bh-frame can be translated to the barycenter of the orbit along the line joining the pulsar and the companion, giving $X'_{bh}Y'_{bh}Z'_{bh}$ frame. Hence, $\hat{S}_{bh}$ can be taken along the Z'_{bh} axis too. We introduce two new angles λ_{bh} and η_{bh} to specify the direction of the Z'_{bh} -axis in the s-frame. The angle λ_{bh} is the angle between the Z_s -axis and the Z'_{bh} -axis while the angle η_{bh} is the angle between the X_s -axis and the projection of the Z'_{bh} -axis on the sky plane ($X_s Y_s$ plane).

As shown in Fig. A.7, by rotating the s-frame about the Y_s -axis by an angle λ_{bh} , we get a new frame $X'_s Y'_s Z'_s$ where the Y'_s -axis is along the Y_s -axis, i.e., the Y'_s -axis is still in the sky plane ($X_s Y_s$ plane). Z'_s and X'_s are in the $Z_s X_s$ plane. The angle between the Z_s -axis and the Z'_s -axis as well as the angle between the X_s -axis and the X'_s -axis is λ_{bh} .

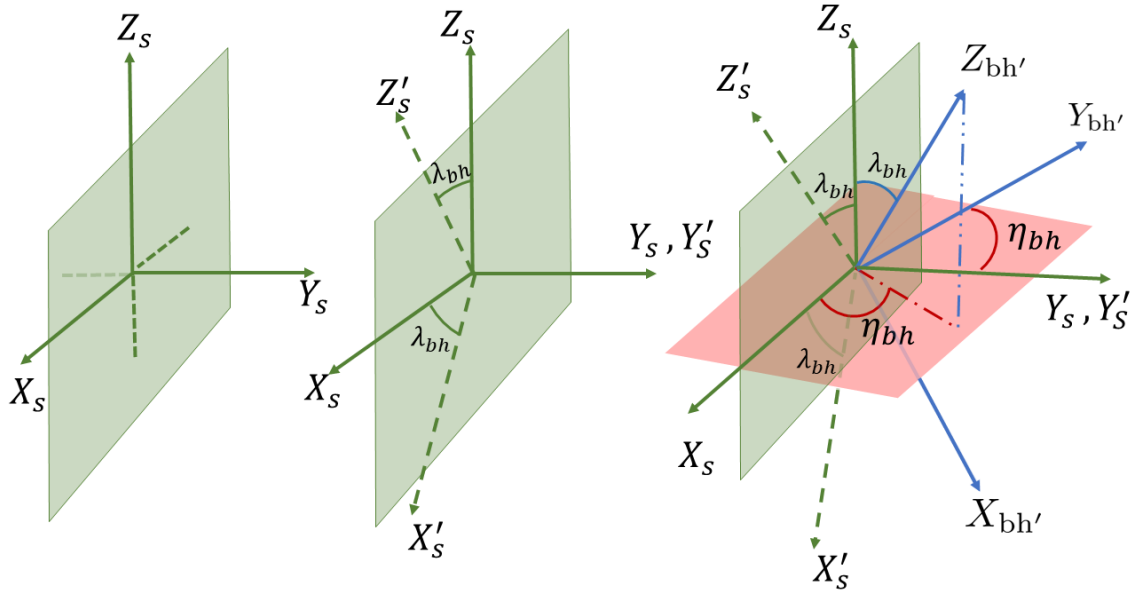


Figure A.7: The transformation from the s-frame to the bh-frame. The green plane is the $Z_s X_s$ plane and the red plane is the $X_s Y_s$ plane.

Next, we rotate the $X'_s Y'_s Z'_s$ frame about the Z'_s -axis by an angle η_{bh} to get the $X'_{bh} Y'_{bh} Z'_{bh}$ frame. As the rotation is about the Z'_s -axis, Y'_{bh} still remains in the $X_s Y_s$ plane, i.e., in the sky-plane.

The above transformation is shown in Fig. A.7 and can be represented as:

$$X_s Y_s Z_s \xrightarrow{R_{Y_s}^P(\lambda_{bh})} X'_s Y'_s Z'_s \xrightarrow{R_{Z_s}^P(\eta_{bh})} X'_{bh} Y'_{bh} Z'_{bh} . \quad (A.22)$$

Hence, the reverse transformation can be written as:

$$X'_{bh} Y'_{bh} Z'_{bh} \xrightarrow{R_{Z_s}^P(-\eta_{bh})} X'_s Y'_s Z'_s \xrightarrow{R_{Y_s}^P(-\lambda_{bh})} X_s Y_s Z_s . \quad (A.23)$$

Thus, if a vector is denoted by $\vec{V}_{bh'}$ in the bh-frame and by $\vec{V}_s$ in the s-frame, then these two would be related as follows:

$$\vec{V}_s = R_Z^A(\eta_{bh}) R_Y^A(\lambda_{bh}) \vec{V}_{bh'} . \quad (A.24)$$

Note that, as the Y_{bh} -axis lies in the sky-plane, if the $X_{bh} Y_{bh} Z_{bh}$ frame is parallelly shifted to any other point as the origin to get a new frame $X'_{bh} Y'_{bh} Z'_{bh}$, Y'_{bh} will still remain in the sky-plane.

A.6 Conversion between the bh' -frame and the b-frame

Using Eqs. (A.5) and (A.8), we write

$$R_Z^A(\eta_{bh}) R_Y^A(\lambda_{bh}) \vec{V}_{bh'} = R_X^A(i) R_Z^A(\omega) \vec{V}_b . \quad (A.25)$$

Hence, the relation between the forms of any given vector in the b-frame and in the bh-frame is given by:

$$\vec{V}_{bh'} = R_Y^A(-\lambda_{bh}) R_Z^A(-\eta_{bh}) R_X^A(i) R_Z^A(\omega) \vec{V}_b . \quad (A.26)$$

A.7 Conversion between bh'-frame and T-frame

Using Eq. (A.18) and (A.26), we get the relation between the forms of any given vector in the bh-frame and in the T-frame as:

$$\vec{V}_{\text{bh}'} = R_Y^A(-\lambda_{\text{bh}}) R_Z^A(-\eta_{\text{bh}}) R_X^A(i) R_Z^A(\omega) R_Z^A(\phi_{\text{Tb}}) R_Y^A(\theta_{\text{Tb}}) \vec{V}_{\text{T}} . \quad (\text{A.27})$$

Appendix B

Details of Calculations for the Schwarzschild Spacetime

B.1 Equations of motion of light rays in the Schwarzschild spacetime

To obtain the path of a light ray in Schwarzschild space time, we need to solve the equation for the null geodesic. This equation takes different forms if the impact parameter, i.e., the perpendicular distance between the light ray's initial direction and the gravitating body, is zero or non-zero. Hence, we will discuss these two cases separately. Moreover to understand the discussion in this chapter, we need to remember the definition of the b-frame (discussed in Chapter 2) and the definition of L-frame (discussed in Chapter 3).

B.2 Geodesics in the Schwarzschild space time

The line element in Schwarzschild space time is given by:

$$ds^2 = \left[- \left(1 - \frac{2GM_c}{c^2 r} \right) c^2 dt^2 + \left(1 - \frac{2GM_c}{c^2 r} \right)^{-1} dr^2 + r^2 d\theta^2 + (r^2 \sin^2 \theta) d\psi^2 \right], \quad (\text{B.1})$$

where (ct, r, θ, ψ) are the time, the radial, the polar, and the azimuthal co-ordinates in the four dimensional spacetime and M_c is the mass of the gravitating object.

The Lagrangian for a particle with mass m is given by

$$\mathcal{L} = \frac{mc^2}{2} g_{ij} \frac{\partial x^i}{\partial \lambda} \frac{\partial x^j}{\partial \lambda} \quad (\text{B.2})$$

where g_{ij} is the metric component and λ is an affine parameter. Then, we can write the Lagrangian of the particle in the metric defined by Eq. (B.1) as

$$\mathcal{L} = \frac{mc^2}{2} \left[- \left(1 - \frac{2GM_c}{c^2 r} \right) c^2 \dot{t}^2 + \left(1 - \frac{2GM_c}{c^2 r} \right)^{-1} \dot{r}^2 + (r^2 \sin^2 \theta) \dot{\psi}^2 \right], \quad (\text{B.3})$$

where the dot denotes the differentiation with respect to the affine parameter λ . The corresponding canonical momenta are:

$$p_t = - \frac{1}{c} \frac{\partial \mathcal{L}}{\partial \dot{t}} = mc^3 \left(1 - \frac{2GM_c}{c^2 r} \right) \dot{t}, \quad (\text{B.4})$$

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = mc^2 \left(1 - \frac{2GM_c}{c^2 r} \right)^{-1} \dot{r}, \quad (\text{B.5})$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mc^2 r^2 \dot{\theta}, \quad (\text{B.6})$$

and

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = mc^2 (r^2 \sin^2 \theta) \dot{\psi}. \quad (\text{B.7})$$

Using Eqs. (B.4), (B.5), (B.6), and (B.7), we can show that

$$-p_t(ct) + (p_r\dot{r} + p_\theta\dot{\theta} + p_\psi\dot{\psi}) = 2\mathcal{L} . \quad (\text{B.8})$$

The definition of the Hamiltonian is

$$\mathcal{H} = -p_t(ct) + (p_r\dot{r} + p_\theta\dot{\theta} + p_\psi\dot{\psi}) - \mathcal{L} . \quad (\text{B.9})$$

Using Eqs. (B.8) and (B.9), we can write

$$\mathcal{H} = \mathcal{L} . \quad (\text{B.10})$$

Hence, $\mathcal{L}$ is a constant of motion.

From Hamilton's equation of motion, we can also write:

$$\frac{dp_t}{d\lambda} = \frac{1}{c} \frac{\partial \mathcal{H}}{\partial t} = \frac{1}{c} \frac{\partial \mathcal{L}}{\partial t} , \quad (\text{B.11a})$$

$$\frac{dp_\psi}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial \psi} = -\frac{\partial \mathcal{L}}{\partial \psi} . \quad (\text{B.11b})$$

From the Lagrangian (Eq. B.3), we see that,

$$\frac{\partial \mathcal{L}}{\partial t} = 0 , \quad (\text{B.12a})$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0 . \quad (\text{B.12b})$$

Using Eq. (B.12a) in Eq. (B.11a), we get $\frac{dp_t}{d\lambda} = 0$ or $p_t = \text{constant}$, which we denote by E . Then, using Eq. (B.4), we can write

$$p_t = -\frac{1}{c} \frac{\partial \mathcal{L}}{\partial \dot{t}} = mc^3 \left(1 - \frac{2GM_c}{c^2 r} \right) \dot{t} = \text{constant} = E, \quad (\text{B.13})$$

Similarly,

Using Eq. (B.12b) in Eq. (B.11b), we get $\frac{dp_\psi}{d\lambda} = 0$ or $p_\psi = \text{constant}$ and using Eq. (B.7), we can write,

$$p_\psi = \frac{\delta \mathcal{L}}{\delta \dot{\psi}} = mc^2 (r^2 \sin^2 \theta) \dot{\psi} = \text{constant} \quad (\text{B.14})$$

It is known that in the Schwarzschild spacetime, if a particle start moving in the $\theta = \frac{\pi}{2}$ plane, it will continue to move in that plane [82]. Due to the spherical symmetry of the spacetime, without lossing any genarality, one can always choose the coordinate system in such a way that the initial value of θ is $\frac{\pi}{2}$. Hence, a particle would always move in the $\theta = \frac{\pi}{2}$ plane making $\dot{\theta} = 0$. Hence, from Eq. (B.6), we get $p_\theta = 0$.

Using $\theta = \frac{\pi}{2}$ in Eq. (B.14), we get

$$p_\psi = \frac{\delta \mathcal{L}}{\delta \dot{\psi}} = mc^2 r^2 \dot{\psi} = \text{constant} = cL. \quad (\text{B.15})$$

The constants E is the energy and L is the angular momentum of the particle. Using Eqs. (B.13), (B.5), (B.15), and the fact that $\dot{\theta} = 0$, in the left hand side of Eq. (B.8), we can construct the equation of motion in Schwarzhild space time as:

$$-\frac{E^2}{mc^2 \left(1 - \frac{2GM_c}{c^2 r} \right)} + \frac{mc^2 \dot{r}^2}{\left(1 - \frac{2GM_c}{c^2 r} \right)} + \frac{L^2}{mr^2} = 2\mathcal{L} \quad (\text{B.16})$$

Rescalling the affine parameter, we can set $2\mathcal{L} = -mc^2$ for the timelike geodesic. Hence

the equation of motion for particle with mass m is

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{m^2 c^2} \left[\frac{E^2}{c^2} - m^2 c^2 \left(1 - \frac{2GM_c}{c^2 r}\right) \left(1 + \frac{L^2}{m^2 c^2 r^2}\right) \right] = \frac{1}{m^2 c^2} \left[\frac{E^2}{c^2} - \frac{V_{\text{eff}}^2}{c^2} \right]. \quad (\text{B.17})$$

Where V_{eff} is known as the effective potential for the particle of mass m moving in Schwarzschild spacetime and given by,

$$V_{\text{eff}}^2 = m^2 c^4 \left(1 - \frac{2GM_c}{c^2 r}\right) \left(1 + \frac{L^2}{m^2 c^2 r^2}\right). \quad (\text{B.18})$$

From Eq. (B.15), we write

$$\frac{d\psi}{d\lambda} = \frac{L}{mcr^2}, \quad (\text{B.19})$$

and From Eq. (B.13), we write

$$\frac{dt}{d\lambda} = \frac{E}{mc^3 \left(1 - \frac{2GM_c}{c^2 r}\right)}. \quad (\text{B.20})$$

B.2.1 Impact parameter D

The impact parameter plays a significant role in the study of geodesics. It is the perpendicular distance between the initial direction of the particle under motion and the gravitating body and is defined as [65]:

$$D = cL/E. \quad (\text{B.21})$$

B.2.2 Geodesics for $D = 0$

From the definition of the impact parameter (Eq. (B.21)), we see that $D = 0$ implies $L = 0$.

Then, from Eq.(B.17), we get

$$\begin{aligned} \left(\frac{dr}{d\lambda}\right)^2 &= \frac{1}{m^2 c^2} \left[\frac{E^2}{c^2} - m^2 c^2 \left(1 - \frac{2GM_c}{c^2 r} \right) \right] \\ \text{or, } \left(\frac{dr}{d\lambda}\right)^2 &= \frac{1}{m^2 c^2} \left[\frac{E^2}{c^2} - V_{\text{eff}}^2|_{L=0} \right]. \end{aligned} \quad (\text{B.22})$$

From Eq. (B.19), we get

$$\frac{d\psi}{d\lambda} = 0. \quad (\text{B.23})$$

Integrating Eq. (B.23), we get

$$\psi = \text{constant}. \quad (\text{B.24})$$

This means that the a particle moves radially when $D = 0$ and this is known as the radial geodesic.

B.2.3 Null Geodesics for $D = 0$

We have seen that when $D = 0$, ψ is constant and the particle moves radially. This is true for photons also, i.e., for null geodesics too. We now see how the radial coordinate varies with the affine parameter in case of a radial null geodesic. In this case, i.e., for photons, $m = 0$ gives $V_{\text{eff}}^2 = 0$ (see Eq. (B.18)).

Then, from Eq. (B.22), we write

$$\begin{aligned} \left(\frac{dr}{d\lambda}\right)^2 &= \lim_{m \rightarrow 0} \frac{1}{m^2 c^2} \left[\frac{E^2}{c^2} \right], \\ \text{giving } \left(\frac{dr}{d\lambda}\right) &= \pm \lim_{m \rightarrow 0} \left(\frac{E}{mc^2} \right). \end{aligned} \quad (\text{B.25})$$

Integrating Eq. (B.25), we get

$$r = \pm \lim_{m \rightarrow 0} \left(\frac{E}{mc^2} \right) \lambda + \text{constant} . \quad (\text{B.26})$$

B.2.4 Null geodesics for $D \neq 0$

From Eqs. (B.17) and (B.19), one can construct the equation of motion for the null geodesic that describes r as a function of ψ in Schwarzschild spacetime. This equation of motion is:

$$\left(\frac{du}{d\psi} \right)^2 = 2 \left(\frac{GM_c}{c^2} \right) u^3 - u^2 + \frac{1}{D^2} = f(u) , \quad (\text{B.27})$$

where $u = 1/r$ is the inverse of the radial co-ordinate of the light rays in the Schwarzschild spacetime. The cartesian coordinate system $X_L Y_L Z_L$ or the ‘L-frame’ which is at the center of the gravitating body and defined in a way that the light ray moves in the $X_L Y_L$ plane and the azimuthal angle ψ is measured from the X_L -axis (see Fig. 3.1).

As $f(u)$ is cubic in u , $f(u) = 0$ must have three roots, say u_1, u_2, u_3 . Now, from the properties of the cubic equation, we can write:

$$u_1 u_2 u_3 = -\frac{c^2}{2GM_c D^2} , \quad (\text{B.28})$$

$$u_1 + u_2 + u_3 = \frac{c^2}{2GM_c} . \quad (\text{B.29})$$

From Eqs. (B.28) and (B.29), we can argue that:

1. All three roots can not be complex, as then $u_1 u_2 u_3$ would be complex, which contradicts Eq. (B.28).
2. It is not possible to have two roots real and one complex as then both $u_1 u_2 u_3$ and $u_1 + u_2 + u_3$ would be complex contradicting both Eqs. (B.28) and (B.29).

3. It is possible to have one root real and the other two complex provided the complex roots are conjugates to each other. Otherwise, both $u_1 u_2 u_3$ and $u_1 + u_2 + u_3$ would be complex contradicting both Eqs. (B.28) and (B.29). When one root is real and the other two are complex and conjugates of each other, we can explore their nature further as below.

- (a) Say the roots are U_d (real), $U_{re} + iU_{im}$, and $U_{re} - iU_{im}$. Giving $u_1 u_2 u_3 = U_d (U_{re}^2 + U_{im}^2)$ and $u_1 + u_2 + u_3 = U_d + 2U_{re}$. Here U_{re} and U_{im} are real.
- (b) If the real root, U_d is positive, then $u_1 u_2 u_3$ would be positive contradicting Eq. (B.28). Hence, the real root, U_d must be negative.
- (c) If U_{re} is negative, then $u_1 + u_2 + u_3$ would be negative contradicting Eq. (B.29), because U_d is also negative (to satisfy Eq. (B.28)). Hence, U_{re} must be positive and $2U_{re} > |U_d|$ to satisfy Eq. (B.29).

Hence, when we have one real and two complex roots, the real root must be negative and the complex roots are conjugate to each other with the real part being positive.

4. All three roots can be real. In this case,

- (a) all three roots can not be negative, as in that case $u_1 + u_2 + u_3$ would be negative, which contradicts Eq. (B.29).
- (b) All three roots can not be positive, as in that case $u_1 u_2 u_3$ would be positive, which contradicts Eq. (B.28).
- (c) It is not possible to have two negative roots and one positive root, As in such a case, $u_1 u_2 u_3$ is positive, which contradicts Eq. (B.28).
- (d) If two roots are positive and one negative, then $u_1 u_2 u_3$ is necessarily negative satisfying Eq. (B.28). Moreover, if the sum of the positive roots are greater than the absolute value of the negative root, then $u_1 + u_2 + u_3$ is positive, which satisfies Eq. (B.29). Hence this is a possible scenario. It is also possible that the two positive roots are equal.

Hence, if all three roots are real, one must be negative and the other two positive. The sum of the values of the positive roots must be larger than the absolute value of the negative root. It is possible that two positive roots are equal.

In Fig. B.1, we have plotted $f(u)$ along the abscissa and u along the ordinate with D as a function of $3\sqrt{3} \frac{GM_c}{c^2}$. We see that, depending on the value of the impact parameter D , three different cases can arise:

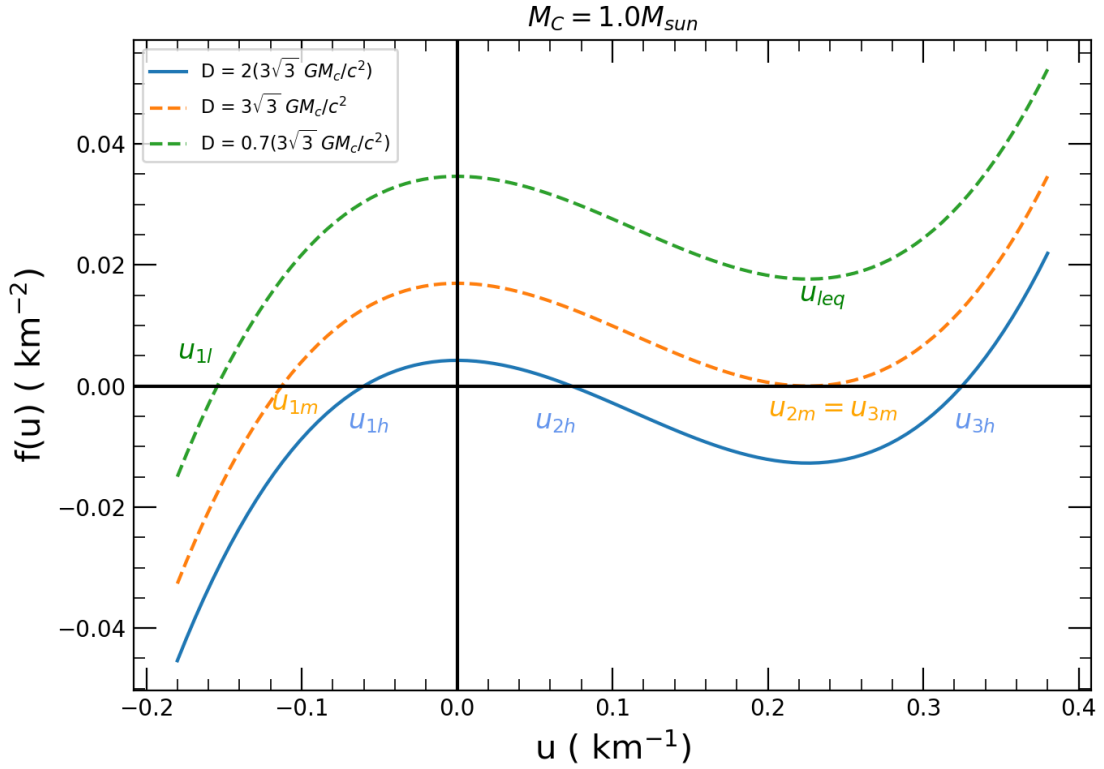


Figure B.1: The plot of $f(u)$ vs u for 3 different values of the impact parameter D in terms of $3\sqrt{3} \frac{GM_c}{c^2}$. The numerical values of u and $f(u)$ depend on the value of M_c . Here we have used $M_c = 1.0 M_\odot$. We have checked that the behaviour of $f(u)$ remain unchanged for a different choice of M_c , only the values of $f(u)$ and roots change.

1. When $D > 3\sqrt{3} \frac{GM_c}{c^2}$, there are three real and distinct roots as seen in Fig. B.1 (solid blue line). Let us call these roots u_{1h} , u_{2h} , u_{3h} in such a way that $u_{1h} < u_{2h} < u_{3h}$, i.e., u_{1h} is negative and u_{2h} and u_{3h} are positive. It is known that $u_{3h} > (3GM_c/c^2)^{-1}$ or $r_{3h} < (3GM_c/c^2)$ [65]. This can be easily understood from Fig. B.1 by remembering the constant $GM_\odot/c^2 = 1.47698$ km.

2. When $D = 3\sqrt{3} \frac{GM_c}{c^2}$, there are three real roots, one negative and other two positive.

We call the positive roots as u_{2m} and u_{3m} and the negative root as u_{1m} . The positive roots are equal to each other. As Fig. B.1 shows, $f(u)$ is minimum at $u = u_{2m} = u_{3m}$.

It is also can be shown that $u_{2m} = u_{3m} = (3GM_c/c^2)^{-1}$, or $r_{2m} = r_{3m} = 3GM_c/c^2$ [65]. In our case, $(3GM_c/c^2)^{-1} = 0.22568 \text{ km}^{-1}$.

3. When $D < 3\sqrt{3} \frac{GM_c}{c^2}$, only the negative root is real, and can be seen Fig.B.1 (marked as u_{1l}). The other two roots are complex and conjugate to each other.

From Eq. (B.27), we see that only the region where $f(u) > 0$ can represent a physical null geodesic as it is a square term, $(du/d\psi)^2$ (see Eq. (B.27)). Moreover, as u is the inverse of the radial position, it cannot be negative, hence the roots u_{1h} , u_{1m} and u_{1l} are unphysical.

B.3 Initial Direction of the light rays

The light emitted from the source can either move towards or away from the gravitation body. If χ is the angle between the initial direction of the light ray ($\hat{n}_{L,\text{in}}$) in the L -frame as defined in Sec. 3.1, see Fig. 3.1 and the radius vector of the pulsar from the companion ($\vec{r}_L$), then, as seen in Fig. B.2, there might be two cases as:

1. Case-I: If $0 \leq \chi \leq \pi/2$, the light ray initially travels away from the companion. The bending is usually weak in this case.
2. Case-II: If $\pi/2 < \chi \leq \pi$, the light ray initially travels towards the companion.

We know that the numerical value of the impact parameter can be written as:

$$D = \frac{r_L \sin(\chi)}{\sqrt{1 - \frac{2GM_c}{c^2 r_L}}}, \quad (\text{B.30})$$

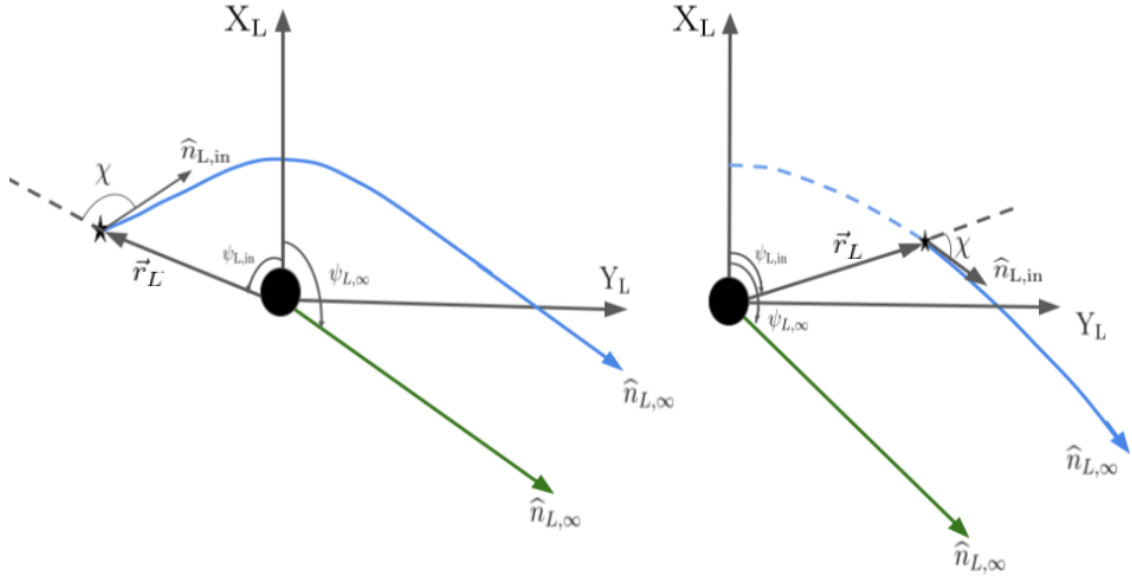


Figure B.2: Schematic diagrams of the light bending phenomenon in the $X_L Y_L$ plane. The left diagram is for the case when the light ray initially moves towards the gravitating body. The right diagram is for the case when the light ray initially moves away from the gravitating body. In both of the cases, the source of the light ray is shown with a black star and the gravitating body is shown with a black filled circle. Here, $\hat{n}_{L,\text{in}}$ denotes the initial direction of the light ray, while $\hat{n}_{L,\infty}$ represents its final direction after bending. The quantity $\psi_{L,\text{in}}$ is the initial azimuthal angle of the light ray (at the time of emission) in the L-frame, i.e., the angle between the X_L axis and the position vector $\vec{r}_L$. Similarly, $\psi_{L,\infty}$ is the azimuthal angle corresponding to the final direction of the light ray, i.e., the angle between the X_L axis and $\hat{n}_{L,\infty}$.

where r_L is the magnitude of the position vector of the pulsar in the L-frame as shown in Fig. (B.2), which is also the initial magnitude of the position vector of the light ray that can be used in the geodesic equation as the origin of the L-frame is chosen at the location of the companion.

We now discuss the physically acceptable geodesics relevant for our study in both of the cases. Note that, in the following discussions, whenever we say that the light ray can reach the observer, we mean that the light ray can reach the distance at which the observer is located. Whether the light ray would actually reach the observer or not depends on whether the final direction of the light ray is towards the observer in the three dimensional space.

Note that, the variable u is measured from the gravitating body, in case of a binary pulsar, it is the companion of the pulsar. If u_{st} is the starting point of the light ray, i.e., the location of the pulsar and u_{obs} is the location of the observer, it is obvious that for any realistic scenario, $0 \leq u_{\text{obs}} \leq u_{\text{st}}$, i.e., $\infty \geq r_{\text{obs}} \geq r_{\text{st}}$.

There is another property of Schwarzschild spacetime, which is the fact that no object can move around a gravitating body closer than the innermost stable circular orbit (ISCO) of the gravitating object. The radius of the ISCO is $r_{\text{isco}} = 6GM_c/c^2$. Hence $r_{\text{st}} \geq 6GM_c/c^2$ or $u_{\text{st}} \leq (6GM_c/c^2)^{-1}$.

B.3.1 The light ray initially travelling away from the companion (u initially decreasing):

Case A: $D > \frac{3\sqrt{3}GM_c}{c^2}$ (solid blue curve in Fig. B.1)

Only two physically acceptable null geodesics satisfy both of the conditions $f(u) > 0$ and $u \geq 0$.

1. The geodesic in the region of $0 \leq u \leq u_{2h}$. If the light ray starts at $u_{\text{st}} \leq u_{2h}$, it can go to $u \rightarrow 0$, i.e., up to infinity. The light ray can come to the observer whenever $0 \leq u_{\text{obs}} \leq u_{\text{st}}$.
2. The geodesic in the region of $u \geq u_{3h}$. If the light ray starts at $u_{\text{st}} > u_{3h}$, it can go up to $u = u_{3h}$, it cannot go further away from the companion (as $f(u)$ becomes negative at $u \leq u_{3h}$). Moreover, $u_{\text{st}} > u_{3h}$ means $r_{\text{st}} < r_{3h}$. We have discussed in Sec. B.2.4 that $r_{3h} < 3GM_c/c^2$, i.e., closer than ISCO. Hence, $r_{\text{st}} < r_{3h}$ is not possible.

Case B: $D = \frac{3\sqrt{3}GM_c}{c^2}$ (orange dashed curve in Fig. B.1)

Here also only two physically acceptable null geodesics satisfy both of the conditions $f(u) > 0$ and $u \geq 0$.

1. The geodesic in the region of $0 \leq u \leq u_{2m}$. If the light ray starts at $u_{st} \leq u_{2m}$, it can go to $u \rightarrow 0$, i.e., up to infinity. The light ray can come to the observer whenever $0 \leq u_{obs} \leq u_{st}$.
2. The geodesic in the region of $u \geq u_{3m}$. However, $u_{st} > u_{3m}$ means $r_{st} < r_{3m}$. We know that $r_{3m} = 3GM_c/c^2$, i.e., closer than ISCO. Hence, $r_{st} < r_{3m}$ is not possible.

Case C: $D < \frac{3\sqrt{3}GM_c}{c^2}$ (green dotted curve in Fig. B.1)

For all real positive values of u , $f(u) > 0$. We also see that $f(u)$ has a minimum at a particular positive value of $u = u_{leq}$.

1. $u_{st} \leq u_{leq}$: In this case the bending will be very weak and the light ray can reach observer regardless of its starting position.
2. $u_{st} > u_{leq}$: In this case light ray initially inspiral around the companion and cross the minimum point $u = u_{leq}$ and then it moves to $u = 0$ almost in a straight line. However, u_{st} must have to be less than $\left(\frac{2GM_c}{c^2}\right)^{-1}$ i.e, r_{st} should be greater than the Schwarzschild radius. Otherwise light ray cannot escape from the event horizon.

Hence, the light ray can reach infinity, i.e., $u \rightarrow 0$ regardless where it starts. However, it cannot start closer to ISCO, i.e, one must have $u_{st} \leq u_{isco}$. The light ray can come to the observer whenever $0 \leq u_{obs} \leq u_{st}$ or $\infty \geq r_{obs} \geq r_{st}$.

B.3.2 The light ray initially travelling towards the companion (u initially increasing):

Case A: $D > \frac{3\sqrt{3}GM_c}{c^2}$ (solid blue curve in Fig. B.1)

Only two physically acceptable null geodesics satisfy both of the conditions $f(u) > 0$ and $u \geq 0$.

1. Geodesic of the first kind: This is the geodesic in the region $0 \leq u \leq u_{2h}$. For this geodesic, if the light starts anywhere in the region $0 \leq u_{st} \leq u_{2h}$, it comes towards the gravitating object and reach the minimum distance at $u = u_{2h}$ and return to the infinity ($u = 0$). Hence, the light ray can come to the observer whenever $0 \leq u_{obs} \leq u_{st}$ or $0 \geq r_{obs} \geq r_{st}$.
2. Geodesic of the second kind: This is the geodesic when $u \geq u_{3h}$. Here, if the the light ray start at $u_{st} \geq u_{3h}$ it will keep on increasing u and eventually reach $u \rightarrow \infty$, i.e., will be trapped by the gravitating body. Moreover, $u_{st} > u_{3h}$ means $r_{st} < r_{3h}$. We have seen that $r_{3h} < 3GM_c/c^2$, i.e., closer than ISCO. Hence, $r_{st} < r_{3h}$ is not possible

Case B: $D = \frac{3\sqrt{3}GM_c}{c^2}$ (dashed orange curve in Fig. B.1)

Here also only two physically acceptable null geodesics satisfy both of the conditions $f(u) > 0$ and $u \geq 0$.

1. Geodesic of the first kind: This is the geodesic in the region $0 \leq u \leq u_{2m}$. If the light ray starts at $u_{st} > 0$, it comes towards the gravitating object and reach the minimum distance at $u = u_{2m}$. However, when light ray reaches close to $u = u_{2m}$, the value of u changes very slowly with respect to ψ and light ray reaches $u = u_{2m}$ when $\psi \rightarrow \infty$ [65]. Hence light ray can reach $u = u_{2m}$ asymptotically by spiraling

around the circle of radius $r = 3\frac{GM_c}{c^2}$ (see Sec. B.2.4). Hence, the light ray can come to the observer only if $u_{\text{st}} \leq u_{\text{obs}} \leq u_{2m}$ or $r_{\text{st}} \geq r_{\text{obs}} \geq r_{2m}$, which is impossible, since the light ray can reach to the observer whenever $0 \leq u_{\text{obs}} \leq u_{\text{st}}$, as explained at the end of Sec. B.3.1. .

2. Geodesic of the second kind: This is the geodesic when $u \geq u_{3m}$. However, $u_{\text{st}} > u_{3m}$ means $r_{\text{st}} < r_{3m}$. We know that $r_{3m} = 3GM_c/c^2$, i.e., closer than ISCO. Hence, $r_{\text{st}} < r_{3m}$ is not possible.

Hence, we conclude that we can ignore the case of $D = \frac{3\sqrt{3}GM_c}{c^2}$ (when u increases initially).

Case C: $D < \frac{3\sqrt{3}GM_c}{c^2}$ (green dotted curve in Fig. B.1)

For all real positive values of u , $f(u) > 0$. We also see that $f(u)$ has a minimum at a particular positive value of $u = u_{\text{leq}}$. When light ray reaches close to the $u = u_{\text{leq}}$, the value of u changes very slowly with respect to ψ , as a result it starts spiraling around the companion and eventually cross this minimum point and goes to $u \rightarrow \infty$ [65].

Hence, regardless of the starting position, the light ray would keep on increasing u and eventually would reach $u \rightarrow \infty$, i.e., the light ray would be trapped by the gravitating body. In this situation, the light ray can reach the observer only if $u_{\text{st}} \leq u_{\text{obs}} \leq \infty$ or $r_{\text{st}} \geq r_{\text{obs}} \geq 0$, which is impossible (explained at the end of Sec. B.3.1).

Hence, we ignore the case of $D < \frac{3\sqrt{3}GM_c}{c^2}$ (when u increases initially).

B.4 Solving the geodesic equation for $D > \frac{3\sqrt{3}GM_c}{c^2}$ (solid blue curve in Fig. B.1), regardless of the initial direction of the light ray

We have seen in Secs. B.3.1 and B.3.2 that when $D > \frac{3\sqrt{3}GM_c}{c^2}$, the only physically acceptable null geodesic is possible in the region $0 \leq u \leq u_{2h}$, and u_{2h} is the inverse of the distance of the closest approach.

It is known that [65], for $D > \frac{3\sqrt{3}GM_c}{c^2}$, the three roots of $f(u) = 0$ where $f(u)$ is the right hand side of Eq. (B.27) can be expressed in a parametric form as:

$$u_{1h} = \frac{P - 2GM_c/c^2 - Q}{4MP}, \quad (\text{B.31})$$

$$u_{2h} = \frac{1}{P}, \quad (\text{B.32})$$

$$u_{3h} = \frac{P - 2GM_c/c^2 + Q}{4PGM_c/c^2}. \quad (\text{B.33})$$

The parameter P is just the distance of the closest approach ($1/u_{2h}$) of the light ray to the gravitating body. The parameter Q is related to P and M_c as

$$Q = [(Pc^2 - 2GM_c)(Pc^2 + 2GM_c)]^{1/2} \quad (\text{B.34})$$

Using these parameters, the solution of Eq. (B.27) can be written in a parametric form as:

$$u = \frac{1}{P} - \frac{(Q - P + 6GM_c/c^2)(1 + \cos \xi)}{8PGM_c/c^2}, \quad (\text{B.35})$$

$$\psi = 2 \left(\frac{P}{Q} \right)^{1/2} \left[K(k) - F \left(\frac{\xi}{2}, k \right) \right]. \quad (\text{B.36})$$

Here K and F are the Jacobi complete and incomplete elliptic integrals, respectively. ξ is

a free parameter and k is defined as:

$$k = \left[\frac{1}{2Q} (Q - P + 6GM_c/c^2) \right]^{1/2}. \quad (\text{B.37})$$

From Eqs. (B.35) and (B.36), it can be shown that for $\xi = \pi$, i.e., for $\cos \xi = -1$, $u = 1/P = u_{2h}$ (the distance of the closest approach), and $\psi = 0$. As ψ is measured from the X_L -axis (Fig. B.2), we can say that the distance of the closest approach happens along the X_L -axis.

We denote the initial value of the azimuthal angle of the light ray (at the time of the emission) as $\psi_{L,\text{in}}$ and the final value of the azimuthal angle of the light ray (after bending) as $\psi_{L,\infty}$. As shown in Fig B.2, $\psi_{L,\text{in}}$ is negative if the light ray initially moves towards the companion and $\psi_{L,\text{in}}$ is positive if the light ray initially moves towards the companion. In the first case, ψ increases and becomes zero at the distance of minimum approach ($u = u_{2h}$) and then becomes positive, i.e., $\psi_{L,\infty}$ is positive. For the second case also, ψ increases and remains positive. These facts are demonstrated in Fig. B.2.

Next we discuss the procedure to calculate $\psi_{L,\text{in}}$. As the masses of the pulsar and the companion, as well as the orbital parameters are known, we first calculate the position vector of the pulsar in the b-frame, i.e., $\vec{r}_b$. As the origin of the b-frame and the L-frame are the same, we write $r_b = r_L$ where $r_b = |\vec{r}_b|$. Then, we calculate the value of D using Eq. (B.30). Using the value of D , we first find the roots of $f(u) = 0$. Then we calculate the value of P using Eq. (B.32), Q using Eq. (B.34) and k using Eq. (B.37). We also know that $u_{\text{in}} = 1/r_L$. Using this value of u_{in} in Eq. (B.35), we get the value of ξ_{in} and using the value of ξ_{in} in Eq. (B.36), we get the value of $\psi_{L,\text{in}}$.

Now, we need to calculate the value of the azimuthal angle after the bending. As this angle does not change after the bending is completed, which happens much before the position of the observer, we can say that the value of the azimuthal angle at the observer is the same as the value of this angle at infinity. Hence, we use $u = 0$ in Eq. (B.35), we

get the value of ξ_∞ , which can be expressed as:

$$\sin^2 \left(\frac{\xi_\infty}{2} \right) = \frac{Q - P + 2GM_c/c^2}{Q - P + 6GM_c/c^2} . \quad (\text{B.38})$$

Using the value of ξ_∞ in Eq. (B.36), we get the value of $\psi_{L,\infty}$. The final direction of the light ray in the L-frame is given by $\hat{n}_{L,\infty} = [\cos \psi_{L,\infty}, \sin \psi_{L,\infty}, 0]$.

Appendix C

Details of Calculations for the Kerr Spacetime

C.1 Equations of motion of light rays in the Kerr spacetime

In this section, we present the analytical derivation of geodesic motion of light rays in Kerr spacetime. Note that, although originally proposed for spinning black holes, the Kerr metric can be used to describe the spacetime around any spinning gravitating object, e.g., a spinning neutron star. However, in the present work, we are using only a black hole as the gravitating body, i.e., the companion of the pulsar.

In Boyer-Lindquist coordinates, the coordinates x^α are $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. In this coordinate system, Kerr metric is defined as:

$$g_{tt} = - \left(\frac{\Delta - \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \sin^2 \theta}{\rho^2} \right), \quad (\text{C.1a})$$

$$g_{rr} = \frac{\rho^2}{\Delta} , \quad (\text{C.1b})$$

$$g_{\theta\theta} = \rho^2 , \quad (\text{C.1c})$$

$$g_{\phi\phi} = \frac{\Sigma^2 \sin^2 \theta}{\rho^2} , \quad (\text{C.1d})$$

$$g_{t\phi} = -\frac{2 \left(\frac{GM_c}{c^2} \right) \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 r \sin^2 \theta}{\rho^2} , \quad (\text{C.1e})$$

where $\rho^2 = r^2 + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \cos^2 \theta$, $\Delta = r^2 - 2 \frac{GM_c}{c^2} r + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2$, $\Sigma^2 = \left[r^2 + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right]^2 - \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \Delta \sin^2 \theta$, $\omega = \frac{2 \left(\frac{GM_c}{c^2} \right) r \left(\tilde{a} \frac{GM_c}{c^2} \right)}{\Sigma^2}$. M_c is the mass of the spinning gravitating object, G is the gravitational constant, and c is the speed of light in vacuum. $|\tilde{a}|$ is a number ranged between 0 to 1. $\tilde{a}$ takes a positive value for a prograde rotation of the spinning gravitating object and a negative value for the retrograde rotation of the spinning gravitating object. It is related to the spin angular momentum $\mathcal{S}_{\text{bh}}$ of the gravitating object by the relation $\mathcal{S}_{\text{bh}} = \tilde{a} GM_c^2 / c$. The non-zero components of the inverse metric are:

$$g^{tt} = -\frac{\Sigma^2}{\rho^2 \Delta} , \quad (\text{C.2a})$$

$$g^{rr} = \frac{\Delta}{\rho^2} , \quad (\text{C.2b})$$

$$g^{\theta\theta} = \frac{1}{\rho^2} , \quad (\text{C.2c})$$

$$g^{\phi\phi} = \frac{\Delta - \left(\tilde{a}\frac{GM_c}{c^2}\right)^2 \sin^2 \theta}{\rho^2 \sin^2 \theta \Delta}, \quad (\text{C.2d})$$

$$g^{t\phi} = -\frac{2(GM_c/c^2) \left(\tilde{a}\frac{GM_c}{c^2}\right) r}{\rho^2 \Delta}. \quad (\text{C.2e})$$

The line element in a spacetime defined by the Kerr metric takes the following form:

$$ds^2 = -\left(\frac{\Delta - \left(\tilde{a}\frac{GM_c}{c^2}\right)^2 \sin^2 \theta}{\rho^2}\right)(cdt)^2 - \frac{4\left(\frac{GM_c}{c^2}\right)\left(\tilde{a}\frac{GM_c}{c^2}\right)^2 r \sin^2 \theta}{\rho^2} cdt d\phi \\ + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} d\phi^2, \quad (\text{C.3})$$

or equivalently,

$$ds^2 = -\frac{\rho^2 \Delta}{\Sigma^2} (cdt)^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} (d\phi - \omega cdt)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \quad (\text{C.4})$$

It is also known that if the value of the radial coordinate r is less than particular value(s) known as the event horizon(s) at which $g^{rr} = 0$, no information from the spinning black hole can reach any observer, more specifically, any information takes an infinite amount of time to cross the event horizon. For a Kerr black hole, there exist two event horizons, the outer event horizon r_H^+ and the inner event horizon r_H^- given by:

$$r_H^+ = \frac{GM_c}{c^2} + \sqrt{\left(\frac{GM_c}{c^2}\right)^2 - \left(\tilde{a}\frac{GM_c}{c^2}\right)^2}, \quad (\text{C.5a})$$

$$r_H^- = \frac{GM_c}{c^2} - \sqrt{\left(\frac{GM_c}{c^2}\right)^2 - \left(\tilde{a}\frac{GM_c}{c^2}\right)^2}. \quad (\text{C.5b})$$

The equation governing the geodesic motion of a particle of mass m with an action $\mathcal{A}$ is

given by:

$$-m^2 c^4 = g^{\alpha\beta} \partial_\alpha \mathcal{A} \partial_\beta \mathcal{A} , \quad (\text{C.6})$$

where, as usual, the indices α and β run from 0 to 3 and Einstein summation convention is implied. Assuming that the variables can be separated, the action can be written as:

$$\mathcal{A} = -Ect + cL_z \phi + \mathcal{A}_r(r) + \mathcal{A}_\theta(\theta) , \quad (\text{C.7})$$

where, E and L_z are the energy and the Z component of the angular momentum and the Z axis is along the spin axis of the gravitating object. $\mathcal{A}_r(r)$, $\mathcal{A}_\theta(\theta)$ are functions of r and θ respectively. Using Eq. (C.7) in Eq. (C.6) we get,

$$\begin{aligned} m^2 c^4 \rho^2 = \frac{1}{\Delta} \left[\left\{ r^2 + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right\} E - \left(\tilde{a} \frac{GM_c}{c^2} \right) cL_z \right]^2 \\ - \frac{1}{\sin^2 \theta} \left(\left(\tilde{a} \frac{GM_c}{c^2} \right) E \sin^2 \theta - cL_z \right)^2 \\ - \Delta \left(\frac{d\mathcal{A}_r(r)}{dr} \right)^2 - \left(\frac{d\mathcal{A}_\theta(\theta)}{d\theta} \right)^2 . \quad (\text{C.8}) \end{aligned}$$

We can rewrite Eq. (C.8) in the form:

$$\begin{aligned} \left[\Delta \left(\frac{d\mathcal{A}_r(r)}{dr} \right)^2 - \frac{1}{\Delta} \left[\left(r^2 + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right) E - \left(\tilde{a} \frac{GM_c}{c^2} \right) cL_z \right]^2 + \left(cL_z - \left(\tilde{a} \frac{GM_c}{c^2} \right) E \right)^2 + m^2 c^4 r^2 \right] \\ + \left[\left(\frac{d\mathcal{A}_\theta(\theta)}{d\theta} \right)^2 + \left(\frac{c^2 L_z^2}{\sin^2 \theta} - \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 E^2 \right) \cos^2 \theta + m^2 c^4 \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \cos^2 \theta \right] = 0 . \quad (\text{C.9}) \end{aligned}$$

From Eq. (C.9), it is clear that the sum of the r dependent part and the θ dependent part is zero. Hence, they can be equated to a constant of equal magnitude but of opposite sign.

Denoting this separation constant as $\mathcal{L}$, we write:

$$\Delta \left(\frac{d\mathcal{A}_r(r)}{dr} \right)^2 = \frac{1}{\Delta} \left[\left(r^2 + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right) E - \left(\tilde{a} \frac{GM_c}{c^2} \right) cL_z \right]^2 - \left[\mathcal{L} + \left(cL_z - \left(\tilde{a} \frac{GM_c}{c^2} \right) E \right)^2 + m^2 c^4 r^2 \right] = \mathcal{R}(r)/\Delta, \quad (\text{C.10})$$

$$\left(\frac{d\mathcal{A}_\theta(\theta)}{d\theta} \right)^2 = \mathcal{L} - \left(\frac{c^2 L_z^2}{\sin^2 \theta} - \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 E^2 + m^2 c^4 \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right) \cos^2 \theta = \Theta(\theta). \quad (\text{C.11})$$

In the above two equations, we define the radial potential $\mathcal{R}(r)$ and polar potential $\Theta(\theta)$ as:

$$\mathcal{R}(r) = \left[\left\{ r^2 + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right\} E - \left(\tilde{a} \frac{GM_c}{c^2} \right) cL_z \right]^2 - \left[\mathcal{L} + \left(cL_z - \left(\tilde{a} \frac{GM_c}{c^2} \right) E \right)^2 + m^2 c^4 r^2 \right], \quad (\text{C.12a})$$

$$\Theta(\theta) = \left(\frac{d\mathcal{A}_\theta(\theta)}{d\theta} \right)^2 = \mathcal{L} - \left[\frac{c^2 L_z^2}{\sin^2 \theta} - \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 E^2 + m^2 c^4 \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right] \cos^2 \theta. \quad (\text{C.12b})$$

Eq. (C.7) can be rewritten as:

$$\mathcal{A} = -Ect + cL_z \phi + \oint_r \frac{\text{sign}(p_r) \sqrt{\mathcal{R}(r)}}{\Delta} dr + \oint_\theta \text{sign}(p_\theta) \sqrt{\Theta(\theta)} d\theta, \quad (\text{C.13})$$

where p_r and p_θ are the r component and the θ component of the momentum, respectively, of the test mass m and $\oint$ indicate that the integrals are to be understood as path integrals. The equations governing the motion can be derived from Eq. (C.13) by the standard procedure of equating the partial derivatives of $\mathcal{A}$ with respect to different constants of motion, $\mathcal{L}$, E , and cL_z to zero.

Setting the partial derivative of $\mathcal{A}$ with respect to $\mathcal{L}$ to zero we get

$$\oint_r \frac{dr}{\text{sign}(p_r) \sqrt{\mathcal{R}(r)}} = \oint_\theta \frac{d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} . \quad (\text{C.14})$$

Setting the partial derivative of $\mathcal{A}$ with respect to cL_z to zero we get

$$\begin{aligned} \phi_o - \phi_s = & \left(\tilde{a} \frac{GM_c}{c^2} \right) \oint_r \left(\left(r^2 + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right) E - \left(\tilde{a} \frac{GM_c}{c^2} \right) cL_z \right) \frac{dr}{\text{sign}(p_r) \sqrt{\mathcal{R}(r)} \Delta} \\ & + \oint_\theta \left(cL_z \csc^2 \theta - \left(\tilde{a} \frac{GM_c}{c^2} \right) E \right) \frac{d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} , \end{aligned} \quad (\text{C.15})$$

and similarly, setting the partial derivative of $\mathcal{A}$ with respect to E to zero we get

$$\begin{aligned} ct_0 - ct_s = E \oint_r & \frac{r^2 \Delta + 2GM_c r / c^2 \left(r^2 - \left(\tilde{a} \frac{GM_c}{c^2} \right) L_z / E + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right)}{\text{sign}(p_r) \sqrt{\mathcal{R}(r)} \Delta} \\ & + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \oint_\theta \frac{\cos^2 \theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} d\theta . \end{aligned} \quad (\text{C.16})$$

In this context x_s^μ and x_o^μ are the source and the observer points, respectively for each x^μ . Rescaling the quantity $\mathcal{R}(r)$ and $\Theta(\theta)$ as $\mathcal{R}(r) \rightarrow \mathcal{R}(r)/E^2$, $\Theta(\theta) \rightarrow \Theta(\theta)/E^2$ and introducing a new set of constants of motion $D_q = \mathcal{L}/E^2$ and $D_\lambda = cL_z/E$ (following [83]), we rewrite the equations of motion from Eq. (C.14), (C.15), (C.16) as:

$$I_r = G_\theta , \quad (\text{C.17})$$

$$\phi_o - \phi_s = I_\phi + D_\lambda G_\phi , \quad (\text{C.18})$$

$$ct_0 - ct_s = I_t + \left(\tilde{a} \frac{GM_c}{c^2} \right) G_t . \quad (\text{C.19})$$

Eqs. C.17, C.18 and C.19 are known as the geodesics equation in Kerr spacetime, where we have defined a set of parameters as follows:

$$I_r = \oint_r \frac{dr}{\text{sign}(p_r) \sqrt{\mathcal{R}(r)}} , \quad (\text{C.20a})$$

$$I_\phi = \int_r \frac{\left(\tilde{a}\frac{GM_c}{c^2}\right) \left((2GM_c/c^2)r - D_\lambda \left(\tilde{a}\frac{GM_c}{c^2}\right)\right)}{\text{sign}(p_r)\sqrt{\mathcal{R}(r)}\Delta} dr, \quad (\text{C.20b})$$

$$I_t = \int_r \frac{r^2\Delta + 2(GM_c/c^2)r \left(r^2 + \left(\tilde{a}\frac{GM_c}{c^2}\right)^2 - D_\lambda \left(\tilde{a}\frac{GM_c}{c^2}\right)\right)}{\text{sign}(p_r)\sqrt{\mathcal{R}(r)}\Delta} dr, \quad (\text{C.20c})$$

$$G_\theta = \int_\theta \frac{d\theta}{\text{sign}(p_\theta)\sqrt{\Theta(\theta)}}, \quad (\text{C.20d})$$

$$G_\phi = \int_\theta \frac{\csc^2 \theta}{\text{sign}(p_\theta)\sqrt{\Theta(\theta)}} d\theta, \quad (\text{C.20e})$$

$$G_t = \int_\theta \frac{\cos^2 \theta}{\text{sign}(p_\theta)\sqrt{\Theta(\theta)}} d\theta, \quad (\text{C.20f})$$

Note that, each I_i involves an integration with respect to r and each G_i represents an integration with respect to θ , where the subscript i goes for t , r and ϕ . Using those new parameters, the radial and the polar potentials take the following forms:

$$\begin{aligned} \mathcal{R}(r) = r^4 + & \left[\left(\tilde{a}\frac{GM_c}{c^2}\right)^2 - D_\lambda^2 - D_q \right] r^2 \\ & + 2 \left(\frac{GM_c}{c^2}\right) \left[D_q + \left\{ D_\lambda - \left(\tilde{a}\frac{GM_c}{c^2}\right) \right\}^2 \right] r \\ & - \left(\tilde{a}\frac{GM_c}{c^2}\right)^2 D_q - \left(\frac{m^2 c^4}{E^2}\right) r^2 \Delta, \quad (\text{C.21a}) \end{aligned}$$

$$\Theta(\theta) = D_q + \left(\tilde{a}\frac{GM_c}{c^2}\right)^2 \cos^2 \theta - D_\lambda^2 \cot^2 \theta - \left(\frac{m^2 c^4}{E^2}\right) \left(\tilde{a}\frac{GM_c}{c^2}\right)^2 \cos^2 \theta. \quad (\text{C.21b})$$

The equations derived in this section describe the motion of a particle of mass m in Kerr spacetime.

C.2 Null geodesics

For null geodesic, we need to set $m = 0$ in the geodesics equations given in Eqs. C.17, C.18 and C.19. All the equations of motion remain the same except $\mathcal{R}(r)$ and $\Theta(\theta)$ take the form as:

$$\begin{aligned} \mathcal{R}(r) = r^4 + \left[\left(\tilde{a} \frac{GM_c}{c^2} \right)^2 - D_\lambda^2 - D_q \right] r^2 \\ + 2 \left(\frac{GM_c}{c^2} \right) \left[D_q + \left\{ D_\lambda - \left(\tilde{a} \frac{GM_c}{c^2} \right) \right\}^2 \right] r - \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 D_q, \quad (\text{C.22a}) \end{aligned}$$

$$\Theta(\theta) = D_q + \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \cos^2 \theta - D_\lambda^2 \cot^2 \theta. \quad (\text{C.22b})$$

From Eqs. (C.22), we see that if we need to trace the path of a light ray along the null geodesic, we need to solve path integrals about r i.e., I_i and θ i.e., G_i . The solution of the path integral I_i can be expressed in terms of the roots of $\mathcal{R}(r)$, while the solution of the path integral G_i can be expressed in terms of the roots of $\Theta(\theta)$. These roots also help identify the regions where $\mathcal{R}(r)$ and $\Theta(\theta)$ are positive. Hence, we discuss these two kinds of integrals next. As we are dealing with the null geodesic only, p_r , p_θ , p_ϕ are the canonical momenta associated with the motion of a light ray in the radial (r), polar (θ) azimuthal (ϕ) directions, respectively and p_t is the time component of the angular momentum of the light ray, in a locally non-rotating frame.

C.2.1 θ integrals

In Eqs. (C.20d), (C.20e) and (C.20f), we can see that the θ integrals depend on the function $\Theta(\theta)$ which is defined in Eq. (C.22b). From Eqs. (C.13), (C.20d), (C.20e), and (C.20f), we see that $\Theta(\theta)$ must be zero or positive, otherwise $\mathcal{A}$, G_θ , G_ϕ , G_t would be imaginary. Moreover, as $\cot^2 \theta$ becomes ∞ at $\theta = 0$ and $\theta = \pi$, Eq. (C.22b) ensures that the light ray can reach to $\theta = 0$ and $\theta = \pi$ only if $D_\lambda = 0$. We exclude this special case of $D_\lambda = 0$ and work on the generic condition $D_\lambda^2 > 0$, so the light ray cannot reach to $\theta = 0$, $\theta = \pi$ in our formalism. Here we distinguish three cases that can arise depending on the value of D_q . For this, we first define a new variable:

$$u = \cos^2 \theta \quad (\text{C.23})$$

Using Eq. (C.23), we re-write Eq. (C.22b) as:

$$\Theta(u) = \begin{cases} \frac{\left(\tilde{a} \frac{GM_c}{c^2}\right)^2}{1-u} (u_+ - u) (u - u_-) & \text{if } \tilde{a} \neq 0, \\ \frac{D_q + D_\lambda^2}{1-u} (u - u_q) & \text{if } \tilde{a} = 0, \end{cases} \quad (\text{C.24})$$

where,

$$\Delta_\theta = \frac{1}{2} \left(1 - \frac{D_q + D_\lambda^2}{\left(\tilde{a} \frac{GM_c}{c^2}\right)^2} \right), \quad (\text{C.25a})$$

$$u_\pm = \Delta_\theta \pm \sqrt{\left(\Delta_\theta^2 + \frac{D_q}{\left(\tilde{a} \frac{GM_c}{c^2}\right)^2} \right)}, \quad (\text{C.25b})$$

$$u_q = \frac{D_q}{D_q + D_\lambda^2}. \quad (\text{C.25c})$$

We also define,

$$\Psi^\pm = \sin^{-1} \left(\sqrt{\frac{\cos^2 \theta}{u_\pm}} \right), \quad (\text{C.25d})$$

This means that $u_{\pm}$ are roots of $\Theta(u)$ when $\tilde{a} \neq 0$ and u_q is the root of $\Theta(u)$ when $\tilde{a} = 0$. We also note that $\theta = \pi/2$ denotes the equatorial plane of the gravitating object. The points with $\theta < \pi/2$ are the points above the equatorial plane and the points with $\theta > \pi/2$ are the points below the equatorial plane.

Case-1: For $\tilde{a} \neq 0$, when $D_q > 0$, the condition $\Theta(u) \geq 0$ gives: $u_- < 0 \leq \cos^2 \theta \leq u_+ < 1$. Hence $\Theta(u) \geq 0$ only between $\theta_1^a = \cos^{-1}(+\sqrt{u_+})$ and $\theta_1^b = \cos^{-1}(-\sqrt{u_+})$. This means that the light ray oscillates between the turning points θ_1^a and θ_1^b , where, $\theta_1^a = \pi - \theta_1^b$.

Case-2: For $\tilde{a} \neq 0$, when $D_q < 0$, the condition $\Theta(u) \geq 0$ gives: $0 < u_- \leq \cos^2 \theta \leq u_+ < 1$. Hence, $\Theta(u) \geq 0$ condition results in two regions, one is above and the other is below the equatorial plane. The region above the equatorial plane has the polar boundary in which null geodesics can exist is between $\theta_{2,f}^a = \cos^{-1}(+\sqrt{u_+})$ and $\theta_{2,n}^a = \cos^{-1}(+\sqrt{u_-})$ and for the region below the equatorial plane the polar boundary is between $\theta_{2,f}^b = \cos^{-1}(-\sqrt{u_+})$ and $\theta_{2,n}^b = \cos^{-1}(-\sqrt{u_-})$.

Case-3: For $\tilde{a} \neq 0$ when $D_q = 0$, the condition $\Theta(u) \geq 0$ gives: $0 = u_- \leq \cos^2 \theta \leq u_+ < 1$. Hence $\Theta(u) \geq 0$ condition results in two regions. The first region is above the equatorial plane, lying between the equatorial plane and $\theta_3^a = \cos^{-1}(+\sqrt{u_+})$. The second region is below the equatorial plane, lying between the equatorial plane and $\theta_3^b = \cos^{-1}(-\sqrt{u_+})$. However, $u_- = 0$ corresponds to a non-integrable singularity, which means that the geodesic motion is barred from reaching the equatorial plane, it can only approach the equatorial plane asymptotically.

Case-4: For $\tilde{a} = 0$, D_q must be positive ($D_q > 0$) to have $\Theta(u) \geq 0$ (see Eq. (C.22b)). That gives $0 \leq \cos^2 \theta \leq u_0 < 1$. Hence, the light ray oscillates between $\theta_4^a = \cos^{-1}(+\sqrt{u_q})$ and $\theta_4^b = \cos^{-1}(-\sqrt{u_q})$.

In the above discussions, a/b in superscripts are correspond to the motion above/below the equatorial plane and the subscripts n, f imply whether the turning point is near or far

respectively from the equatorial plane. The numbers 1, 2, 3, and 4 in the subscripts stand for Case-1, Case-2, Case-3, and Case-4, respectively.

The nature of the function $\Theta(\theta)$ for Case-1, Case-2 and Case-3 are shown in the Fig. (C.1a) and for Case-4, it is shown in Fig. (C.1b) for some arbitrarily chosen values of M_c, D_λ and D_q . From the above four cases, we can divide the polar motion of the light ray into two types:

Type-A: When the light ray oscillates about the equatorial plane between $\theta^a \in (0, \pi/2)$ and $\theta^b \in (\pi/2, \pi)$ with $\theta^b = \pi - \theta^a$, the motion is known as a ‘Type-A’ motion (Case-1 and Case-4).

Type-B: On the other hand, when the light ray can have the motion in the θ direction only between the turning points $0 < \theta_f^a < \theta_n^a < \pi/2$ or $\pi/2 < \theta_n^b < \theta_f^b < \pi$, the motion is known as a ‘Type-B’ motion (Case-2). The Case-3 is the singular limit of Type-B motion where light ray moves towards the equatorial plane asymptotically.

For a Type-A motion, the path integral of any given function $f(\theta)$ on θ can be written as:

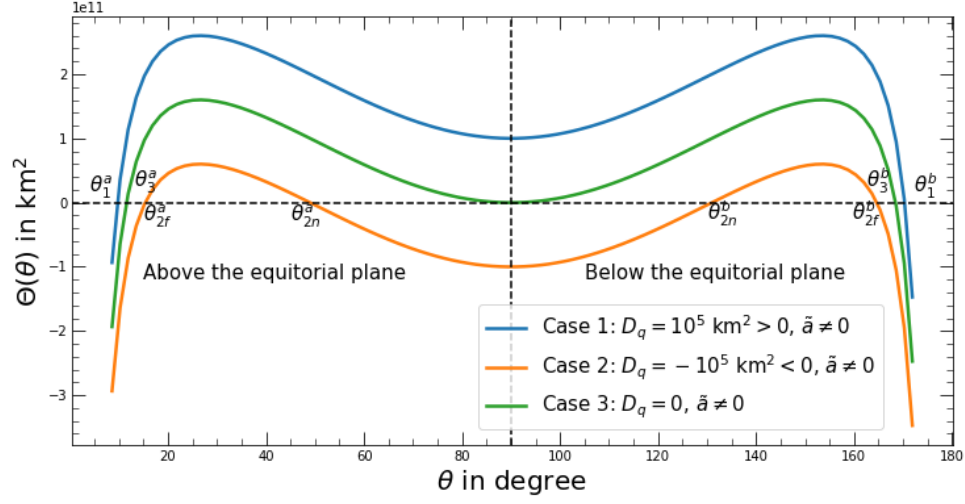
$$\int_{\theta_s}^{\theta_o} f(\theta) d\theta = 2\mu \left| \int_{\pi/2}^{\theta^{a/b}} f(\theta) d\theta \right| + \eta_s \left| \int_{\pi/2}^{\theta_s} f(\theta) d\theta \right| - \eta_o \left| \int_{\pi/2}^{\theta_o} f(\theta) d\theta \right|, \quad (\text{C.26})$$

whereas for a Type-B motion, the path integral on θ is given by:

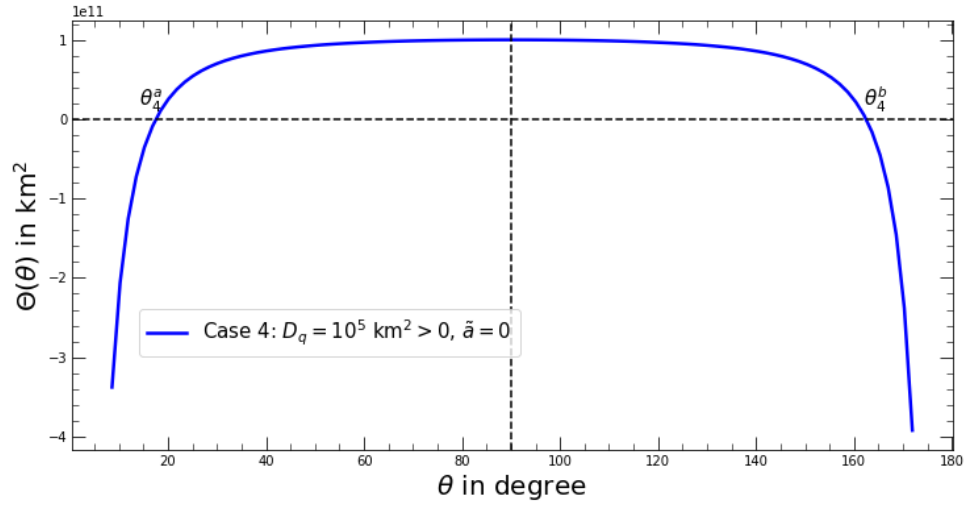
$$\int_{\theta_s}^{\theta_o} f(\theta) d\theta = \left(\mu \pm \eta_o \frac{\{1 - (-1)^\mu\}}{2} \right) \left| \int_{\theta_f}^{\theta_n} f(\theta) d\theta \right| \pm \eta_s \left| \int_{\theta_{n/f}}^{\theta_s} f(\theta) d\theta \right| \mp \eta_o \left| \int_{\theta_{n/f}}^{\theta_o} f(\theta) d\theta \right|, \quad (\text{C.27})$$

In Eqs. (C.26) and (C.27), θ_s is the value of θ of the light ray at the source and θ_o is the value of θ of the light ray at the observer. μ is the number of turning points during the motion. η_s and η_o are defined as follows:

$$\eta_o = \text{sign}(p_o^\theta) \text{sign}(\cos \theta_o) = (-1)^\mu \text{sign}(p_s^\theta) \text{sign}(\cos \theta_o), \quad (\text{C.28a})$$



(a)



(b)

Figure C.1: The function $\Theta(\theta)$ for four different cases depending on value of D_q as discussed in the text. Here we choose $GM_c/c^2 = 1000$ km (corresponding to $M_c = 677.09M_\odot$) and $D_\lambda = 100$ km. Panel-a shows Case-1, Case-2, and Case-3, each for $\tilde{a} = 0.5$, but for different values of D_q . Panel-b is for Case-4 where $\tilde{a} = 0$ and $D_q > 0$.

$$\eta_s = \text{sign}(p_s^\theta) \text{sign}(\cos \theta_s) = (-1)^\mu \text{sign}(p_o^\theta) \text{sign}(\cos \theta_s) . \quad (\text{C.28b})$$

where p_s^θ and p_o^θ denote the polar momentum at the starting point (source) and at the end point (observer) of the motion, respectively, giving

$$\text{sign}(p_s^\theta) = (-1)^\mu \text{sign}(p_o^\theta) . \quad (\text{C.29})$$

Note that,

$$\eta_o = \frac{1}{\eta_o} , \quad (\text{C.30a})$$

$$\eta_s = \frac{1}{\eta_s} . \quad (\text{C.30b})$$

We see that the Eqs. (C.26) and (C.27) do not depend on the choice of the roots either above or below the equatorial plane.

In Eq. (C.27), we see signs like $\pm$ and $\mp$. We need to choose the signs consistently, either the ones on the top or the ones on the bottom. The first choice would represent the root near the equatorial plane (meaning n from the subscript $n|f$) and the second choice would represent the root far from the equatorial plane (meaning f from the subscript $n|f$). Both of the choices are equivalent.

Note that, when we solve the path integrals about θ as we are discussing in this subsection, our aim is to find the value of the angle θ at the observer ('o') or θ_o . Hence, Eq. (C.17) becomes very useful to eliminate θ from one side of the expressions. Moreover, as in many of the equations in the next subsections, the left hand sides contain the absolute value of a variable (written within $||$ symbol), we need to take only the positive square root of the quantities appearing on the right hand sides. We use this convention throughout.

θ integrals given in G_θ, G_ϕ, G_t as functions of $p_s^\theta, \cos(\theta_s)$ and I_r for Case-1

When $\tilde{a} \neq 0$ and $D_q > 0$, the motion is Type-A, and using Eq. (C.26), we can write the θ integrals given in Eqs. (C.20d), (C.20e), and (C.20f) as:

$$\oint_{\theta_s}^{\theta_o} \frac{d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} = 2\mu \left| \int_{\pi/2}^{\theta_1^{a|b}} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| + \eta_s \left| \int_{\pi/2}^{\theta_s} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| - \eta_o \left| \int_{\pi/2}^{\theta_o} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.31a})$$

$$\oint_{\theta_s}^{\theta_o} \frac{\csc^2 \theta d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} = 2\mu \left| \int_{\pi/2}^{\theta_1^{a|b}} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| + \eta_s \left| \int_{\pi/2}^{\theta_s} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| - \eta_o \left| \int_{\pi/2}^{\theta_o} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.31b})$$

$$\oint_{\theta_s}^{\theta_o} \frac{\cos^2 \theta d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} = 2\mu \left| \int_{\pi/2}^{\theta_1^{a|b}} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| + \eta_s \left| \int_{\pi/2}^{\theta_s} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| - \eta_o \left| \int_{\pi/2}^{\theta_o} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right|. \quad (\text{C.31c})$$

The meanings of the superscripts and subscripts in the integration limits have been already explained where we explained different cases, as well as after we introduced Eqs. (C.26) and (C.27). Using the Eq. (C.24), we write:

$$\left| \int_{\pi/2}^{\theta_j} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| = \frac{1}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)} \int_0^{u_j} \frac{du}{\sqrt{u(u_+ - u)(u - u_-)}} = \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{-u_-}} \mathcal{F} \left(\Psi_j^+, \frac{u_+}{u_-} \right), \quad (\text{C.32a})$$

$$\left| \int_{\pi/2}^{\theta_j} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| = \frac{1}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)} \int_0^{u_j} \frac{du}{(1-u) \sqrt{u(u_+ - u)(u - u_-)}} = \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{-u_-}} \Pi \left(\Psi_j^+, \frac{u_+}{u_-} \right), \quad (\text{C.32b})$$

$$\left| \int_{\pi/2}^{\theta_j} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| = \frac{1}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)} \int_0^{u_j} \frac{u du}{\sqrt{u(u_+ - u)(u - u_-)}} = \frac{2u_+}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{-u_-}} \mathcal{E}' \left(\Psi_j^+, \frac{u_+}{u_-} \right), \quad (\text{C.32c})$$

where, $\mathcal{F}$, $\mathcal{E}$, and Π are the incomplete elliptical integrals of the first, the second, and the third kind, respectively, and $\mathcal{E}'(x, k) = \partial_k \mathcal{E}(x, k) = [\mathcal{E}(x, k) - \mathcal{F}(x, k)] / (2k)$ where ∂_k means differentiation with respect to the variable k . $\mathcal{E}'(x, k)$ is always negative. Ψ_j is as defined in Eq. (C.25d), with the subscript j was introduced to denote either the observer o or the source s . Using Eqs. (C.31a), (C.32a), (C.31b), (C.32b), and (C.31c), (C.32c) in Eqs. (C.20d), (C.20e), and (C.20f), respectively, we get

$$G_\theta = \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2}\right) \sqrt{-u_-}} \left[2\mu \mathcal{F}\left(\pi/2, \frac{u_+}{u_-}\right) + \eta_s \mathcal{F}\left(\Psi_s, \frac{u_+}{u_-}\right) - \eta_o \mathcal{F}\left(\Psi_o, \frac{u_+}{u_-}\right) \right], \quad (\text{C.33a})$$

$$G_\phi = \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2}\right) \sqrt{-u_-}} \left[2\mu \Pi\left(u_+, \pi/2, \frac{u_+}{u_-}\right) + \eta_s \Pi\left(u_+, \Psi_s, \frac{u_+}{u_-}\right) - \eta_o \Pi\left(u_+, \Psi_o, \frac{u_+}{u_-}\right) \right], \quad (\text{C.33b})$$

$$G_t = \frac{2u_+}{\left(\tilde{a} \frac{GM_c}{c^2}\right) \sqrt{-u_-}} \left[2\mu \mathcal{E}'\left(\pi/2, \frac{u_+}{u_-}\right) + \eta_s \mathcal{E}'\left(\Psi_s, \frac{u_+}{u_-}\right) - \eta_o \mathcal{E}'\left(\Psi_o, \frac{u_+}{u_-}\right) \right]. \quad (\text{C.33c})$$

From Eqs. (C.17), (C.28a), (C.28b) and (C.33a) we can write

$$\begin{aligned} (-1)^\mu \text{sign}(p_s^\theta) \text{sign}(\cos \theta_o) \mathcal{F}\left(\Psi_o, \frac{u_+}{u_-}\right) &= -I_r \left(\tilde{a} \frac{GM_c}{c^2}\right) \sqrt{-u_-} + 2\mu \mathcal{F}\left(\pi/2, \frac{u_+}{u_-}\right) \\ &\quad + \text{sign}(p_s^\theta) \text{sign}(\cos \theta_s) \mathcal{F}\left(\Psi_s, \frac{u_+}{u_-}\right). \end{aligned} \quad (\text{C.34})$$

Multiplying Eq. (C.34) by $(-1)^\mu \text{sign}(p_s^\theta)$, we get

$$\begin{aligned} \text{sign}(\cos \theta_o) \mathcal{F} \left(\Psi_o, \frac{u_+}{u_-} \right) &= (-1)^\mu \text{sign}(p_s^\theta) \left[-I_r \left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{-u_-} + 2\mu \mathcal{F} \left(\pi/2, \frac{u_+}{u_-} \right) \right] \\ &\quad + (-1)^\mu \text{sign}(\cos \theta_s) \mathcal{F} \left(\Psi_s, \frac{u_+}{u_-} \right). \end{aligned} \quad (\text{C.35})$$

From Eq. (C.25d), we write

$$\text{sign}(\cos \theta_j) \mathcal{F} \left(\Psi_j, \frac{u_+}{u_-} \right) = \mathcal{F} \left(\sin^{-1} \left(\frac{\cos \theta_j}{\sqrt{u_+}} \right), \frac{u_+}{u_-} \right). \quad (\text{C.36})$$

Inserting Eq. (C.36) in Eq. (C.35) we get,

$$\begin{aligned} \mathcal{F} \left(\sin^{-1} \left(\frac{\cos \theta_o}{\sqrt{u_+}} \right), \frac{u_+}{u_-} \right) &= (-1)^\mu \text{sign}(p_s^\theta) \left[-I_r \left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{-u_-} + 2\mu \mathcal{F} \left(\pi/2, \frac{u_+}{u_-} \right) \right] \\ &\quad + (-1)^\mu \mathcal{F} \left(\sin^{-1} \left(\frac{\cos \theta_s}{\sqrt{u_+}} \right), \frac{u_+}{u_-} \right) \\ &= (-1)^\mu X_\mu. \end{aligned} \quad (\text{C.37})$$

In the last step, we defined a new function

$$\begin{aligned} X_\mu &= \text{sign}(p_s^\theta) \left[-I_r \left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{-u_-} + 2\mu \mathcal{F} \left(\pi/2, \frac{u_+}{u_-} \right) \right] \\ &\quad + \mathcal{F} \left(\sin^{-1} \left(\frac{\cos \theta_s}{\sqrt{u_+}} \right), \frac{u_+}{u_-} \right). \end{aligned} \quad (\text{C.38})$$

Putting $\mu - 1$ in the place of μ on the left hand side of Eq. (C.38), and after doing some algebra and using Eq. (C.38) again, one can show that,

$$X_\mu = X_{\mu-1} + 2 \text{sign}(p_s^\theta) \mathcal{F} \left(\pi/2, \frac{u_+}{u_-} \right). \quad (\text{C.39})$$

The Jacobi elliptic function ‘sn’ is defined as

$$\text{sn}(\mathcal{F}(\sin^{-1}(x), k), k) = x, \quad (\text{C.40})$$

where x, k are any variables. From Eq. (C.37)) we get,

$$\frac{\cos \theta_o}{\sqrt{u_+}} = \text{sn} \left((-1)^\mu X_\mu, \frac{u_+}{u_-} \right) . \quad (\text{C.41})$$

We will use two known identities:

$$\text{sn}(x \pm 2\mathcal{F}(\pi/2, k)k) = -\text{sn}(x, k) , \quad (\text{C.42a})$$

$$\text{sn}(-x, k) = -\text{sn}(x, k) , \quad (\text{C.42b})$$

Using the above identities in Eq. (C.39), through an iterative process, we get

$$\text{sn} \left((-1)^\mu X_\mu, \frac{u_+}{u_-} \right) = \text{sn} \left((-1)^{\mu-1} X_{\mu-1}, \frac{u_+}{u_-} \right) = \dots = \text{sn} \left(X_0, \frac{u_+}{u_-} \right) . \quad (\text{C.43})$$

Using Eq. (C.43) into Eq. (C.41) we get,

$$\cos \theta_o = \sqrt{u_+} \text{sn} \left(X_0, \frac{u_+}{u_-} \right) , \quad (\text{C.44})$$

where Eq. (C.38) gives

$$X_0 = \text{sign}(p_s^\theta) \left[-I_r \left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{-u_-} \right] + \mathcal{F} \left(\sin^{-1} \left(\frac{\cos \theta_s}{\sqrt{u_+}} \right), \frac{u_+}{u_-} \right) . \quad (\text{C.45})$$

Note that, as u_- is negative, $\sqrt{-u_-}$ appears in various equations instead of $\sqrt{u_-}$.

θ integrals given in G_θ, G_ϕ, G_t as functions of $p_s^\theta, \cos(\theta_s)$ and I_r for Case-2

When $\tilde{a} \neq 0$ and $D_q < 0$, the motion is Type-B, and using Eq. (C.27), we can write the θ integrals given in Eqs. (C.20d), (C.20e), and (C.20f) as:

$$\begin{aligned} \int_{\theta_s}^{\theta_o} \frac{d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} &= \left[\mu \pm \eta_o \frac{\{1 - (-1)^\mu\}}{2} \right] \left| \int_{\theta_{2,f}}^{\theta_{2,n}} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| \\ &\quad \pm \eta_s \left| \int_{\theta_{2,n|f}}^{\theta_s} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| \mp \eta_o \left| \int_{\theta_{2,n|f}}^{\theta_o} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.46a}) \end{aligned}$$

$$\begin{aligned} \int_{\theta_s}^{\theta_o} \frac{\csc^2 \theta d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} &= \left[\mu \pm \eta_o \frac{\{1 - (-1)^\mu\}}{2} \right] \left| \int_{\theta_{2,f}}^{\theta_{2,n}} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| \\ &\quad \pm \eta_s \left| \int_{\theta_{2,n|f}}^{\theta_s} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| \mp \eta_o \left| \int_{\theta_{2,n|f}}^{\theta_o} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.46b}) \end{aligned}$$

$$\begin{aligned} \int_{\theta_s}^{\theta_o} \frac{\cos^2 \theta d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} &= \left[\mu \pm \eta_o \frac{\{1 - (-1)^\mu\}}{2} \right] \left| \int_{\theta_{2,f}}^{\theta_{2,n}} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| \\ &\quad \pm \eta_s \left| \int_{\theta_{2,n|f}}^{\theta_s} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| \mp \eta_o \left| \int_{\theta_{2,n|f}}^{\theta_o} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right|. \quad (\text{C.46c}) \end{aligned}$$

The meanings of the superscripts and subscripts in the integration limits, as well as the choice of either the top or the bottom signs from $\pm$ and $\mp$ have been already explained where we explained different cases, as well as after we introduced Eqs. (C.26) and (C.27). In summary, the symbol $\theta_{2, n|f}$ represents either $\theta_{2,n}$ or $\theta_{2,f}$ depending on whether the top or the bottom sign is chosen, respectively. Next, using Eq. (C.24), we write

$$\left| \int_{\theta_{2n|f}}^{\theta_j} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| = \frac{1}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)} \int_{u_\mp}^{u_j} \frac{du}{\sqrt{u(u_+ - u)(u - u_-)}} = \pm \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{u_\mp}} \mathcal{F} \left(\Gamma_j^\pm, 1 - \frac{u_\pm}{u_\mp} \right), \quad (\text{C.47a})$$

$$\begin{aligned}
\left| \int_{\theta_{2n|f}}^{\theta_j} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| &= \frac{1}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)} \int_{u_{\mp}}^{u_j} \frac{du}{(1-u) \sqrt{u(u_+ - u)(u - u_-)}} \\
&= \pm \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{u_{\mp}} (1 - u_{\mp})} \Pi \left(\Gamma_j^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right), \quad (\text{C.47b})
\end{aligned}$$

$$\left| \int_{\theta_{2n|f}}^{\theta_j} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| = \frac{1}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)} \int_{u_{\pm}}^{u_j} \frac{u du}{\sqrt{u(u_+ - u)(u - u_-)}} = \pm \frac{\sqrt{u_{\mp}}}{\left(\tilde{a} \frac{GM_c}{c^2} \right)} \mathcal{E} \left(\Gamma_j^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right), \quad (\text{C.47c})$$

where we defined

$$\Gamma_j^{\pm} = \pm \sin^{-1} \sqrt{\pm \left(\frac{\cos^2 \theta_j - u_{\mp}}{u_+ - u_-} \right)}, \quad (\text{C.48})$$

with the subscript j meaning either the source ‘ s ’ or the observer ‘ o ’. Using Eqs. (C.46a), (C.47a), (C.46b), (C.47b), and (C.46c), (C.47c) in Eqs. (C.20d), (C.20e) and (C.20f) respectively, we get

$$\begin{aligned}
G_{\theta} &= \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{u_{\mp}}} \left[\left(\mu \pm \eta_o \frac{1 - (-1)^{\mu}}{2} \right) \mathcal{F} \left(\pi/2, 1 - \frac{u_{\pm}}{u_{\mp}} \right) \right. \\
&\quad \left. + \eta_s \mathcal{F} \left(\Gamma_s^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right) - \eta_o \mathcal{F} \left(\Gamma_o^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right) \right], \quad (\text{C.49a})
\end{aligned}$$

$$\begin{aligned}
G_{\phi} &= \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{u_{\mp}}} \left[\left(\mu \pm \eta_o \frac{1 - (-1)^{\mu}}{2} \right) \Pi \left(\frac{u_{\pm} - u_{\mp}}{1 - u_{\mp}}, \pi/2, 1 - \frac{u_{\pm}}{u_{\mp}} \right) \right. \\
&\quad \left. + \eta_s \Pi \left(\frac{u_{\pm} - u_{\mp}}{1 - u_{\mp}}, \Gamma_s^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right) - \eta_o \Pi \left(\frac{u_{\pm} - u_{\mp}}{1 - u_{\mp}}, \Gamma_o^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right) \right], \quad (\text{C.49b})
\end{aligned}$$

$$G_t = \frac{\sqrt{u_{\pm}}}{\left(\tilde{a}\frac{GM_c}{c^2}\right)} \left[\left(\mu \pm \eta_o \frac{1 - (-1)^\mu}{2} \right) \mathcal{E} \left(\pi/2, 1 - \frac{u_{\pm}}{u_{\mp}} \right) + \eta_s \mathcal{E} \left(\Gamma_s^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right) - \eta_o \mathcal{E} \left(\Gamma_o^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right) \right]. \quad (\text{C.49c})$$

From Eq. (C.17) and (C.49a) we can write

$$\sqrt{u_{\mp}} \left(\tilde{a}\frac{GM_c}{c^2} \right) I_r = \left(\mu \pm \eta_o \frac{\{1 - (-1)^\mu\}}{2} \right) \mathcal{F} \left(\pi/2, 1 - \frac{u_{\pm}}{u_{\mp}} \right) + \eta_s \mathcal{F} \left(\Gamma_s^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right) - \eta_o \mathcal{F} \left(\Gamma_o^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right). \quad (\text{C.50})$$

Using Eqs. (C.28a), (C.28b), (C.30a), and (C.30b), Eq. (C.50) can be rewritten as

$$\mathcal{F} \left(\Gamma_o^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right) = (-1)^\mu \text{sign}(\cos \theta_o) Y_\mu^{\pm}, \quad (\text{C.51})$$

where,

$$Y_\mu^{\pm} = \text{sign}(\cos \theta_s) \mathcal{F} \left(\Gamma_s^{\pm}, 1 - \frac{u_{\pm}}{u_{\mp}} \right) + \text{sign}(p_s^\theta) \left[\left(\mu \pm \eta_o \frac{1 - (-1)^\mu}{2} \right) \mathcal{F} \left(\pi/2, 1 - \frac{u_{\pm}}{u_{\mp}} \right) - \sqrt{u_{\mp}} \left(\tilde{a}\frac{GM_c}{c^2} \right) I_r \right]. \quad (\text{C.52})$$

After doing some algebra and using $\mu - 1$ in the place of μ in Eq. (C.52), we can show that,

$$Y_\mu^{\pm} = Y_{\mu-1}^{\pm} + [1 \pm \eta_o (-1)^{\mu-1}] \text{sign}(p_s^\theta) \mathcal{F} \left(\pi/2, 1 - \frac{u_{\pm}}{u_{\mp}} \right). \quad (\text{C.53})$$

The Jacobi elliptic function ‘dn’ is defined as

$$\text{dn}(\mathcal{F}(\sin^{-1}(x), k)) = \left| \sqrt{1 - kx^2} \right|, \quad (\text{C.54})$$

which satisfy the following identities:

$$\operatorname{dn}(-x, k) = \operatorname{dn}(x, k), \quad (\text{C.55})$$

$$\operatorname{dn}(x \pm 2\mathcal{F}(\pi/2, k)) = \operatorname{dn}(x, k). \quad (\text{C.56})$$

Form Eq. (C.54), we can write,

$$|y| = \operatorname{dn} \left(\mathcal{F} \left(\sin^{-1} \sqrt{\frac{1-y^2}{k}}, k \right), k \right), \quad (\text{C.57})$$

where y is any variable. If we set $y = \sin \Psi_o^\mp$ and $k = 1 - u_\pm/u_\mp$, we can write

$$|\sin \Psi_o^\mp| = \operatorname{dn} \left(\mathcal{F} \left(\sin^{-1} \sqrt{\frac{1 - \sin^2 \Psi_o^\mp}{1 - \frac{u_\pm}{u_\mp}}}, 1 - \frac{u_\pm}{u_\mp} \right), 1 - \frac{u_\pm}{u_\mp} \right). \quad (\text{C.58})$$

Next, we derive a relation between $\Psi_j^\mp$ and $\Gamma_j^\pm$ using Eqs. (C.25d) and (C.48) as:

$$1 - \sin^2 \Psi_j^\mp = \left(1 - \frac{u_\pm}{u_\mp}\right) \sin^2 \Gamma_j^\pm, \quad (\text{C.59a})$$

$$\pm \Gamma_j^\pm = \sin^{-1} \sqrt{\frac{1 - \sin^2 \Psi_j^\mp}{1 - \frac{u_\pm}{u_\mp}}}. \quad (\text{C.59b})$$

The details of the derivation of Eq. (C.59a) is given in Sec. C.4.3. Using Eq. (C.59b) in Eq. (C.58), and using $j = o$ on the right hand side, we get

$$|\sin \Psi_o^\mp| = \operatorname{dn} \left(\mathcal{F} \left(\pm \Gamma_o^\pm, 1 - \frac{u_\pm}{u_\mp} \right), 1 - \frac{u_\pm}{u_\mp} \right). \quad (\text{C.60})$$

Using Eq. (C.51) in Eq. (C.60), we get

$$|\sin \Psi_o^\mp| = \operatorname{dn} \left(\pm (-1)^\mu \operatorname{sign}(\cos \theta_o) Y_\mu^\pm, 1 - \frac{u_\pm}{u_\mp} \right). \quad (\text{C.61})$$

The identity Eq. (C.55) helps rewrite Eq. (C.61) as:

$$|\sin \Psi_o^\mp| = \text{dn} \left(Y_\mu^\pm, 1 - \frac{u_\pm}{u_\mp} \right). \quad (\text{C.62})$$

Next, from Eq. (C.25d), we write

$$|\sin \Psi_o^\pm| = \text{sign}(\cos \theta_o) \frac{\cos \theta_o}{\sqrt{u_\pm}}. \quad (\text{C.63})$$

Using Eq. (C.63) in Eq. (C.62), we write:

$$\frac{\cos \theta_o}{\sqrt{u_\mp}} = \text{sign}(\cos \theta_o) \text{dn} \left(Y_\mu^\pm, 1 - \frac{u_\pm}{u_\mp} \right) = \text{sign}(\cos \theta_s) \text{dn} \left(Y_\mu^\pm, 1 - \frac{u_\pm}{u_\mp} \right), \quad (\text{C.64})$$

where at the last step, we used the fact $\text{sign}(\cos \theta_o) = \text{sign}(\cos \theta_s)$ for Type-B motion.

As the term $(1 \pm \eta_o(-1)^{\mu-1})$ can take any one of the three values 0, ± 2 , using the identity Eq. (C.56) in Eq. (C.53), through an iterative process, we get

$$\text{dn} \left(Y_\mu^\pm, 1 - \frac{u_\pm}{u_\mp} \right) = \text{dn} \left(Y_{\mu-1}, 1 - \frac{u_\pm}{u_\mp} \right) = \dots = \text{dn} \left(Y_0, 1 - \frac{u_\pm}{u_\mp} \right). \quad (\text{C.65})$$

Using Eq. (C.65) in Eq. (C.64), we get

$$\cos \theta_o = \text{sign}(\cos \theta_s) \sqrt{u_\mp} \text{dn} \left(Y_0^\pm, 1 - \frac{u_\pm}{u_\mp} \right). \quad (\text{C.66})$$

The expression for $Y_0^\pm$ can be obtained from Eq. (C.52) as:

$$Y_0^\pm = \text{sign}(\cos \theta_s) \mathcal{F} \left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp} \right) - \text{sign}(p_s^\theta) \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r. \quad (\text{C.67})$$

θ integrals (G_θ, G_ϕ, G_t) as a function of $P_s(\theta)$, $\cos(\theta_s)$ and I_r for Case-3

When $\tilde{a} \neq 0$ and $D_q = 0$, the motion is appear to be of Type-B with turning points at equatorial plane and $\theta_3^{a|b} = \cos^{-1}(\pm \sqrt{u_\mp})$, where the upper/lower sign indicates the motion is

above/below the equatorial plane. However, as discussed earlier, in this case $u_- = 0$ corresponds to a non-integrable singularity, which means that the geodesic motion is barred from reaching the equatorial plane, it can only approach the equatorial plane asymptotically. Hence, the number turning points, μ can be either 1 or 0. In this special situation, we can write path integral over θ of a function $f(\theta)$ as:

$$\oint_{\theta_s}^{\theta_o} f(\theta) d\theta = \eta_o \left| \int_{\theta_3^{a|b}}^{\theta_o} f(\theta) d\theta \right| - \eta_s \left| \int_{\theta_3^{a|b}}^{\theta_s} f(\theta) d\theta \right|, \quad (\text{C.68})$$

which is a special case of Eq. (C.27). In Eq. (C.68), $\theta^{a|b}$ means that we can choose either θ^a or θ^b . Note that, we have chosen the lower signs from $\pm$ and $\mp$ in Eq. (C.27), as the turning point (if exist) is far from the equatorial plane as the motion asymptotically reaches that plane.

We can write the θ integrals from Eqs. (C.20d), (C.20e) and (C.20f),

$$\oint_{\theta_s}^{\theta_o} \frac{d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} = \eta_o \left| \int_{\theta_3^{a|b}}^{\theta_o} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| - \eta_s \left| \int_{\theta_3^{a|b}}^{\theta_s} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.69a})$$

$$\oint_{\theta_s}^{\theta_o} \frac{\csc^2 \theta d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} = \eta_o \left| \int_{\theta_3^{a|b}}^{\theta_o} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| - \eta_s \left| \int_{\theta_3^{a|b}}^{\theta_s} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.69b})$$

$$\oint_{\theta_s}^{\theta_o} \frac{\cos^2 \theta d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} = \eta_o \left| \int_{\theta_3^{a|b}}^{\theta_o} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| - \eta_s \left| \int_{\theta_3^{a|b}}^{\theta_s} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.69c})$$

Using Eq. (C.24), we write

$$\left| \int_{\theta_3^{a|b}}^{\theta_j} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| = \frac{1}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)} \int_{u_j}^{u_+} \frac{du}{u \sqrt{(u_+ - u)}} = \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{u_+}} \tanh^{-1} \left(\sqrt{1 - \frac{u_j}{u_+}} \right), \quad (\text{C.70a})$$

$$\begin{aligned}
\left| \int_{\theta_3^{a|b}}^{\theta_j} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| &= \frac{1}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)} \int_{u_j}^{u_+} \frac{du}{(1-u)u\sqrt{(u_+-u)}} \\
&= \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{u_+}} \left[\tanh^{-1} \left(\sqrt{1 - \frac{u_j}{u_+}} \right) + \sqrt{\frac{u_+}{1-u_+}} \tanh^{-1} \left(\sqrt{\frac{u_+-u_j}{1-u_+}} \right) \right], \quad (C.70b)
\end{aligned}$$

$$\left| \int_{\theta_3^{a|b}}^{\theta_j} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| = \frac{1}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)} \int_{u_j}^{u_+} \frac{du}{\sqrt{(u_+-u)}} = \sqrt{\frac{u_+-u_j}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2}}. \quad (C.70c)$$

Using Eqs. (C.69a), (C.70a), (C.69b), (C.70b), and (C.69c), (C.70c) in Eqs. (C.20d), (C.20e) and (C.20f) respectively, we get

$$G_\theta = \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{u_+}} \left[\eta_o \tanh^{-1} \left(\sqrt{1 - \frac{u_o}{u_+}} \right) - \eta_s \tanh^{-1} \left(\sqrt{1 - \frac{u_s}{u_+}} \right) \right], \quad (C.71a)$$

$$G_\phi = G_\theta + \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{1-u_+}} \left[\eta_o \tanh^{-1} \left(\sqrt{\frac{u_+-u_o}{1-u_+}} \right) - \eta_s \tanh^{-1} \left(\sqrt{\frac{u_+-u_s}{1-u_+}} \right) \right], \quad (C.71b)$$

$$G_t = \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right)} \left[\eta_o \sqrt{u_+-u_o} - \eta_s \sqrt{u_+-u_s} \right]. \quad (C.71c)$$

From Eq. (C.17) and (C.71a) we can write,

$$I_r = \frac{1}{\left(\tilde{a} \frac{GM_c}{c^2} \right) \sqrt{u_+}} \left[\eta_o \tanh^{-1} \left(\sqrt{1 - \frac{u_o}{u_+}} \right) - \eta_s \tanh^{-1} \left(\sqrt{1 - \frac{u_s}{u_+}} \right) \right], \quad (C.72)$$

Using Eq.(C.28a) and (C.28b) we can write,

$$\tanh^{-1} \left(\sqrt{1 - \frac{u_o}{u_+}} \right) = (-1)^\mu \text{sign}(\cos \theta_o) Z, \quad (\text{C.73})$$

where Z is defined as:

$$Z = \text{sign}(\cos \theta_s) \tanh^{-1} \sqrt{1 - \frac{u_s}{u_+}} + \text{sign}(p_s^\theta) \sqrt{u_+} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r. \quad (\text{C.74})$$

As $\tanh(-x) = -\tanh(x)$ and $(-1)^\mu \text{sign}(\cos \theta_o)$ can either be $+1$ or -1 , from Eq. (C.73), we write,

$$\sqrt{1 - \frac{u_o}{u_+}} = (-1)^\mu \text{sign}(\cos \theta_o) \tanh(Z), \quad (\text{C.75})$$

which gives,

$$u_o = u_+ \text{sech}^2(Z). \quad (\text{C.76})$$

Using the fact that $\text{sign}(\cos \theta_o) = \text{sign}(\cos \theta_s)$ for a type-B motion, and remembering the definition of u as given in Eq. (C.23), Eq. (C.76) gives the expression for θ_o as:

$$\cos \theta_o = \text{sign}(\cos \theta_s) |\sqrt{u_+} \text{sech } Z|. \quad (\text{C.77})$$

θ integrals, i.e., G_θ, G_ϕ, G_t as functions of $P_s(\theta)$, $\cos(\theta_s)$, and I_r for Case-4

When $\tilde{a} = 0$ requiring $D_q > 0$, the motion is appear to be of Type-A with turning points at $\theta_4^{a|b} = \cos^{-1}(\pm \sqrt{u_q})$. Using Eq. (C.26), the θ integrals from Eqs. (C.20d), (C.20e), and (C.20f) can be written as:

$$\oint_{\theta_s}^{\theta_o} \frac{d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} = 2\mu \left| \int_{\pi/2}^{\theta_4^{a|b}} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| + \eta_s \left| \int_{\pi/2}^{\theta_s} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| - \eta_o \left| \int_{\pi/2}^{\theta_o} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.78a})$$

$$\int_{\theta_s}^{\theta_o} \frac{\csc^2 \theta d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} = 2\mu \left| \int_{\pi/2}^{\theta_4^{a|b}} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| + \eta_s \left| \int_{\pi/2}^{\theta_s} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| - \eta_o \left| \int_{\pi/2}^{\theta_o} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.78b})$$

$$\int_{\theta_s}^{\theta_o} \frac{\cos^2 \theta d\theta}{\text{sign}(p_\theta) \sqrt{\Theta(\theta)}} = 2\mu \left| \int_{\pi/2}^{\theta_4^{a|b}} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| + \eta_s \left| \int_{\pi/2}^{\theta_s} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| - \eta_o \left| \int_{\pi/2}^{\theta_o} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right|, \quad (\text{C.78c})$$

where, $a|b$ implies that we can choose the root either above or below the equatorial plane.

Using the Eq. (C.24), we write

$$\left| \int_{\theta_{\pi/2}}^{\theta_j} \frac{d\theta}{\sqrt{\Theta(\theta)}} \right| = \frac{1}{2} \sqrt{\frac{u_q}{D_q}} \int_0^{u_j} \frac{du}{\sqrt{u(u_q - u)}} = \sqrt{\frac{u_q}{D_q}} \sin^{-1} \left(\sqrt{\frac{u_j}{u_q}} \right), \quad (\text{C.79a})$$

$$\begin{aligned} \left| \int_{\theta_{\pi/2}}^{\theta_j} \frac{\csc^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| &= \frac{1}{2} \sqrt{\frac{u_q}{D_q}} \int_0^{u_j} \frac{du}{(1-u) \sqrt{u(u_q - u)}} \\ &= \frac{1}{\sqrt{D_q}} \sqrt{\frac{u_q}{1-u_q}} \sin^{-1} \left(\sqrt{\frac{u_j}{u_q} \frac{1-u_q}{1-u_j}} \right), \end{aligned} \quad (\text{C.79b})$$

$$\left| \int_{\pi/2}^{\theta_j} \frac{\cos^2 \theta d\theta}{\sqrt{\Theta(\theta)}} \right| = \frac{1}{2} \sqrt{\frac{u_q}{D_q}} \int_{u_0}^{u_j} \frac{u du}{\sqrt{u(u_q - u)}} = \frac{1}{2} \sqrt{\frac{u_q}{D_q}} \left[u_q \sin^{-1} \left(\sqrt{\frac{u_j}{u_q}} \right) - \sqrt{u_j(u_q - u_j)} \right]. \quad (\text{C.79c})$$

Using Eqs. (C.78a), (C.79a), (C.78b), (C.79b), and (C.78c), (C.79c) in Eqs. (C.20d), (C.20e), and (C.20f), respectively, we get

$$G_\theta = \sqrt{\frac{u_q}{D_q}} \left[\pi\mu + \eta_s \sin^{-1} \left(\sqrt{\frac{u_s}{u_q}} \right) - \eta_o \sin^{-1} \left(\sqrt{\frac{u_o}{u_q}} \right) \right], \quad (\text{C.80a})$$

$$G_\phi = \frac{1}{\sqrt{D_q}} \sqrt{\frac{u_q}{1-u_q}} \left[\pi\mu + \eta_s \sin^{-1} \left(\sqrt{\frac{u_s}{u_q} \frac{1-u_q}{1-u_s}} \right) - \eta_o \sin^{-1} \left(\sqrt{\frac{u_o}{u_q} \frac{1-u_q}{1-u_o}} \right) \right], \quad (\text{C.80b})$$

$$G_t = \frac{1}{2} \left[u_q G_\theta - \sqrt{\frac{u_q}{D_q}} \left\{ \eta_s \sqrt{u_s(u_q - u_s)} - \eta_o \sqrt{u_o(u_q - u_o)} \right\} \right]. \quad (\text{C.80c})$$

From Eq. (C.17) and (C.80a) we write,

$$\sqrt{\frac{D_q}{u_q}} I_r = \pi\mu + \eta_s \sin^{-1} \left(\sqrt{\frac{u_s}{u_q}} \right) - \eta_o \sin^{-1} \left(\sqrt{\frac{u_o}{u_q}} \right). \quad (\text{C.81})$$

Note, than in Eq. (C.79c), we have an absolute value on the right hand side and a $\sqrt{u_j/u_q}$ on the left hand side. Moreover, $\sin^{-1}(-x) = -\sin^{-1}(x)$. Hence, we need to choose the positive value of $\sqrt{u_j/u_q}$. Hence, we will use the following relation

$$\text{sign}(\cos \theta_j) \sin^{-1} \left(\sqrt{\frac{u_j}{u_q}} \right) = \sin^{-1} \left(\frac{\cos \theta_j}{\sqrt{u_q}} \right). \quad (\text{C.82})$$

Using Eq. (C.82) with $j = o$ in Eq. (C.81), we get

$$\sin^{-1} \left(\frac{\cos \theta_o}{\sqrt{u_q}} \right) = (-1)^\mu \left[\sin^{-1} \left(\frac{\cos \theta_s}{\sqrt{u_q}} \right) + \text{sign}(p_s^\theta) \left(\pi\mu - \sqrt{\frac{D_q}{u_q}} I_r \right) \right]. \quad (\text{C.83})$$

From Eq. (C.83), we can write

$$\cos \theta_o = \sqrt{u_q} \sin \left[(-1)^\mu W_\mu \right], \quad (\text{C.84})$$

where we defined a new variable:

$$W_\mu = \sin^{-1} \left(\frac{\cos \theta_s}{\sqrt{u_q}} \right) + \text{sign}(p_s^\theta) \left(\pi\mu - \sqrt{\frac{D_q}{u_q}} I_r \right). \quad (\text{C.85})$$

Using the fact that $\sin(-x) = -\sin(x)$, from Eq. (C.85), it can be easily shown that,

$$W_\mu = W_{\mu-1} + \text{sign}(p_s^\theta)\pi. \quad (\text{C.86})$$

Using the fact that $\sin(x \pm \pi) = -\sin x$, from Eq. (C.86), we get

$$\sin [(-1)^\mu W_\mu] = \sin [(-1)^{\mu-1} W_{\mu-1}]. \quad (\text{C.87})$$

Hence, using an iterative process, we can write

$$\sin [(-1)^\mu W_\mu] = \sin [(-1)^{\mu-1} W_{\mu-1}] = \dots = \sin W_0. \quad (\text{C.88})$$

Using Eq. (C.88) in Eq. (C.84), we get

$$\cos \theta_o = \sqrt{u_q} \sin W_0, \quad (\text{C.89})$$

Putting $\mu = 0$ in Eq. (C.85), we get

$$W_0 = \sin^{-1} \left(\frac{\cos \theta_s}{\sqrt{u_q}} \right) - \text{sign}(p_s^\theta) \sqrt{\frac{D_q}{u_q}} I_r. \quad (\text{C.90})$$

We will use Eq. (C.90) while evaluating Eq. (C.89).

C.2.2 r integrals

All the physical scenarios described in this section are valid only when the gravitating object is a blackhole, as the concept of an event horizon exists only for blackholes. The function that drives the motion in the r coordinate along a null geodesic is $\mathcal{R}(r)$ given in Eq. (C.22a). From Eqs. (C.13), (C.20a), (C.20b), and (C.20c), we see that $\mathcal{R}(r)$ must be zero or positive, otherwise $\mathcal{A}$, I_r , I_ϕ , and I_t would be imaginary. The radial coordinate r

is always positive. Moreover, as $\mathcal{R}(r)$ is a quartic polynomial (see Eq. (C.22a)), it can have four roots. If r_1, r_2, r_3 and r_4 are the roots of $\mathcal{R}(r)$ and r_H^+, r_H^- are the outer and the inner event horizons, respectively, as defined in Eqs. (C.5a) and (C.5b), then, depending on the values of D_q and D_λ , the roots can be classified into 4 cases that we discuss with the help of a demonstrative Fig. C.2.

All other possible cases are excluded by the condition satisfied by D_q and D_λ :

$$D_q \geq - \left(|D_\lambda| - \tilde{a} \frac{GM_c}{c^2} \right)^2 \text{ for any value of } |D_\lambda|. \quad (\text{C.91a})$$

$$\text{Additionally, } D_q \geq 0, \text{ if } |D_\lambda| \geq \left(\tilde{a} \frac{GM_c}{c^2} \right). \quad (\text{C.91b})$$

We discuss these conditions in Sec. C.5.1.

The plots of the radial potential with the radial position is shown in the Fig. C.2. As already mentioned, only the regions where both $\mathcal{R}(r) \geq 0$ and $r \geq 0$ are physical. Note that, the light ray can change its direction in r only if it reaches one of the roots of $\mathcal{R}(r)$, which are the turning points. Moreover, as the observers on the earth are significantly far away from the source and the companion, the situations when the light ray can reach to infinity, i.e., can have $r = \infty$, are the only relevant situations for us. From Fig. C.2, we can see the basic features of various cases as:

Case-1: Case-1 is defined as the case when there are four real roots, only two outside the outer horizon:

$r_1 < r_2 < r_H^- < r_H^+ < r_3 < r_4$. The radial function $\mathcal{R}(r)$ is positive in two regions with positive values of r ,

- The first region is $r_2 \leq r \leq r_3$. As no observer can be located inside r_H^+ , only the region $r_H^+ < r \leq r_3$ can lead to observable light rays. However, no light ray can reach to the observer at infinity from this region as $\mathcal{R}(r) = 0$ at $r = r_3$.
- The second region is $r_4 \leq r \leq \infty$. The light ray with any value of $r \geq r_4$ can

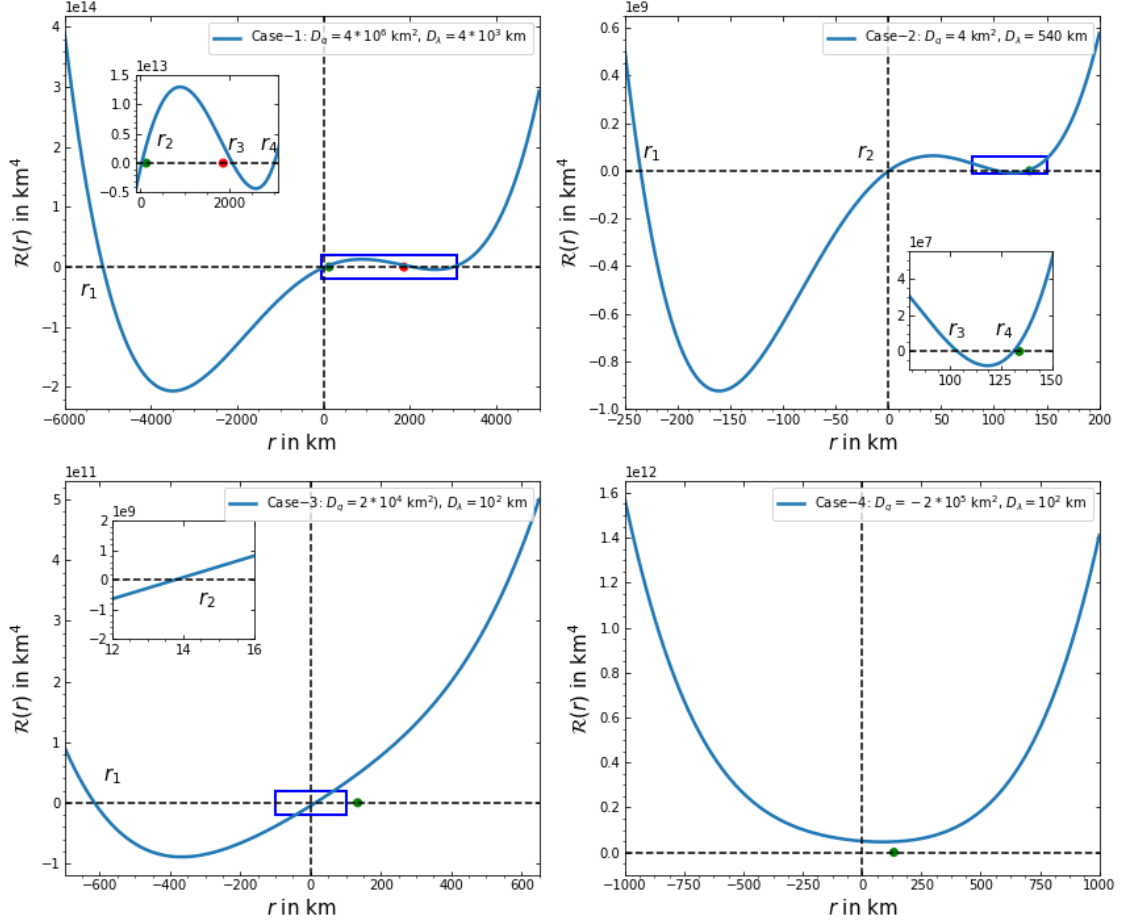


Figure C.2: Four possible cases that can arise for the function $\mathcal{R}(r)$ depending on the value of D_q and D_λ are shown in the figures. In each case, we plot $\mathcal{R}(r)$ along the abscissa and r along the ordinate. We choose $GM_c/c^2 = 1000$ km (corresponding to $M_c = 677.09 M_\odot$) and $\tilde{a} = 0.5$. The outer event horizon is located at the value of r_+ as 1866 km and is shown with a red dot. The inner event horizon is located at the value of r_- as 134 km and is shown with a green dot in each of the figures. The upper left figure is for Case-1, the upper right figure is for Case-2, the lower left figure is for Case-3 and the lower right figure is for Case-4. See text for classifications of the cases. In Case-2, Case-3, and Case-4, the outer event horizon is outside the range of r shown in the figures. The regions containing the roots have zoomed in for Case-1, Case-2, Case-3.

reach to the observer at infinity. The light bending effect due to the gravitating object is significant when light rays are initially directed toward the object. On the other hand, the light bending effect becomes negligible when the light rays are initially directed away from the gravitating object.

Case-2: Case-2 is defined as the case when there are four real roots, all inside the inner horizon:

$r_1 < r_2 < r_3 < r_4 < r_H^- < r_H^+$, The radial function $\mathcal{R}(r)$ is positive in two regions with positive values of r ,

- The first region is either $0 \leq r \leq r_3$ if r_2 is negative, or $r_2 \leq r \leq r_3$ if r_2 is positive. As no observer can be located inside r_H^- , this region does not have any physical relevance.
- The second region is $r_4 \leq r \leq \infty$. From the definition of Case-2, $r_H^+ > r_4$. Hence, when a light ray start from $r > r_H^+$ and is initially directed to the gravitating body, it can reach the observer only if the observer is located between r_H^+ and the starting point of the light ray (the source), which is a very unrealistic situation. Hence, usually, the light ray approaches the outer event horizon and eventually falls onto it.

Conversely, when a light ray starts from $r > r_H^+$ and is initially directed away from the gravitating object, it can reach the observer at infinity without crossing the gravitating object. In this scenario, the light bending effect due to the gravitating object is negligible.

Case-3: Case-3 is defined as the case when there are two real roots, both inside the inner horizon:

$r_1 < r_2 < r_H^- < r_H^+$. The complex roots are conjugate to each other, i.e., $r_3 = \tilde{r}_4$,

- The radial function $\mathcal{R}(r)$ is positive in the region $r_2 \leq r \leq \infty$. We consider only the cases when the light ray starts at $r > r_H^+$. In such a case, if the light

ray is initially directed toward the gravitating body, it can reach the observer only if the observer is located between r_H^+ and the starting point of the light ray (the source), which is a very unrealistic situation. Hence, usually, the light ray approaches the outer event horizon and eventually falls onto it. On the other hand, if the light ray is initially directed away from the black hole (gravitating body), it can reach the observer at infinity without crossing the gravitating object. In this scenario, the light bending effect due to the gravitating object is negligible.

Case-4: Case-4 is defined as the case when there is no real root. There are two pairs of complex conjugate roots, i.e., $r_1 = \tilde{r}_2$ and $r_3 = \tilde{r}_4$.

- The radial function $\mathcal{R}(r)$ is positive for every value of r . We can consider the light ray starting from any positive value of r . However, if a light ray is initially directed towards the gravitating object, it will reach the observer only if the observer is located between r_H^+ and the starting point of the light ray (the source), which is a very unrealistic situation. Hence, usually, the light ray approaches the outer event horizon and eventually falls onto it.

Conversely, when a light ray starts from $r > r_H^+$ and is initially directed away from the gravitating object, it can reach the observer at infinity without crossing the black hole (gravitating object). In this scenario, the light bending effect due to the gravitating object is negligible.

Next, we present a detailed analysis of the geodesic described in Case-1 within the region $r_4 \leq r \leq \infty$, where the light ray is initially directed towards the gravitating object and the light bending effect is strong.

Calculating the r integrals, i.e., I_r, I_ϕ, I_t for the Case-1 in the region $r_4 \leq r \leq \infty$ when the light ray initially moves towards the companion

In this section we write the path integrals defined in Eqs. (C.20a), (C.20b) and (C.20c) as functions of elliptic integrals. First, we re-write the expressions of the r integrals of Eqs. (C.20a), (C.20b) and (C.20c) as follows:

$$I_r = I_0 , \quad (C.92a)$$

$$I_\phi = \frac{2 \left(\frac{GM_c}{c^2} \right) \left(\tilde{a} \frac{GM_c}{c^2} \right)}{r_H^+ - r_H^-} \left[\left(r_H^+ - \frac{\tilde{a} \left(\frac{GM_c}{c^2} \right) D_\lambda}{2 \left(\frac{GM_c}{c^2} \right)} \right) I_+ - \left(r_H^- - \frac{\tilde{a} \left(\frac{GM_c}{c^2} \right) D_\lambda}{2 \left(\frac{GM_c}{c^2} \right)} \right) I_- \right] , \quad (C.92b)$$

$$I_t = \frac{\left(2 \frac{GM_c}{c^2} \right)^2}{r_H^+ - r_H^-} \left[r_H^+ \left(r_H^+ - \frac{\tilde{a} \frac{GM_c}{c^2} D_\lambda}{2 \frac{GM_c}{c^2}} \right) I_+ - r_H^- \left(r_H^- - \frac{\tilde{a} \frac{GM_c}{c^2} D_\lambda}{2 \frac{GM_c}{c^2}} \right) I_- \right] + \left(2 \frac{GM_c}{c^2} \right)^2 I_0 + \left(2 \frac{GM_c}{c^2} \right) I_1 + I_2 , \quad (C.92c)$$

where,

$$I_\pm = \oint_{r_s}^{r_o} \frac{dr}{\text{sign}(r) (r - r_H^+) \sqrt{\mathcal{R}}} , \quad (C.93a)$$

$$I_l = \oint_{r_s}^{r_o} \frac{r^l dr}{\text{sign}(r) \sqrt{\mathcal{R}}} \text{ where } l \text{ can take any one value from } 0, 1, 2 . \quad (C.93b)$$

For Case-1, in the region $r_4 \leq r \leq \infty$, a light ray has $p_r < 0$ when it travels towards the gravitating object. In such a situation, it can reach close to the gravitating object up to the turning point r_4 and then it flies away to the infinity with a positive p_r . Hence, any radial path integral can be split into two parts, the first is in the region $r_4 \leq r \leq r_s$ with a negative p_r and the second is in the region $r_4 \leq r \leq \infty$ with a positive p_r . Rewriting the radial function $\mathcal{R}(r)$ as $\mathcal{R}(r) = (r - r_1)(r - r_2)(r - r_3)(r - r_4)$, we can write the radial path integrals for $I_\pm$ and I_l as:

$$I_l = \int_{r_s}^{r_0} \frac{r^l dr}{\text{sign}(r) \sqrt{\mathcal{R}(r)}} = \int_{r_s}^{r_4} \frac{r^l dr}{-\sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}} + \int_{r_4}^{r_o} \frac{r^l dr}{+\sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}}, \quad (\text{C.94a})$$

$$I_{\pm} = \int_{r_s}^{r_o} \frac{dr}{\text{sign}(r) (r-r_H^{\pm}) \sqrt{\mathcal{R}(r)}} = \int_{r_s}^{r_4} \frac{dr}{-(r-r_H^{\pm}) \sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}} + \int_{r_4}^{r_o} \frac{dr}{+(r-r_H^{\pm}) \sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}}. \quad (\text{C.94b})$$

We now define a new function $\varpi(r)$ and a new variable ς as:

$$\varpi(r) = \sqrt{\frac{r-r_4}{r-r_3} \frac{r_3-r_1}{r_4-r_1}} \quad (\text{C.95})$$

$$\varsigma = \frac{(r_3-r_2)(r_4-r_1)}{(r_3-r_1)(r_4-r_2)}. \quad (\text{C.96})$$

Using the algebraic properties of elliptical integrals presented in Sec. C.3, we get

$$I_0 = \frac{2}{\sqrt{(r_3-r_1)(r_4-r_2)}} \left[\mathcal{F}(\sin^{-1}(\varpi(r_s)), \varsigma) + \mathcal{F}(\sin^{-1}(\varpi(r_o)), \varsigma) \right], \quad (\text{C.97a})$$

$$I_1 = \frac{2}{\sqrt{(r_3-r_1)(r_4-r_2)}} \times \left[\left\{ r_3 \mathcal{F}(\sin^{-1}(\varpi(r_s)), \varsigma) + (r_4-r_3) \Pi\left(\frac{r_4-r_1}{r_3-r_1}, \sin^{-1}(\varpi(r_s)), \varsigma\right) \right\} + \left\{ r_3 \mathcal{F}(\sin^{-1}(\varpi(r_o)), \varsigma) + (r_4-r_3) \Pi\left(\frac{r_4-r_1}{r_3-r_1}, \sin^{-1}(\varpi(r_o)), \varsigma\right) \right\} \right], \quad (\text{C.97b})$$

$$\begin{aligned}
I_2 = & \left[\frac{\sqrt{\mathcal{R}(r_s)}}{r_s - r_3} - \frac{r_1 r_4 + r_2 r_3}{2} \times \frac{2}{(r_3 - r_1)(r_4 - r_2)} \mathcal{F}(\sin^{-1}(\varpi(r_s)), \varsigma) \right. \\
& \left. - \sqrt{(r_3 - r_1)(r_4 - r_2)} \mathcal{E}(\sin^{-1}(\varpi(r_s)), \varsigma) \right] \\
& + \left[\frac{\sqrt{\mathcal{R}(r_o)}}{r_s - r_3} - \frac{r_1 r_4 + r_2 r_3}{2} \times \frac{2}{(r_3 - r_1)(r_4 - r_2)} \mathcal{F}(\sin^{-1}(\varpi(r_o)), \varsigma) \right. \\
& \left. - \sqrt{(r_3 - r_1)(r_4 - r_2)} \mathcal{E}(\sin^{-1}(\varpi(r_o)), \varsigma) \right], \quad (\text{C.97c})
\end{aligned}$$

$$\begin{aligned}
I_{\pm} = & \frac{2}{(r_H^{\pm} - r_3) \sqrt{(r_3 - r_1)(r_4 - r_2)}} \times \\
& \left[\left\{ -\frac{r_4 - r_3}{r_H^{\pm} - r_4} \Pi\left(\frac{(r_H^{\pm} - r_3)(r_4 - r_1)}{(r_H^{\pm} - r_4)(r_3 - r_1)}, \sin^{-1}(\varpi(r_s)), \varsigma\right) - \mathcal{F}(\sin^{-1}(\varpi(r_s)), \varsigma) \right\} \right. \\
& \left. + \left\{ -\frac{r_4 - r_3}{r_H^{\pm} - r_4} \Pi\left(\frac{(r_H^{\pm} - r_3)(r_4 - r_1)}{(r_H^{\pm} - r_4)(r_3 - r_1)}, \sin^{-1}(\varpi(r_o)), \varsigma\right) - \mathcal{F}(\sin^{-1}(\varpi(r_o)), \varsigma) \right\} \right], \quad (\text{C.97d})
\end{aligned}$$

If the observer is at infinity then $r_o = \infty$ and $\varpi(\infty) = \sqrt{\frac{r_3 - r_1}{r_4 - r_1}}$.

Using the elliptic integrals of Eqs.(C.97) in Eq.(C.92), we can calculate the radial integrals I_r , I_{ϕ} , and I_t in terms of elliptic integral as a function of r_s and r_o .

C.3 Elliptical integrals in the form of algebraic integration

In this section we write down the algebraic integrations (which are useful for the study of the null geodesic) in terms of elliptic integrals.

Let us consider a, b, c, d are the roots of an equation $f(x) = 0$ where x is any independent

variable and $a > b > c > d$. We also define a new variable v and a parameter q as:

$$v = \sin^{-1} \sqrt{\frac{(b-d)(x-a)}{(a-d)(x-b)}}, \quad (\text{C.98a})$$

$$q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}. \quad (\text{C.98b})$$

Then, in terms of v and q , we can write the following expressions:

$$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \mathcal{F}(v, q), \quad (\text{C.99})$$

$$\begin{aligned} \int_a^u \frac{xdx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \\ \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-b) \Pi \left(\frac{a-d}{b-d}, v, q \right) + b \mathcal{F}(v, q) \right\}, \quad (\text{C.100}) \end{aligned}$$

$$\begin{aligned} \int_a^u \frac{x^2 dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \\ \frac{\sqrt{f(u)}}{u-b} - \frac{ad+bc}{\sqrt{(a-c)(b-d)}} \mathcal{F}(v, q) - \sqrt{(b-d)(a-c)} \mathcal{E}(v, q), \quad (\text{C.101}) \end{aligned}$$

$$\begin{aligned} \int_a^u \frac{dx}{(p-x)\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \\ \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \\ \times \left[(a-b) \Pi \left(\frac{(a-d)(p-b)}{(b-d)(p-a)}, v, q \right) + (p-a) \mathcal{F}(v, q) \right], \quad (\text{C.102}) \end{aligned}$$

where p and u are any arbitrary value of the variable x and $p \neq a$.

C.4 Derivations of useful expressions needed for θ integrations

C.4.1 Derivation of Eq. (C.43)

In Sec. C.2.1, we have seen

$$X_\mu = X_{\mu-1} + 2 \operatorname{sign}(p_s^\theta) \mathcal{F}\left(\pi/2, \frac{u_+}{u_-}\right). \quad (\text{C.39})$$

Multiplying both sides of Eq. (C.39) by $(-1)^\mu$, we get

$$(-1)^\mu X_\mu = - \left[(-1)^{\mu-1} X_{\mu-1} - 2(-1)^\mu \operatorname{sign}(p_s^\theta) \mathcal{F}\left(\pi/2, \frac{u_+}{u_-}\right) \right]. \quad (\text{C.103})$$

The term $2(-1)^\mu \operatorname{sign}(p_s^\theta)$ can take any one of the two values ± 1 . Hence, identities Eqs. (C.42a) and (C.42b) give

$$\operatorname{sn}\left((-1)^\mu X_\mu, \frac{u_+}{u_-}\right) = \operatorname{sn}\left((-1)^{\mu-1} X_{\mu-1}, \frac{u_+}{u_-}\right) = \dots = \operatorname{sn}\left(X_0, \frac{u_+}{u_-}\right) \quad (\text{C.43})$$

C.4.2 Derivation of Eq. (C.53)

In Sec. C.2.1, we have also seen

$$\begin{aligned} \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r = & \left(\mu \pm \eta_o \frac{\{1 - (-1)^\mu\}}{2} \right) \mathcal{F}\left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp}\right) + \eta_s \mathcal{F}\left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp}\right) \\ & - \eta_o \mathcal{F}\left(\Gamma_o^\pm, 1 - \frac{u_\pm}{u_\mp}\right), \quad (\text{C.50}) \end{aligned}$$

or,

$$\eta_o \mathcal{F} \left(\Gamma_o^\pm, 1 - \frac{u_\pm}{u_\mp} \right) = -\sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r + \left(\mu \pm \eta_o \frac{\{1 - (-1)^\mu\}}{2} \right) \mathcal{F} \left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp} \right) + \eta_s \mathcal{F} \left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp} \right), \quad (\text{C.104})$$

or,

$$\mathcal{F} \left(\Gamma_o^\pm, 1 - \frac{u_\pm}{u_\mp} \right) = -\frac{1}{\eta_o} \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r + \left(\frac{\mu}{\eta_o} \pm \frac{\{1 - (-1)^\mu\}}{2} \right) \mathcal{F} \left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp} \right) + \frac{\eta_s}{\eta_o} \mathcal{F} \left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp} \right). \quad (\text{C.105})$$

Using $\eta_o = \frac{1}{\eta_o}$ from Eq. (C.30a), we write

$$\mathcal{F} \left(\Gamma_o^\pm, 1 - \frac{u_\pm}{u_\mp} \right) = -\eta_o \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r + \eta_o \left(\mu \pm \eta_o \frac{1 - (-1)^\mu}{2} \right) \mathcal{F} \left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp} \right) + \eta_s \eta_o \mathcal{F} \left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp} \right) \quad (\text{C.106})$$

Using the expression for η_s , η_o , i.e., Eqs. (C.28a) and (C.28b) in Eq. (C.106), we get

$$\mathcal{F} \left(\Gamma_o^\pm, 1 - \frac{u_\pm}{u_\mp} \right) = (-1)^\mu \text{sign}(\cos \theta_o) \left\{ -\text{sign}(p_s^\theta) \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r + \text{sign}(p_s^\theta) \left(\mu \pm \eta_o \frac{1 - (-1)^\mu}{2} \right) \mathcal{F} \left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp} \right) + \text{sign}(\cos \theta_s) \mathcal{F} \left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp} \right) \right\}, \quad (\text{C.107})$$

giving

$$\mathcal{F} \left(\Gamma_o^\pm, 1 - \frac{u_\pm}{u_\mp} \right) = (-1)^\mu \text{sign}(\cos \theta_o) \left\{ \text{sign}(\cos \theta_s) \mathcal{F} \left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp} \right) + \text{sign}(p_s^\theta) \left[\left(\mu \pm \eta_o \frac{1 - (-1)^\mu}{2} \right) \mathcal{F} \left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp} \right) - \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r \right] \right\}, \quad (\text{C.108})$$

or,

$$\mathcal{F}\left(\Gamma_o^\pm, 1 - \frac{u_\pm}{u_\mp}\right) = (-1)^\mu \text{sign}(\cos \theta_o) Y_\mu^\pm. \quad (\text{C.109})$$

As we already defined $Y_\mu^\pm$ as:

$$Y_\mu^\pm = \text{sign}(\cos \theta_s) \mathcal{F}\left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp}\right) + \text{sign}(p_s^\theta) \left[\left(\mu \pm \eta_o \frac{1 - (-1)^\mu}{2} \right) \mathcal{F}\left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp}\right) - \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r \right]. \quad (\text{C.52})$$

Next, we derive a relation between $Y_\mu^\pm$ and $Y_{\mu-1}^\pm$. First, from Eq. (C.52), we write,

$$Y_\mu^\pm = \text{sign}(\cos \theta_s) \mathcal{F}\left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp}\right) + \text{sign}(p_s^\theta) \left[\left((\mu - 1) + 1 \pm \eta_o \frac{1 + (-1)^{\mu-1}}{2} \right) \mathcal{F}\left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp}\right) - \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r \right], \quad (\text{C.110})$$

or,

$$Y_\mu^\pm = \text{sign}(\cos \theta_s) \mathcal{F}\left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp}\right) + \text{sign}(p_s^\theta) \left[\left((\mu - 1) + 1 \pm \eta_o \frac{1 - (-1)^{\mu-1} + 2(-1)^{\mu-1}}{2} \right) \mathcal{F}\left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp}\right) - \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r \right], \quad (\text{C.111})$$

or,

$$Y_\mu^\pm = \text{sign}(\cos \theta_s) \mathcal{F}\left(\Gamma_s^\pm, 1 - \frac{u_\pm}{u_\mp}\right) + \text{sign}(p_s^\theta) \left[\left((\mu - 1) \pm \eta_o \frac{1 - (-1)^{\mu-1}}{1} \right) \mathcal{F}\left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp}\right) - \sqrt{u_\mp} \left(\tilde{a} \frac{GM_c}{c^2} \right) I_r \right] + (1 \pm \eta_o (-1)^{\mu-1}) \text{sign}(p_s^\theta) \mathcal{F}\left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp}\right). \quad (\text{C.112})$$

The terms within the square bracket of Eq. (C.112) gives the expression of $Y_{\mu-1}$ if we

replace μ of Eq. (C.52) by $\mu - 1$. Hence, we can write

$$Y_\mu^\pm = Y_{\mu-1}^\pm + (1 \pm \eta_o(-1)^{\mu-1}) \text{sign}(p_s^\theta) \mathcal{F}\left(\frac{\pi}{2}, 1 - \frac{u_\pm}{u_\mp}\right). \quad (\text{C.53})$$

C.4.3 Derivation of Eq. (C.59a), i.e., the relation between Ψ_j and Γ_j

Eq. (C.25d) can be written as:

$$\Psi_j^\pm = \sin^{-1} \left(\sqrt{\frac{\cos^2 \theta_j}{u_\pm}} \right), \quad (\text{C.113})$$

which gives

$$\sin^2 \Psi_j^\pm = \frac{\cos^2 \theta_j}{u_\pm}. \quad (\text{C.114})$$

From the definition of Γ_j in Eq. (C.48), we can write

$$\sin^2 \Gamma_j^\pm = \pm \left(\frac{\cos^2 \theta_j - u_\mp}{u_+ - u_-} \right), \quad (\text{C.115})$$

which gives,

$$\sin^2 \Gamma_j^+ = + \left(\frac{\cos^2 \theta_j - u_-}{u_+ - u_-} \right), \quad (\text{C.116a})$$

$$\sin^2 \Gamma_j^- = - \left(\frac{\cos^2 \theta_j - u_+}{u_+ - u_-} \right), \quad (\text{C.116b})$$

As discussed earlier, in the context of null geodesics in Kerr spacetime, $\Theta(\theta_j)$ remains non-negative (which is a necessity), only when $u_- \leq \cos^2 \theta_j \leq u_+$, which in turn confirms that $\sin^2 \Gamma_j^\pm \geq 0$ in Eqs. (C.116a) and (C.116b). That means,

$$\sin^2 \Gamma_j^+ = + \left(\frac{(\cos^2 \theta_j / u_-) - 1}{(u_+ / u_-) - 1} \right) = + \left(\frac{1 - (\cos^2 \theta_j / u_-)}{1 - (u_+ / u_-)} \right), \quad (\text{C.117a})$$

$$\sin^2 \Gamma_j^- = - \left(\frac{(\cos^2 \theta_j / u_+) - 1}{1 - (u_- / u_+)} \right) = + \left(\frac{1 - (\cos^2 \theta_j / u_+)}{1 - (u_- / u_+)} \right), \quad (\text{C.117b})$$

Eqs. (C.117a) and (C.117b) can be rewritten as,

$$\left(1 - \frac{u_{\pm}}{u_{\mp}} \right) \sin^2 \Gamma_j^{\pm} = \left(1 - \frac{\cos^2 \theta_j}{u_{\mp}} \right) = 1 - \sin^2 \Psi_j^{\mp}, \quad (\text{C.59a})$$

where in the last step we have used Eq. (C.114).

C.5 Calculation methods of p_r/p_t , p_{θ}/p_t and p_{ϕ}/p_t using the initial position and the initial direction of the light ray

We compute the position vector of the pulsar in the b-frame ($\vec{r}_b$) with the help of the given values of the masses of the pulsar and its companion as well as the orbital parameters. We can convert $\vec{r}_b$ to $\vec{r}_T$ using Eq. (A.18). It is obvious that $\vec{r}_T$ will be along the Y_T -axis, which is by definition along $\vec{r}_b$, while $\vec{r}_b$ and $\vec{r}_T$ are the same vector expressed in different frames. See Fig. 2.1 and Sec. 2.2 for the definitions of the I-frame, b-frame, T-frame, bh-frame, and bh'-frame.

On the other hand, the model of the emission geometry make us get the initial direction of the light ray in the I-frame ($\hat{n}_I$). This direction is then transformed into the b-frame via conversion matrices, resulting in $\hat{n}_b$. We define p_{r_T} , p_{θ_T} , and p_{ϕ_T} as the momenta associated with the light ray's motion in the radial (r_T), polar (θ_T), and azimuthal (ϕ_T) directions, respectively. The spherical coordinates r_T , θ_T , ϕ_T are related to the T-frame coordinates x_T , y_T , z_T by the conventional relations:

$$x_T = r_T \sin \theta_T \cos \phi_T, \quad (\text{C.118a})$$

$$y_T = r_T \sin \theta_T \sin \phi_T , \quad (\text{C.118b})$$

$$z_T = r_T \cos \theta_T . \quad (\text{C.118c})$$

Additionally, let p_{t_T} denote the temporal component of the angular momentum of the light ray in the T-frame. Hence, we can write

$$-p_{t_T}^2 + p_{r_T}^2 + p_{\phi_T}^2 + p_{\theta_T}^2 = 0 , \quad (\text{C.119})$$

which means,

$$p_{t_T}^2 = p_{r_T}^2 + p_{\theta_T}^2 + p_{\phi_T}^2 = |\vec{p}_T|^2 , \quad (\text{C.120})$$

where, $\vec{p}_T$ is the 3-dimensional spatial component of the angular momentum of the light ray measured in the T-frame.

We denote the initial values of p_{t_T} , p_{r_T} , p_{θ_T} , and p_{ϕ_T} as $p_{t_{T,s}}$, $p_{r_{T,s}}$, $p_{\theta_{T,s}}$, and $p_{\phi_{T,s}}$, respectively, as the light ray was initially at the source ‘s’. The coordinate of the light ray at that moments (at source, in the T frame) are $r_{T,s}$, $\theta_{T,s}$ and $\phi_{T,s}$. As Eq. (A.11) reveals the fact that the Z_T -axis is always perpendicular to the initial direction of the light ray ($\hat{n}_b$), implying that the light ray is initially confined to the $X_T Y_T$ plane giving $p_{\theta_{T,s}} = 0$. If χ is the angle between the $\hat{n}_b$ and $\vec{r}_b$, then we can write (see Fig. C.3),

$$p_{\phi_{T,s}}/|\vec{p}_{T,s}| = p_{\phi_{T,s}}/p_{t_{T,s}} = -\sin \chi , \quad (\text{C.121a})$$

$$p_{\theta_{T,s}}/|\vec{p}_{T,s}| = p_{\theta_{T,s}}/p_{t_{T,s}} = 0 , \quad (\text{C.121b})$$

$$p_{r_{T,s}}/|\vec{p}_{T,s}| = p_{r_{T,s}}/p_{t_{T,s}} = \cos \chi , \quad (\text{C.121c})$$

where, in Eq. (C.121a) we use $-ve$ sign for our choice of the T-frame in Eqs. (A.11) and (A.10), and used the fact that r_T is along the Y_T -axis. We have also used the fact that $p_{\theta_{T,s}} = 0$ as the light ray is initially confined in the $X_T Y_T$ plane.

Note that, the equations for the null geodesic are known in the frame of the gravitating object, i.e., in the bh-frame, as defined in Chapter 2 (see Fig. 2.1) . Hence, we need to first get the values of the components of the linear momentum in the bh-frame. However, the transformation relations between different frames involving rotation matrices are known (Appendix A) for Cartesian components of vectors. Hence, we use the following mathematical trick. We first find the Cartesian components of the angular momentum of the light ray at the location of the source in the T-frame, then we transform those into the Cartesian components of the angular momentum of the light ray at the location of the source in the bh-frame. After this, we get the spherical components of the linear momentum of the light ray (when it is at the source) in the bh-frame. These steps are outlined in details below.

From the definition of angular momentum $\vec{L} = \vec{r} \times \vec{p}$, we can write (in any frame)

$$L_X = -r [p_\phi \cos \theta \cos \phi + p_\theta \sin \phi] , \quad (\text{C.122a})$$

$$L_Y = -r [p_\phi \cos \theta \sin \phi - p_\theta \cos \phi] , \quad (\text{C.122b})$$

$$L_Z = r p_\phi \sin \theta , \quad (\text{C.122c})$$

where L_X, L_Y, L_Z are the components of the angular momentum $\vec{L}$ along the Cartesian X, Y, and Z axes, respectively, and p_θ and p_ϕ are the polar and the azimuthal components of the linear momentum $\vec{p}$. We now define a quantity called the ‘scaled angular momentum’ which is defined as $\vec{\tilde{L}} = \vec{L} / (r p_t)$ where p_t is the temporal component of the linear momentum. Hence, from Eqs. (C.122), the Cartesian components of $\vec{\tilde{L}}$ become:

$$\tilde{L}_X = - \left[\frac{p_\phi}{p_t} \cos \theta \cos \phi + \frac{p_\theta}{p_t} \sin \phi \right] , \quad (\text{C.123a})$$

$$\tilde{L}_Y = - \left[\frac{p_\phi}{p_t} \cos \theta \sin \phi - \frac{p_\theta}{p_t} \cos \phi \right] , \quad (\text{C.123b})$$

$$\tilde{L}_Z = \frac{p_\phi}{p_t} \sin \theta . \quad (\text{C.123c})$$

Hence, in the T-frame, for the initial position of a light ray, i.e., when the light ray is at the source, we can write

$$\tilde{L}_{X_{T,s}} = - \left[\frac{p_{\phi_{T,s}}}{p_{t_{T,s}}} \cos \theta_{T,s} \cos \phi_{T,s} + \frac{p_{\theta_{T,s}}}{p_{t_{T,s}}} \sin \phi_{T,s} \right], \quad (\text{C.124a})$$

$$\tilde{L}_{Y_{T,s}} = - \left[\frac{p_{\phi_{T,s}}}{p_{t_{T,s}}} \cos \theta_{T,s} \sin \phi_{T,s} - \frac{p_{\theta_{T,s}}}{p_{t_{T,s}}} \cos \phi_{T,s} \right], \quad (\text{C.124b})$$

$$\tilde{L}_{Z_{T,s}} = \frac{p_{\phi_{T,s}}}{p_{t_{T,s}}} \sin \theta_{T,s}, \quad (\text{C.124c})$$

where $\tilde{L}_{X_{T,s}}$, $\tilde{L}_{Y_{T,s}}$, $\tilde{L}_{Z_{T,s}}$ are the components of the angular momentum $\vec{\tilde{L}}$ of the light ray when it is at the source along the X_T , Y_T , and Z_T axes.

By construction of the T-frame, we have, $\theta_{T,s} = \pi/2$ and $\phi_{T,s} = \pi/2$ (see Fig.C.3). Using Eqs. (C.121) and (C.124), we can construct the initial angular momentum vector for the light ray, scaled by the magnitude of the spatial angular momentum $p_{t_{T,s}}$, as:

$$\vec{\tilde{L}}_{T,s} = \begin{pmatrix} \tilde{L}_{X_{T,s}} \\ \tilde{L}_{Y_{T,s}} \\ \tilde{L}_{Z_{T,s}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\sin \chi \end{pmatrix}. \quad (\text{C.125})$$

$\vec{\tilde{L}}_{T,s}$, obtained in the T-frame, can be transformed into $\vec{\tilde{L}}_{\text{bh}',s}$, its equivalent representation in the bh'-frame, using the transformation described by Eq. (A.27). Since $\vec{\tilde{L}}_{\text{bh}',s}$ remains invariant under transformation to the bh-frame, we can write:

$$\vec{\tilde{L}}_{\text{bh},s} = R_Y^A(-\lambda_{\text{bh}}) R_Z^A(-\eta_{\text{bh}}) R_X^A(i) R_Z^A(\omega) R_Z^A(\varphi_{\text{Tb}}) R_Y^A(\vartheta_{\text{Tb}}) \vec{\tilde{L}}_{T,s}. \quad (\text{C.126})$$

Here, $\vec{\tilde{L}}_{\text{bh},s} = (\vec{\tilde{L}}_{\text{bh},s})/(|\vec{\tilde{L}}_{\text{bh},s}| p_{t,\text{bh},s})$ is the ‘scaled angular momentum’ of the light ray in the bh-frame when it is at the source, following Eq. (C.123), the Cartesian components of $\vec{\tilde{L}}_{\text{bh},s}$ are known in terms of initial position and initial momentum of the light ray.

Similarly, following the transformations given in Eq. (A.26), $\vec{r}_b$ can be transformed from

b -frame to bh-frame ($\vec{r}_{\text{bh}}$) as follows:

$$\vec{r}_{\text{bh}} = R_Y^A(-\lambda_{\text{bh}}) R_Z^A(-\eta_{\text{bh}}) R_X^A(i) R_Z^A(\omega) \left(1 + \frac{M_p}{M_c}\right) \vec{r}_b, \quad (\text{C.127})$$

where, M_p and M_c are the masses of the pulsar and companion respectively. Following Eq. (C.123), we can write

$$\tilde{L}_{X_{\text{bh},s}} = - \left[\frac{p_{\phi_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \cos \theta_{\text{bh},s} \cos \phi_{\text{bh},s} + \frac{p_{\theta_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \sin \phi_{\text{bh},s} \right], \quad (\text{C.128a})$$

$$\tilde{L}_{Y_{\text{bh},s}} = - \left[\frac{p_{\phi_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \cos \theta_{\text{bh},s} \sin \phi_{\text{bh},s} - \frac{p_{\theta_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \cos \phi_{\text{bh},s} \right], \quad (\text{C.128b})$$

$$\tilde{L}_{Z_{\text{bh},s}} = \frac{p_{\phi_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \sin \theta_{\text{bh},s}. \quad (\text{C.128c})$$

From Eqs. (C.128) we get

$$p_{\theta_{\text{bh},s}}/p_{t_{\text{bh},s}} = \tilde{L}_{Y_{\text{bh},s}} \cos \phi_{\text{bh},s} - \tilde{L}_{X_{\text{bh},s}} \sin \phi_{\text{bh},s}, \quad (\text{C.129a})$$

$$p_{\phi_{\text{bh},s}}/p_{t_{\text{bh},s}} = \frac{\tilde{L}_{Z_{\text{bh},s}}}{\sin \theta_{\text{bh},s}}, \quad (\text{C.129b})$$

where, $\tilde{L}_{X_{\text{bh},s}}$, $\tilde{L}_{Y_{\text{bh},s}}$ and $\tilde{L}_{Z_{\text{bh},s}}$ are the components of the vector $\vec{\tilde{L}}_{\text{bh},s}$ along the X_{bh} , Y_{bh} , and Z_{bh} directions. Moreover, $p_{t_{\text{bh}}} = p_{t_T}$ as the bh-frame and T-frame are connected by spatial rotations only. The T-frame and the bh-frame, both are the locally non rotating frame.

Eqs. (C.129) contain the polar and the azimuthal angles of the location of the light ray at the source, i.e., $\theta_{\text{bh},s}$ and $\phi_{\text{bh},s}$, respectively. These angles can be calculated as:

$$\theta_{\text{bh},s} = \cos^{-1} \left(\frac{(\vec{r}_{\text{bh}})_z}{|\vec{r}_{\text{bh}}|} \right), \quad (\text{C.130a})$$

$$\phi_{\text{bh},s} = \cos^{-1} \left(\frac{(\vec{r}_{\text{bh}})_x}{|\vec{r}_{\text{bh}}| \sin \theta_{\text{bh},s}} \right), \quad (\text{C.130b})$$

where $\vec{r}_{\text{bh}}$ is the position vector of the pulsar in bh-frame, and $(\vec{r}_{\text{bh}})_z$ and $(\vec{r}_{\text{bh}})_x$ are its components along the X_{bh} and the Y_{bh} axes, respectively.

Next, we use the expressions given in [84] to find the constants of the motion D_q and D_λ and write

$$D_\lambda = \frac{\sin \theta_{\text{bh}} (p_{\phi_{\text{bh}}}/p_{t_{\text{bh}}})}{-\sqrt{\Delta} \rho^2 / \Sigma^2 + \omega \sin \theta_{\text{bh}} (p_{\phi_{\text{bh}}}/p_{t_{\text{bh}}})}, \quad (\text{C.131a})$$

$$D_q = \left[\frac{D_\lambda^2}{\sin^2 \theta_{\text{bh}}} - \left(\tilde{a} \frac{GM_c}{c^2} \right)^2 \right] \cos^2 \theta_{\text{bh}} + \left[\frac{p_{\theta_{\text{bh}}}}{p_{t_{\text{bh}}}} (1 - D_\lambda \omega) \right]^2 \frac{\Sigma^2}{\Delta}, \quad (\text{C.131b})$$

where ρ^2 , Σ^2 , Δ and ω are the usual elements of the Kerr metric defined after Eqs. (C.1). We first calculate the values of ρ^2 , Σ^2 , Δ and ω at the position of the source. Then using those values in Eqs. (C.131a) and (C.131b), we get the constants of the motion (D_λ and D_q) from the initial position and the initial direction of the light ray.

Note that, if we choose $\tilde{a} = 0$, then metric elements at the position of the source are $\rho^2 = |\vec{r}_b|^2$, $\Sigma^2 = |\vec{r}_b|^4$, $\Delta = |\vec{r}_b|^2 - 2 \left(\frac{GM_c}{c^2} \right) |\vec{r}_b|$, and $\omega = 0$. If we put these in Eqs. (C.131a) and (C.131b), we get

$$D_\lambda = \frac{\sin \theta_{\text{bh},s} \left(\frac{p_{\phi_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \right)}{- \left[1 - \left(\frac{2GM_c}{c^2 |\vec{r}_b|} \right) \right]^{1/2}}, \quad (\text{C.132a})$$

$$D_q = \frac{D_\lambda^2}{\sin^2 \theta_{\text{bh},s}} \cos^2 \theta_{\text{bh},s} + \left(\frac{p_{\theta_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \right)^2 \frac{|\vec{r}_b|^2}{1 - 2 \frac{GM_c}{c^2 |\vec{r}_b|}}. \quad (\text{C.132b})$$

From Eqs. (C.132a) and (C.132b), we get

$$D_q + D_\lambda^2 = \frac{D_\lambda^2}{\sin^2 \theta_{\text{bh},s}} + \left(\frac{p_{\theta_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \right)^2 \frac{|\vec{r}_b|^2}{1 - 2 \frac{GM_c}{c^2 |\vec{r}_b|}}. \quad (\text{C.133})$$

Using Eq. (C.132a) on the right hand side of Eq. (C.133), we get

$$D_q + D_\lambda^2 = \left[\left(\frac{p_{\theta_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \right)^2 + \left(\frac{p_{\theta_{\text{bh},s}}}{p_{t_{\text{bh},s}}} \right)^2 \right] \frac{|\vec{r}_b|^2}{1 - 2 \frac{GM_c}{c^2 |\vec{r}_b|}}. \quad (\text{C.134})$$

C.5.1 Conditions satisfied by $|D_q|$

In Sec. C.2.2, we stated

$$D_q \geq - \left(|D_\lambda| - \tilde{a} \frac{GM_c}{c^2} \right)^2 \text{ for any value of } |D_\lambda|. \quad (\text{C.91a})$$

$$\text{Additionally, } D_q \geq 0, \text{ if } |D_\lambda| \geq \left(\tilde{a} \frac{GM_c}{c^2} \right). \quad (\text{C.91b})$$

We aim to prove the above relations now.

In Sec. C.2.1, we had

$$\Delta_\theta = \frac{1}{2} \left(1 - \frac{D_q + D_\lambda^2}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right), \quad (\text{C.25a})$$

$$u_\pm = \Delta_\theta \pm \sqrt{\left(\Delta_\theta^2 + \frac{D_q}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right)}, \quad (\text{C.25b})$$

where

$$u = \cos^2 \theta \quad (\text{C.23})$$

and $u_\pm$ are roots of $\Theta(u)$ when $\tilde{a} \neq 0$. We also have a necessary condition $\Theta(u) \geq 0$ for integrations terms G_θ , G_ϕ , and G_t to be real (see Eqs. (C.20d), (C.20e), and (C.20f)).

From Eq. (C.25b), $u_\pm$ would be real only if

$$\Delta_\theta^2 + \frac{D_q}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \geq 0 \quad (\text{C.140})$$

Using (C.25a) in the inequality (C.140), we get

$$D_q^2 + 2 \left(\left(\tilde{a} \frac{GM_c}{c^2} \right)^2 + D_\lambda^2 \right) D_q + \left(\left(\tilde{a} \frac{GM_c}{c^2} \right)^2 - D_\lambda^2 \right)^2 \geq 0 \quad (\text{C.141})$$

The equality in (C.141) is a quadratic equation of D_q with two negative roots

$$D_q^+ = - \left[\left(\tilde{a} \frac{GM_c}{c^2} \right) - |D_\lambda| \right]^2, \quad (\text{C.142a})$$

$$D_q^- = - \left[\left(\tilde{a} \frac{GM_c}{c^2} \right) + |D_\lambda| \right]^2. \quad (\text{C.142b})$$

Now, the right hand side of inequality (C.141) is a function of D_q , say $f(D_q)$. We know that $f(D_q)$ would have an extremum with respect to a variation in D_q only when $f'(D_q) = 0$, and the extremum would be a maximum if $f''(D_q) < 0$ and minimum if $f''(D_q) > 0$. Here, one prime denotes differentiation once with respect to D_q and double prime denotes differentiation twice with respect to D_q . Using these conditions, we see that $f(D_q)$ has a minimum of a negative value of D_q , i.e., $D_{q,min,f} = - \left[\left(\tilde{a} \frac{GM_c}{c^2} \right)^2 + D_\lambda^2 \right]$. This minimum must lie between the roots of $f(D_q) = 0$, i.e., between D_q^+ and D_q^- . Hence, $f(D_q) > 0$ only when $D_q > D_q^+$ or $D_q < D_q^-$. The generic nature of such a function is shown in Fig. C.4.

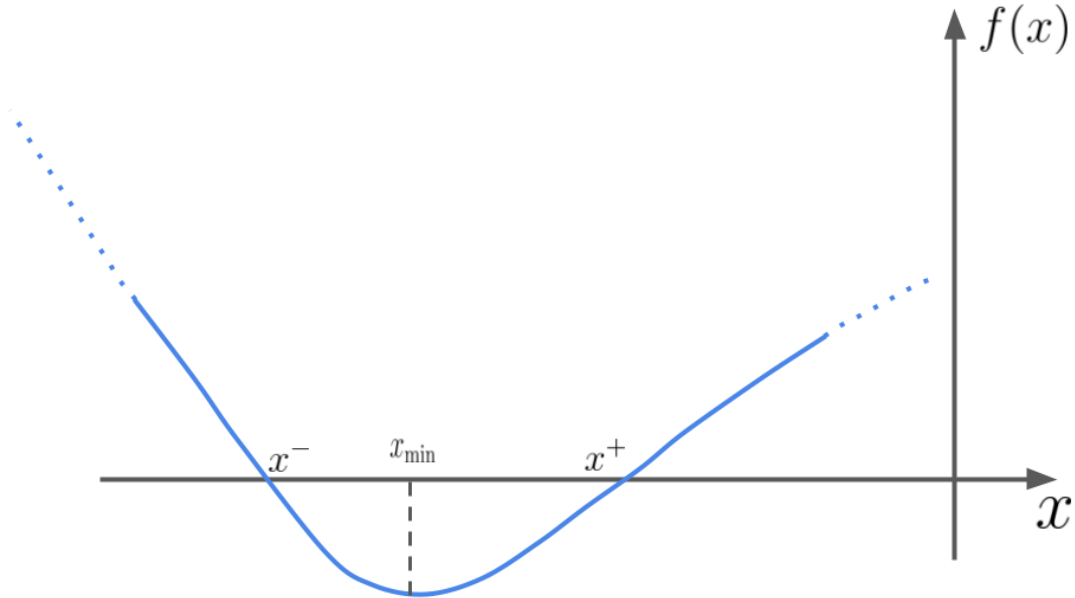


Figure C.4: A hypothetical function $f(x)$, which has two negative roots x^+ and x^- , and $x^+ > x^-$. The function has a minimum value at $x^- < x_{\min} < x^+$. Then, from the schematic diagram, it is clear that $f(x) > 0$ possible only for $x > x^+$ and $x < x^-$.

Using Eq. (C.142a), the conditions $D_q > D_q^+$ can be simply written as:

$$D_q \geq - \left(|D_\lambda| - \tilde{a} \frac{GM_c}{c^2} \right)^2. \quad (\text{C.143})$$

Similarly, Eq. (C.142b), the conditions $D_q < D_q^-$ can be simply written as:

$$D_q \leq - \left(|D_\lambda| + \tilde{a} \frac{GM_c}{c^2} \right)^2. \quad (\text{C.144})$$

From Eq. (C.144), we can show that,

$$\Delta_\theta \geq \left(1 + \frac{|D_\lambda|}{\tilde{a} \frac{GM_c}{c^2}} \right), \quad (\text{C.145})$$

where we have used the definition of Δ_θ given in Eq. (C.25a), which implies that for $D_\lambda \neq 0$, we have $\Delta_\theta > 1$, a result that is not physically permissible, as will be discussed (see Eq. (C.147)). Hence, we reject Eq. (C.144). On the other hand, Eq. (C.143) is a valid condition and permits both positive and negative values for D_q .

When $|D_\lambda| < |\tilde{a}| \frac{GM_c}{c^2}$, we do not find any additional condition on D_q other than Eq. (C.143). However the case when $|D_\lambda| \geq \left(\tilde{a} \frac{GM_c}{c^2} \right)$ is interesting and we describe it next. From the definition of $u = \cos^2 \theta$, we have $u_+ \leq 1$. Hence, from Eq. (C.25b),

$$\Delta_\theta + \sqrt{\left(\Delta_\theta^2 + \frac{D_q}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right)} \leq 1. \quad (\text{C.146})$$

As $\sqrt{\left(\Delta_\theta^2 + \frac{D_q}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right)} \geq 0$ (for u_+ to be real), Eq. (C.146) gives

$$\Delta_\theta \leq 1. \quad (\text{C.147})$$

Using Eq. (C.25a) in inequality (C.147), we write

$$\frac{1}{2} \left(1 - \frac{D_\lambda^2}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} - \frac{D_q}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right) \leq 1. \quad (\text{C.148})$$

Now it seems that there are two possibilities and we discuss the consequences of these one by one

- **Possibility-I:** $D_q \geq 0$

This is a valid possibility where D_q takes positive values, and it does not contradict any other conditions.

- **Possibility-II:** $D_q < 0$

Let us now choose a negative value for D_q (as Eq. (C.143) allows both positive and negative values for D_q) and investigate whether it is possible or not when $|D_\lambda| \geq \left(\tilde{a} \frac{GM_c}{c^2} \right)$. The inequality (C.148) can be re-written as:

$$\frac{1}{2} \left(1 - \frac{D_\lambda^2}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} + \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right) \leq 1. \quad (\text{C.149})$$

Now, as we have selected $|D_\lambda| \geq \left| \tilde{a} \right| \frac{GM_c}{c^2}$, i.e., $\frac{D_\lambda^2}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \geq 1$, we can write

$$\left(1 - \frac{D_\lambda^2}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right) \leq 0. \quad (\text{C.150})$$

Inequality (C.150) means

$$\left(1 - \frac{D_\lambda^2}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right) + \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \leq \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2}, \quad (\text{C.151})$$

which gives

$$\frac{1}{2} \left[\left(1 - \frac{D_\lambda^2}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right) + \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right] \leq \frac{|D_q|}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \quad (\text{C.152})$$

Using Eq. (C.25a), inequality (C.152) can be re-written as:

$$\Delta_\theta \leq \frac{|D_q|}{2 \left(\tilde{a} \frac{GM_c}{c^2} \right)^2}, \quad (\text{C.153})$$

which gives

$$\Delta_\theta^2 \leq \frac{D_q^2}{4 \left(\tilde{a} \frac{GM_c}{c^2} \right)^4}. \quad (\text{C.154})$$

Subtracting $\frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2}$ from both sides of the inequality (C.154) we get

$$\Delta_\theta^2 - \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \leq \frac{D_q^2}{4 \left(\tilde{a} \frac{GM_c}{c^2} \right)^4} - \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2}, \quad (\text{C.155})$$

Using the inequality (C.140), we can write,

$$0 \leq \Delta_\theta^2 - \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \leq \frac{D_q^2}{4 \left(\tilde{a} \frac{GM_c}{c^2} \right)^4} - \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2}, \quad (\text{C.156})$$

or,

$$\frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \left[\frac{|D_q|}{4 \left(\tilde{a} \frac{GM_c}{c^2} \right)^2} - 1 \right] \geq 0, \quad (\text{C.157})$$

which leads to

$$\left[\frac{|D_q|}{4 \left(\tilde{a} \frac{GM_c}{c^2} \right)^2} - 1 \right] \geq 0, \quad (\text{C.158})$$

or,

$$\frac{|D_q|}{4 \left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \geq 1, \quad (\text{C.159})$$

which means

$$\frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \geq 4. \quad (\text{C.160})$$

From the definition of u as given in Eq. (C.23) and the property of $\cos^2 \theta$, we can write

$$0 \leq u_+ \leq 1. \quad (\text{C.161})$$

Using Eq. (C.25b), the inequality (C.161) can be re-written as

$$0 \leq \Delta_\theta + \sqrt{\left(\Delta_\theta^2 + \frac{D_q}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right)} \leq 1. \quad (\text{C.162})$$

As we are dealing with the case of $D_q < 0$, we can re-write:

$$0 \leq \Delta_\theta + \sqrt{\left(\Delta_\theta^2 - \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right)} \leq 1. \quad (\text{C.163})$$

As the $\pm$ sign of the square root has been already taken care of while defining $u_\pm$ separately, to ensure the fact that u_+ must be real, the following condition is to be satisfied:

$$\sqrt{\left(\Delta_\theta^2 - \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \right)} \geq 0, \quad (\text{C.164})$$

or

$$\Delta_\theta^2 - \frac{|D_q|}{\left(\tilde{a} \frac{GM_c}{c^2} \right)^2} \geq 0. \quad (\text{C.165})$$

Using inequality (C.164) in inequality (C.163), we write

$$\Delta_\theta \leq 1. \quad (\text{C.166})$$

Squaring both sides of the inequality (C.166), we get

$$\Delta_\theta^2 \leq 1. \quad (\text{C.167})$$

Now, from inequality (C.165), we get

$$\Delta_\theta^2 \geq \frac{|D_q|}{\left(\tilde{a}\frac{GM_c}{c^2}\right)^2}. \quad (\text{C.168})$$

Combining inequalities (C.167) and (C.168), we write

$$1 \geq \Delta_\theta^2 \geq \frac{|D_q|}{\left(\tilde{a}\frac{GM_c}{c^2}\right)^2}, \quad (\text{C.169})$$

or simply,

$$\frac{|D_q|}{\left(\tilde{a}\frac{GM_c}{c^2}\right)^2} \leq 1. \quad (\text{C.170})$$

We see that inequality (C.160) contradicts with inequality (C.170). Hence, something in our assumptions was wrong. We can say that D_q cannot be chosen negative if we set $|D_\lambda| \geq \left(\tilde{a}\frac{GM_c}{c^2}\right)$.

Hence, we summarise as:

$$D_q \geq -\left(|D_\lambda| - \tilde{a}\frac{GM_c}{c^2}\right)^2 \text{ for any value of } |D_\lambda|. \quad (\text{C.91a})$$

$$\text{Additionally, } D_q \geq 0, \text{ if } |D_\lambda| \geq \left(\tilde{a}\frac{GM_c}{c^2}\right). \quad (\text{C.91b})$$

Appendix D

Geometrical Explanation for The Pulse Profile Distortion Near The Superior Conjunction

An explanation of the distortion of the pulse profile near the superior conjunction is presented below using a simplified geometry, as illustrated in Fig. D.1.

In this figure, we define a straight line YY' connecting the pulsar (denoted as ‘PSR’) and the black hole (denoted as ‘BH’) and extending throughout the space. Three planes, labelled Y_1 , Y_2 , and Y_3 , are shown in a light blue colour and all perpendicular to the line YY' . The plane Y_1 contains the black hole, Y_2 is an intermediate plane located along the path of the light propagation slightly away from the black hole and towards the observer in such a way that the bending of the rays are completed before entering the Y_2 -plane, i.e., the directions of the light rays do not change after they enter the Y_2 -plane. Y_3 is a plane on which the observer is located, it can be considered at infinity for the purpose of geometry. Note that, this geometry would not be valid for all configuration, e.g., for almost face on orbits (low values of i), the pulsar and the black hole would be in the same plane as seen from the observer. Even for $i = 90^\circ$, for a range of the values of the orbital phase, the

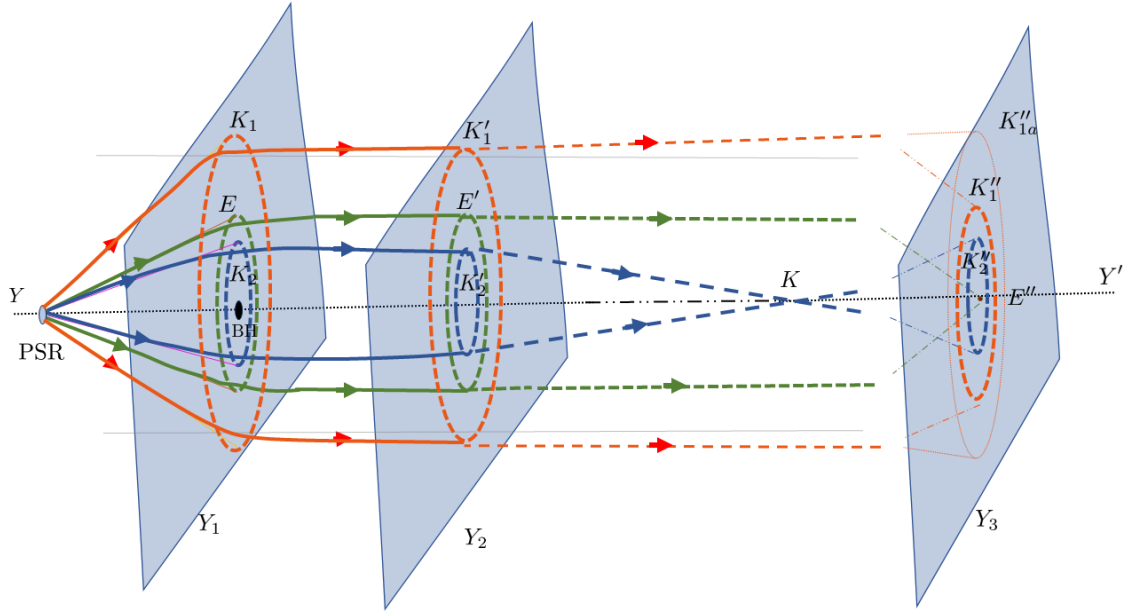


Figure D.1: A schematic diagram representation of the bending and propagation of light rays at the superior conjunction configuration.

pulsar would be between the black hole and the observer. However, such cases are simply explained with the ray tracing and the passage of the Line of sight (LoS) through the beam as explained in detailed in Chapter 3.

Let us consider a set of light rays emitted from the pulsar such that in the absence of the gravitational bending, they would intersect the plane Y_1 on a circle E . In Fig. D.1, only two such light rays are shown for the sake of simplicity. After the experiencing the gravitational bending, these rays become parallel to the line YY' and intersect the plane Y_2 in a circle denoted by E' . This circle E' is smaller than E , as it results from the light rays affected by the bending. However, the size of E' does not change with the position of the plane Y_2 . As we can assume that parallel light rays converge at infinity, let us assume that the rays forming the circle E' meet at the point E'' on the Y_3 -plane. This is a valid assumption as the distance of the observer is much larger than the size of the orbit of the binary. The point E'' is located on the line YY' . In short, all the light rays initially directed towards the circle E converge to the point E'' after bending. The circle E corresponds to the well-known Einstein ring in gravitational lensing theory. The radius of E is known as

the Einstein radius R_E , defined in Chapter 2 (see Eq. (2.48b)).

Next, we consider light rays that are initially directed such that in the absence of bending, they would intersect the plane Y_1 on a circle denoted by K_1 which is larger than E . As before, only two such rays are illustrated in figure. After bending, these rays intersect the plane Y_2 on a circle denoted by K'_1 . Since light rays falling on K_1 are farther from the gravitating object (the black hole) compared to the light rays falling on E , the rays emerging from K'_1 would still be diverging (but less diverging than the original rays that fell on K_1). As a result, the size of K'_1 increases with the increasing distance between the planes Y_1 and Y_2 , and K'_1 is always larger than E' . On the Y_3 -plane, these light rays form a circle K''_{1a} around the point E'' . It is clear that K''_{1a} is larger than K'_1 and K'_1 is larger than K_1 . However, when we assume the Y_3 -plane to be located at the infinity, the light rays diverging from K'_1 can be assumed to be converging at a smaller radius K''_1 (in the same logic parallel rays from E' have been assumed to be converging at the point E'').

Now, we consider another set of light rays that are initially directed such that, in the absence of bending, they would intersect the plane Y_1 on a circle denoted by K_2 which is smaller than E . As before, only two such rays are shown in the figure. After undergoing gravitational bending, these rays intersect the plane Y_2 on the circle K'_2 . Since these light rays are closer to the black hole than the rays that fell on E , they experience stronger bending and thus become converging. As a result, the size of K'_2 decreases with increasing distance between the planes Y_1 and Y_2 . K'_2 remains smaller than E' . These converging light rays meet at a point K on the line YY' such that, K lies between the Y_2 -plane and the Y_3 -plane. After passing through K , these rays become diverging and cut the Y_3 -plane on the circle K''_2 around the point E'' . As the rays that were above the YY' line before K are below the YY' line in the Y_3 -plane (and vice versa), K''_2 might be termed as an inverted image of K_2 . If the size of K_2 is very close to E , then K''_2 would be a small circle around the point E'' . In this situation K''_2 is smaller than K_2 . However, if K_2 is much smaller than E , then the point K would be closer to the Y_2 -plane and K''_2 would be larger than K_2 as the

distance between the Y_2 -plane and the Y_3 -plane is much larger than the distance between the Y_1 -plane and the Y_2 -plane. If K_2 is too small corresponding to an impact parameter smaller than $3\sqrt{3}GM_c/c^2$, the associated light rays would be trapped by the black hole.

We can summarise the above discussions as follows:

1. Rays initially directed towards E become parallel after bending and appear to converge to the point E'' on the plane containing the observer.
2. Rays initially more diverging than those forming E , form a circle K_1'' around E'' on the plane containing the observer.
3. Rays initially less diverging than those forming E form an inverted circle K_2'' around E'' on the plane containing the observer.

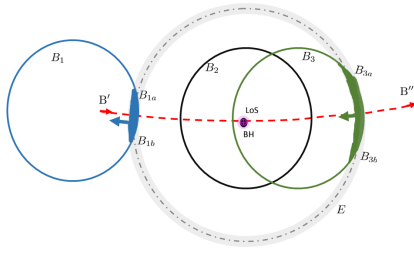
The location of the observer on the Y_3 -plane depends on both the inclination of the orbit and the orbital phase of the pulsar. For example, the observer lies at E'' when the pulsar is exactly at the superior conjunction, i.e., when the pulsar, the black hole, and the observer lie along the straight line YY' . At other orbital phases and different orbital inclinations, the location of the observer shifts away from E'' on the Y_3 -plane.

We are now ready to explain the pulse profiles near superior conjunction based on the motion of the pulsar beam's cross-section on the Y_1 -plane (plane of the paper with the Y_3 -plane lying above it - towards the reader) as illustrated in Fig. D.2. The line of sight (LoS), which is from the pulsar to the observer, intersects the Y_1 plane at the position marked by a small unfilled magenta circle labeled as 'LoS'. In our work, while constructing a pulse profile, instead of counting light rays that falls onto a single point LoS, we consider all the light rays that make an angle less than 0.09° with the LoS on the $X_m Y_m$ -plane. This translates to a finite size of the LoS on the Y_3 plane too, which is projected with a finite size on the Y_1 -plane in Figs. D.2. Hence, the circles K_1 and K_2 becomes annular rings of finite widths as shown in various panels of Fig. D.2.

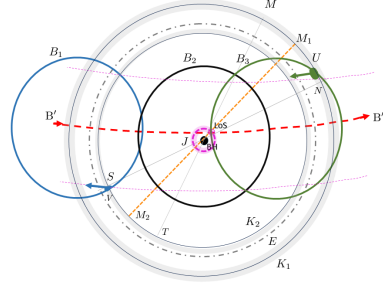
Different panels in Fig. D.2 correspond to different combinations of i and $A_T + \omega$, all chosen to be not very far from the superior conjunction. For these combinations, the value of Einstein ring radius R_E remains fixed at 5.94×10^7 km for the pulsar - super massive black hole binary we are studying. In the observer's rest frame, the centre of the beam of the pulsar moves (due to the spin of the pulsar) along the red dashed line labeled ' $B'B''$ '. The arrows in this line represent the motion of the beam. We call this line as the Line-of-Beam-Rotation (LoBR). Three near-by positions of the cross-section of the beam on the Y_1 -plane are shown with blue (B_1), black (B_2), and green (B_3) circles. We have chosen our parameters α and ζ_L (see Fig. 1.1) in such a way that the LoS falls on the LoBR. However, depending on the values of i and $A_T + \omega$, both the LoBR and LoS shift on the Y_1 -plane with respect to the black hole.

The beam is generated near the pulsar with the radius of the outer cone being 34.49 km (see Chapter 1)). As this conal beam propagates outward, it intersects the Y_1 -plane with a larger radius. For example, in the case pulsar super-massive black hole binary, radius of the beam on the Y_1 -plane is 4.7×10^7 km when the pulsar is at and near the superior conjunction. This implies that the size of the beam's cross-section is slightly smaller than E which has a radius of 5.94×10^7 km for the case of the pulsar - super-massive black hole binary at and near the superior conjunction configuration. If the configuration deviates from the superior conjunction, the radius of the beam on the Y_1 -plane decreases slightly but the radius of E remains the same (as the changes of a_R , e and ω within one orbit is negligible).

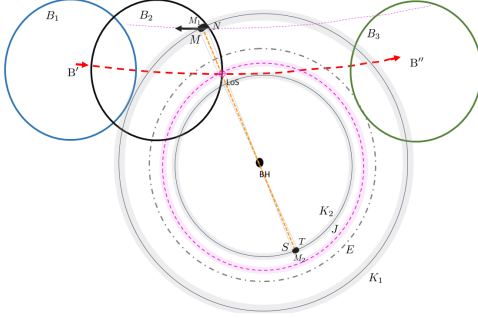
First, we consider the case of the superior conjunction, i.e., $i = 90^\circ$ and $A_T + \omega = 90^\circ$, as shown in Fig. D.2a. At this configuration, the observer is at the point E'' on the Y_3 -plane, i.e., the LoS coincides with the black hole on the Y_1 -plane. The light rays from the portion of the beam intersecting the circle E bend towards the observer (who is at E''). B_1 and B_3 are the two illustrative positions of the beam on two different sides of the circle E . The beam intersects the circle E along the arc $B_1a - B_1b$ for the position



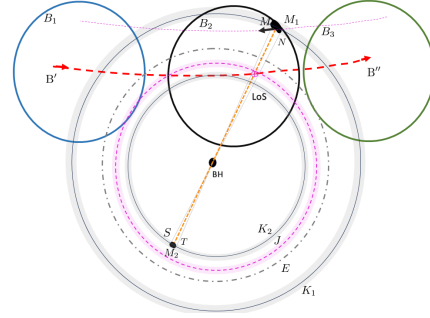
(a) Schematic diagram for $i = 90^\circ$, $A_T + \omega = 90^\circ$



(b) Schematic diagram for $i = 89^\circ$, $A_T + \omega = 90^\circ$.



(c) Schematic diagram for $i = 90^\circ$, $A_T + \omega = 90^\circ$.



(d) Schematic diagram for $i = 85^\circ$, $A_T + \omega = 90^\circ$.

Figure D.2: Understanding strong bending for a pulsar - super massive black hole binary system when the pulsar is near the superior conjunction for different combinations of i and $A_T + \omega$ in the observer's frame, where the centre of the beam moves along the red line ' $B'B''$ ' that passes through the Line-of-Sight ('LoS'). The arrows on ' $B'B''$ ' denotes the direction of motion of the beam. Three positions of beam B_1 , B_2 , and B_3 have been shown in each panel. The magenta ring marked J is the ring on which the light rays from the rings K_1 and K_2 can come due to strong bending as explained in the text. As the magenta ring is chosen to pass through LoS, the observer can receive light rays from the arcs MN on K_1 and ST on K_2 , or more specifically from the segments of those arcs that fall on the beam. The direction of the motion of the segments have been shown with small arrows of the same colour used for that segment. The area of the beam traversed by such a segment results the pulse profile.

B1. All light rays originating from this segment are bent toward the LoS and are received by the observer. As the beam continues to move, this arc sweeps through the beam's width, and the observer receives a highly intense pulse profile due to contributions from the entire beam. When the cross-section of the beam is fully inside E , the observer does not receive any signal (as the light rays fall onto K_2'' on the Y_3 plane which is away from the observer E''). As the beam continues to move, it starts to cut E again. One such position of the beam is $B3$, where the beam intersects the circle E along the arc $B3a - B3b$ from where the observer receives the signal. As the beam keeps on moving, this arc sweeps through the beam's width, again producing a strong signal. As a result of the above facts, at the superior conjunction, the observer detects two highly intense pulse components, symmetrically located around the center of the unbent pulse profile. This explains the structure observed in the pulse profile shown in Fig. 4.15a.

Next, we consider the situation where the pulsar is not at the superior conjunction, but also not very far from it, i.e., the geometry of Fig. D.1 is still valid. In such a situation, the observer would be at a point different than E'' on the Y_3 -plane implying that the point LoS would be shifted from the location of the black hole on the Y_1 -plane. However, as we have already mentioned, we have chosen parameters (α and ζ_L) in such a way that the LoS always falls on the LoBR. The rings K_1 (larger than E) and K_2 (smaller than E) on the Y_1 -plane can be chosen in such a way that the circles K_1'' and K_2'' on the Y_3 -plane coincide. As we aim to explain the pulse profile seen by the observer, we choose K_1 and K_2 rings in such a way that the observer lies on this common ring. As a result, the observer can receive light rays from both K_1 and K_2 provided these rings fall on the beam of the pulsar. The projection of this common ring on the Y_1 -plane is denoted by the ring J ($= K_2'' = K_1''$) which contains the LoS. The radius of J increases as the pulsar moves away from the superior conjunction. However, all the configurations shown in Figs. D.2, J remains smaller than E . Additionally, J is always smaller than K_1 , but can be either larger or smaller than K_2 . Each point on the ring J can receive light from a point on K_1 and a point on K_2 provided those points are on the beam of the pulsar. When this particular

point on J is the LoS, the corresponding point on K_1 is called M_1 and the point on K_2 is called M_2 . The locations of M_1 and M_2 on those rings depend on the location of the LoS on the Y_1 -plane. The finite size of the LoS results in finite sizes of M_1 and M_2 , too.

Now, as we have already seen that the impact of the the black hole's spin on the bending of the light rays is negligible, the light rays remain confined to a plane defined by the the pulsar (the source), the black hole (the gravitating body), and the initial direction vector of the light ray; just as in the case of a Schwarzschild black hole as the binary companion [85]. In Fig. D.1, this plane for any chosen light ray would be the plane containing the YY' -line and the initial direction of the ray, which would intersect the Y_1 -plane. In each panel of Fig. D.2, the saffron dashed line represents this intersection. It is obvious that M_1 , M_2 , LoS, BH, all lie on this line. As we have seen in Fig. D.1, the light rays do not change the side of the YY' -line when they travel from K_1 to K_1'' , M_1 and LoS is located on the same side of the BH in Figs. D.2. On the other hand, as K_2'' is an inverted image, i.e., the light rays swap sides of the YY' -line when they travel from K_2 to K_2'' , M_2 and LoS are located on opposite sides of the BH in Figs. D.2.

In addition to replacing the circles K_1 and K_2 by annular rings, the finite size of the LoS would also make the observer receive signals from an arc MN on K_1 symmetric around M_1 and an arc ST on K_2 symmetric around M_2 . The length of the arc MN depends on the difference in sizes of J and K_1 . Similarly, the length of the arc ST depends on the difference in sizes of J and K_2 . The differences in the sizes of J and K_1 as well as in the sizes of J and K_2 depend on the values of i and $A_T + \omega$, and will be discussed later.

Note that, as the arc ST is far from the LoS (it falls on the part of K_2 located on the other side of the BH), it rarely traverses the beam, contributing insignificantly to the profile. On the other hand, as the arc MN is closer to the LoS which falls on the LoBR, it traverses comparatively larger area of the beam. Hence, the observer at the near superior conjunction receives signal mostly from the arc MN .

When we calculate the values of the bending delay, we consider only the point LoS, and

hence use only M_1 not M_2 as the second one rarely falls on the beam. The difference in the longitudes of M_1 and the LoS in the I -frame of the pulsar (as defined in Chapter 1) gives $\Delta\phi$ needed to calculate the longitudinal bending delay (See Sec. 2.5). Similarly, the difference in the co-latitudes of M_1 and the LoS in the I -frame of the pulsar gives $\Delta\zeta$ needed to calculate the latitudinal bending delay (See Sec. 2.5). This definition of bending delays fail exactly at the superior conjunction when the LoS receives light rays from multiple points at a time (all lying on the arc of the circle E that traverses the beam as seen in D.2a.

1. The geometry for $i = 89^\circ$, $A_T + \omega = 90^\circ$ has been shown in Fig.D.2b. This is closest to the superior conjunction configuration among our three choices of near superior conjunction configurations. In this configuration, light rays from both of the arcs MN and ST can reach the observer (LoS on the Y_1 -plane). When the beam is at the position B_1 , it intersects the arc ST in a small segment labeled SV , from which light rays reach the observer. As the beam moves, the segment SV moves on the beam - traversing a part on the bottom portion of the beam. When the beam is at the location B_2 , it neither cuts K_1 nor K_2 . Hence, the observer does not receive any light. Afterward, when the beam is at the position B_3 , it intersects the arc MN in a segment UN from which the observer receives the rays. As the beam moves, the segment UN moves on the beam - traversing a part on the top portion of the beam. As the length of the segment SV is much smaller than the length of the arc UN , we see a weak component at the beginning of the profile due to the beam area traversed by SV , then zero intensity (when the beam is at B_2) and then a strong single component due to the beam area traversed by UN .
2. The geometry for $i = 90^\circ$, $A_T + \omega = 85^\circ$ has been shown in Fig. D.2c. In this configuration, light rays from both of the arcs MN and ST can reach the observer (LoS on the Y_1 -plane). When the pulsar beam is at the position B_1 , neither the arc MN nor the arc ST falls on the beam, hence the observer does not receive any

signal. After some time, when the beam is at the position B_2 , the arc MN falls on it and the observer receives light from that arc. As the beam moves, the arc MN moves on the beam - traversing a part on the top portion of the beam, from which the observer continues to receive light. When the beam moves to the position B_3 , it stops intersecting the arc MN and the observer does not receive any light anymore. The arc ST never falls on the beam. Hence, we see a shifted pulse profile as shown in Fig. 4.15b. We see a four-component profile as the arc MN travels through the outer and the inner cones of the beam (but not the core).

3. The geometry for $i = 85^\circ$, $A_T + \omega = 90^\circ$ has been shown in Fig. D.2d. It is very similar to the above case, only the location of the LoS with respect to the black hole is different. Here also, when the pulsar beam is at the position B_1 , neither the arc MN nor the arc ST falls on the beam, hence the observer does not receive any signal. After some time, when the beam is the position B_2 , the arc MN falls on it and the observer receives light from that arc. As the beam moves, the arc MN moves on the beam - traversing a part on the top portion of the beam, from which the observer continues to receive light. When the beam moves to the position B_3 , it stops intersecting the arc MN and the observer does not receive any light anymore. The arc ST never falls on the beam. Hence, we see a shifted pulse profile as shown in Fig. 4.16. We see a four-component profile as the arc MN travels through the outer and the inner cones of the beam (but not the core).

D.1 Configurations Other Than Superior Conjunction

Let us now try to understand how the situation changes as the configuration deviates more from the superior conjunction configuration, but only moderately so that the geometry of Fig. D.1 is still valid. As the deviation from the superior conjunction increases, the observer on the Y_3 -plane shifts away from E'' , leading to a larger ring J on the Y_1 -plane.

As we have need to consider $K_2'' = L$ for the LoS to be able to receive light rays from K_2 , the increase in the size of J implies the increase of the size of K_2'' . This is possible if the point K comes closer to the Y_2 -plane, or in other words, if both K_2' and K_2 becomes smaller. As we already mentioned, being on the far side of K_2 from the LoS, the arc ST rarely traverses the beam resulting very little contribution to the profile. This little contribution decreases even more when the size of K_2 itself decreases.

On the other hand, an increase in $L = K_1''$ is accompanied by an increase in the sizes of K_1' and K_1 . However, as the phenomenon of bending is responsible for K_1' being smaller than K_1 and K_1'' being smaller K_1' , the differences in sizes of K_1 , K_1' , K_1'' decreases as K_1 becomes larger. In short, on the Y_1 -plane, the ring K_1 starts coming closer to the ring J . Hence, the length of the arc MN decreases as it comes closer to the LoS, leading to a decrease in the area this arc traverses on the beam (if any). Hence, the contribution of K_1 on the resulting pulse profile also decreases.

The above discussions confirm the fact that as the configuration keeps on deviating from the superior conjunction configuration, the ‘strong bending’ keeps on becoming weaker and the enhancement of the profile strength decreases and the profile starts to show similarity with the unbent profile. Eventually, the strong bending as explained with the help of Fig. D.1 is no longer valid. Little enhancement in the pulse profile due to the weak bending might be still observable, which is the case always for the stellar mass black hole as discussed next.

D.2 Superior Conjunction Configuration with a Stellar-Mass Black Hole

Next we try to understand why such intriguing features are not seen when the binary companion of the pulsar is a stellar mass black hole instead of a super-massive black

hole.

As we have mentioned earlier, in case of the stellar mass black hole, the radius of the circle E is 3.61×10^4 km at and near the superior conjunction as the contribution from varying i slightly is very small and in one orbit, ω is almost constant (see Eq. (2.48b)). Similarly, the radius of the pulsar beam on the Y_1 -plane remains around 1.2×10^6 km at and near the superior conjunction. Hence, in a figure like Fig. D.2, one would basically have E and K_2 as tiny rings around the black hole. On the other hand, J becomes a point (we make it of a finite size of computation purposes) in case of a superior conjunction, however, with a slight deviation from the superior conjunction J becomes larger than E . As an example, for $i = 89^\circ$, $A_T + \omega = 90^\circ$, the radius of J is 2.65×10^5 km.

At the superior conjunction (when the LoS coincides with the black hole), the observer would always receive the light rays from a tiny (with respect to the beam) circle E around the LoS as the beam is larger than E .

At near superior conjunction configurations, the observer would receive almost no light rays from K_2 as it is very tiny. On the other hand, K_1 is larger than E , but as we have already mentioned K_1'' (or J) is smaller than K_1 . As gravitational bending is responsible for this difference in sizes of K_1'' and K_1 , for stellar mass black hole, we can consider them to be of nearly equal size. As a result, the length of the arc MN would be very small (as it is very close to the LoS) and even when it traverses the beam, it would sweep only a very small part, contributing insignificantly to the profile.

Hence, strong bending never happens. Only if the LoS passes close enough to the black hole, it receives some extra light rays due to the weak bending. This explains the difference in the profile shapes of Fig. 4.14 and Fig. 4.15a, for both of the cases, the pulsar is at the superior conjunction, however, for the first case (stellar mass black hole), the profile is much similar to the unbent profile than it is for the second case (super massive black hole).

Bibliography

- [1] Oleg V Doroshenko, Sergei M Kopeikin, and (DK95). Relativistic effect of gravitational deflection of light in binary pulsars. *MNRAS*, 274(4):1029–1038, 1995.
- [2] Roman R. Rafikov, Dong Lai, and (RL06). Effects of pulsar rotation on timing measurements of the double pulsar system J0737-3039. *ApJ*, 641:438–446, 2006. doi: 10.1086/500346.
- [3] J. Schneider. Gravitational beam deviation effects in the timing formula of binary pulsars. *A&A*, 232:62, June 1990.
- [4] Debatri Chattopadhyay, Simon Stevenson, Jarrod R Hurley, Matthew Bailes, and Floor Broekgaarden. Modelling neutron star–black hole binaries: future pulsar surveys and gravitational wave detectors. *Monthly Notices of the Royal Astronomical Society*, 504(3):3682–3710, 04 2021. ISSN 0035-8711. doi: 10.1093/mnras/stab973. URL <https://doi.org/10.1093/mnras/stab973>.
- [5] J. M. Rankin. Toward an empirical theory of pulsar emission. I. Morphological taxonomy. *The Astrophysical Journal*, 274:333–358, November 1983. doi: 10.1086/161450.
- [6] I. Vidaña. A short walk through the physics of neutron stars. *The European Physical Journal Plus*, 133, 10 2018. doi: 10.1140/epjp/i2018-12329-x.
- [7] Amodio Carleo, Delphine Perrodin, and Andrea Possenti. Toward an exact approach

- to pulsar timing. *Phys. Rev. D*, 111(8):083055, April 2025. doi: 10.1103/PhysRevD.111.083055.
- [8] T. Kimpson, K. Wu, and S. Zane. Radio timing in a millisecond pulsar - extreme/intermediate mass ratio binary system. *A&A*, 644:A167, December 2020. doi: 10.1051/0004-6361/202038561.
- [9] Tom Kimpson, Kinwah Wu, and Silvia Zane. Pulsar timing in extreme mass ratio binaries: a general relativistic approach. *MNRAS*, 486(1):360–377, June 2019. doi: 10.1093/mnras/stz845.
- [10] Eva Hackmann and Arnab Dhani. The propagation delay in the timing of a pulsar orbiting a supermassive black hole. *General Relativity and Gravitation*, 51(3):37, March 2019. doi: 10.1007/s10714-019-2517-2.
- [11] Bilel Ben-Salem and Eva Hackmann. Propagation time delay and frame dragging effects of lightlike geodesics in the timing of a pulsar orbiting SgrA*. *MNRAS*, 516(2):1768–1780, October 2022. doi: 10.1093/mnras/stac2337.
- [12] Riccardo Della Monica, Ivan De Martino, and Mariafelicia De Laurentis. Testing space-time geometries and theories of gravity at the Galactic centre with pulsar’s time delay. *MNRAS*, 524(3):3782–3796, September 2023. doi: 10.1093/mnras/stad2125.
- [13] Riccardo Della Monica and Ivan de Martino. Pulsar timing in the Galactic Center. *arXiv e-prints*, art. arXiv:2501.03912, January 2025. doi: 10.48550/arXiv.2501.03912.
- [14] John David Jackson. *Classical Electrodynamics*. Wiley, 1998. ISBN 978-0-471-30932-1.
- [15] D. R. Lorimer and M. Kramer. *Handbook of Pulsar Astronomy*, volume 4 of *Cam-*

bridge Observing Handbooks for Research Astronomers. Cambridge University Press, Cambridge, 2004.

- [16] Arnim J. Deutsch. The electromagnetic field of an idealized star in rigid rotation in vacuo. *Annales d'Astrophysique*, 18:1, January 1955.
- [17] Peter Goldreich and William H. Julian. Pulsar electrodynamics. *The Astrophysical Journal*, 157:869–880, August 1969. doi: 10.1086/150119.
- [18] M. A. Ruderman and P. G. Sutherland. Theory of pulsars: Polar gaps, sparks, and coherent microwave radiation. *The Astrophysical Journal*, 196:51–72, February 1975. doi: 10.1086/153393.
- [19] P. A. Sturrock. A model of pulsars. *The Astrophysical Journal*, 164:529, March 1971. doi: 10.1086/150865.
- [20] J. Arons. Pair creation above pulsar polar caps: Geometrical structure and energetics of slot gaps. *The Astrophysical Journal*, 266:215–241, March 1983. doi: 10.1086/160771.
- [21] J. A. Hirschman and J. Arons. Pair-production multiplicities in rotation-powered pulsars. *The Astrophysical Journal*, 560:871–884, 2001. doi: 10.1086/323069.
- [22] Jr. Arendt, Paul N. and Jean A. Eilek. Pair creation in the pulsar magnetosphere. *The Astrophysical Journal*, 581(1):451–469, December 2002. doi: 10.1086/344133.
- [23] J. M. Cordes. Observational limits on the location of pulsar emission regions. *The Astrophysical Journal*, 222:1006–1011, June 1978. doi: 10.1086/156218.
- [24] Jaroslaw Kijak and Janusz Gil. Radio emission regions in pulsars. *Monthly Notices of the Royal Astronomical Society*, 299(3):855–861, September 1998. doi: 10.1046/j.1365-8711.1998.01832.x.
- [25] J. Kijak and J. Gil. Radio emission altitude in pulsars. *Astronomy & Astrophysics*, 397:969–972, January 2003. doi: 10.1051/0004-6361:20021583.

- [26] K. S. Cheng, C. Ho, and M. Ruderman. Energetic Radiation from Rapidly Spinning Pulsars. I. Outer Magnetosphere Gaps. *ApJ*, 300:500, January 1986. doi: 10.1086/163829.
- [27] G. Srinivasan. Recycled pulsars. *New Astronomy Reviews*, 54(3-6):93–100, March 2010. doi: 10.1016/j.newar.2010.09.026.
- [28] V. Radhakrishnan and D. J. Cooke. Magnetic poles and the polarization structure of pulsar radiation. *Astrophysics Letters*, 3:225, January 1969.
- [29] D. I. Jones. Pulsar state switching, timing noise and free precession. *Monthly Notices of the Royal Astronomical Society*, 420(3):2325–2338, March 2012. doi: 10.1111/j.1365-2966.2011.20238.x.
- [30] J. Gil, P. Gronkowski, and W. Rudnicki. Geometry of the emission region of PSR 0950+08. *Astronomy and Astrophysics*, 132:312–316, March 1984.
- [31] M. M. Komesaroff. Possible mechanism for the pulsar radio emission. *Nature*, 225(5233):612–614, February 1970. doi: 10.1038/225612a0.
- [32] D. C. Backer. Pulsar average wave forms and hollow-cone beam models. *The Astrophysical Journal*, 209:895–907, November 1976. doi: 10.1086/154788.
- [33] A. G. Lyne and R. N. Manchester. The shape of pulsar radio beams. *Monthly Notices of the Royal Astronomical Society*, 234:477–508, October 1988. doi: 10.1093/mnras/234.3.477.
- [34] H. G. Wang, F. P. Pi, X. P. Zheng, C. L. Deng, S. Q. Wen, F. Ye, K. Y. Guan, Y. Liu, and L. Q. Xu. A Fan Beam Model for Radio Pulsars. I. Observational Evidence. *The Astrophysical Journal*, 789(1):73, July 2014. doi: 10.1088/0004-637X/789/1/73.
- [35] J. A. Gil, J. Kijak, and J. H. Seiradakis. On the two-dimensional structure of pulsar beams. *Astronomy & Astrophysics*, 272:268–276, May 1993.

- [36] J. H. Seiradakis, J. A. Gil, D. A. Graham, A. Jessner, M. Kramer, V. M. Malofeev, W. Sieber, and R. Wielebinski. Pulsar profiles at high frequencies. I. The data. *Astronomy and Astrophysics Supplement Series*, 111:205, June 1995.
- [37] Krzysztof Belczynski, Tomasz Bulik, and Charles Bailyn. The Fate of Cyg X-1: An Empirical Lower Limit on Black-hole-Neutron-star Merger Rate. *The Astrophysical Journal Letters*, 742(1):L2, November 2011. doi: 10.1088/2041-8205/742/1/L2.
- [38] J. G. Hills. The effects of sudden mass loss and a random kick velocity produced in a supernova explosion on the dynamics of a binary star of arbitrary orbital eccentricity. Applications to X-ray binaries and to the binary pulsars. *The Astrophysical Journal*, 267:322–333, April 1983. doi: 10.1086/160871.
- [39] T. M. Tauris and R. J. Takens. Runaway velocities of stellar components originating from disrupted binaries via asymmetric supernova explosions. *Astronomy & Astrophysics*, 330:1047–1059, February 1998.
- [40] Niel Brandt and Philipp Podsiadlowski. The effects of high-velocity supernova kicks on the orbital properties and sky distributions of neutron-star binaries. *Monthly Notices of the Royal Astronomical Society*, 274(2):461–484, May 1995. doi: 10.1093/mnras/274.2.461.
- [41] A.G. Lyne, R.S. Pritchard, and F. Graham-Smith. Pulsar polarimetry and the geometry of the pulsar magnetosphere. *Monthly Notices of the Royal Astronomical Society*, 268:845–854, 1994.
- [42] P.A. Sturrock and S. Shibata. A model of the pulsar polar cap with non-corotational magnetic fields. *The Astrophysical Journal*, 588(1):405–413, 2003.
- [43] J. Pétri, J. Pétri, and R.W. Romani. Pulsar current closure and acceleration in general relativistic magnetospheres. *The Astrophysical Journal*, 632(2):1152–1165, 2005.

- [44] O. Kargaltsev, G.G. Pavlov, and J. Hare. Pulsar-wind nebulae in the chandra era. *Proceedings of the International Astronomical Union*, 4(S259):91–104, 2009.
- [45] D.A. Frail, E. Clausen-Brown, and A.N. Lommen. Pulsar emission geometry from wide-band polarization profiles. *The Astrophysical Journal*, 739(2):93–102, 2011.
- [46] I. Contopoulos, C. Kalapotharakos, and D. Kazanas. The pulsar synchrotron model revisited. *The Astrophysical Journal*, 765(2):131–140, 2013.
- [47] A.A. Abdo, M. Ackermann, M. Ajello, W.B. Atwood, L. Baldini, M.G. Baring, P.N. Bhat, S. Brandt, P. Bruel, J. Chiang, and et al. A decade of fermi-lat observations of pulsars. *The Astrophysical Journal*, 896(2):91, 2020.
- [48] G. Ramesh, W.-f. Fong, and P. Ghosh. High-energy pulsar emission: Models and observations. *Journal of High Energy Astrophysics*, 33:42–54, 2022.
- [49] X. Chen, H. Feng, and N. Rea. Observational insights into pulsar magnetospheres. *The Astrophysical Journal*, 951(1):87, 2023.
- [50] J.L. Han, Y. Wang, and Y. Lu. Magnetospheric models of pulsars and their emission mechanisms. *The Astrophysical Journal*, 908(2):123, 2021.
- [51] L. Pan, X. Li, and Y. Xie. Pulsar timing and geometry: Current status and future prospects. *Monthly Notices of the Royal Astronomical Society*, 505(3):3792–3805, 2021.
- [52] A.K. Harding, A. Petrukhin, and M.G. Baring. Electromagnetic emission from pulsar magnetospheres. *The Astrophysical Journal*, 790(2):128, 2014.
- [53] A. González-Galán, E. Kuulkers, P. Kretschmar, S. Larsson, K. Postnov, A. Kochetkova, and M. H. Finger. Spin period evolution of GX 1+4. *A&A*, 537:A66, January 2012. doi: 10.1051/0004-6361/201117893.
- [54] A. La Barbera, A. Baushev, C. Ferrigno, S. Piraino, A. Santangelo, A. Segreto, M. Orlandini, P. Kretschmar, I. Kreykenbohm, J. Wilms, R. Staubert, W. Coburn,

and W. A. Heindl. A Study of Cen X-3 as Seen by INTEGRAL. In V. Schoenfelder, G. Lichti, and C. Winkler, editors, *5th INTEGRAL Workshop on the INTEGRAL Universe*, volume 552 of *ESA Special Publication*, page 337, October 2004.

- [55] A. E. Roy. *Orbital Motion*. Institute of Physics Publishing, 3rd revised and enlarged edition, 1988. URL <https://ui.adsabs.harvard.edu/abs/1988ah1...book.....R>. Provided by the SAO/NASA Astrophysics Data System.
- [56] P. C. Peters. Gravitational Radiation and the Motion of Two Point Masses. *Physical Review*, 136(4B):1224–1232, November 1964. doi: 10.1103/PhysRev.136.B1224.
- [57] B. M. Barker and R. F. O’Connell. Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments. *Phys. Rev. D*, 12(2):329–335, July 1975. doi: 10.1103/PhysRevD.12.329.
- [58] Manjari Bagchi. Prospects of Constraining the Dense Matter Equation of State from Timing Analysis of Pulsars in Double Neutron Star Binaries: The Cases of PSR J0737 - 3039A and PSR J1757 - 1854. *Universe*, 4(2):36, 2018. doi: 10.3390/universe4020036.
- [59] Manjari Bagchi. Rotational parameters of strange stars in comparison with neutron stars. *Nature*, 15(1):126–134, January 2010. doi: 10.1016/j.newast.2009.07.003.
- [60] T. Damour and G. Schafer. Higher-order relativistic periastron advances and binary pulsars. *Nuovo Cimento B Serie*, 101B(2):127–176, January 1988. doi: 10.1007/BF02828697.
- [61] Michele Maggiore. *Gravitational Waves. Vol. 1: Theory and Experiments*. Oxford University Press, 2007. ISBN 978-0-19-171766-6, 978-0-19-852074-0. doi: 10.1093/acprof:oso/9780198570745.001.0001.
- [62] Roman R. Rafikov and Dong Lai. Effects of gravitational lensing and companion

motion on the binary pulsar timing. *Phys. Rev. D*, 73(6):063003, March 2006. doi: 10.1103/PhysRevD.73.063003.

- [63] H. Hu, M. Kramer, D. J. Champion, N. Wex, A. Parthasarathy, T. T. Pennucci, N. K. Porayko, W. van Straten, V. Venkatraman Krishnan, M. Burgay, P. C. C. Freire, R. N. Manchester, A. Possenti, I. H. Stairs, M. Bailes, S. Buchner, A. D. Cameron, F. Camilo, and M. Serylak. Gravitational signal propagation in the double pulsar studied with the MeerKAT telescope. *A&A*, 667:A149, November 2022. doi: 10.1051/0004-6361/202244825.
- [64] M. Kramer, I. H. Stairs, R. N. Manchester, N. Wex, A. T. Deller, W. A. Coles, M. Ali, M. Burgay, F. Camilo, I. Cognard, T. Damour, G. Desvignes, R. D. Ferdman, P. C. C. Freire, S. Grondin, L. Guillemot, G. B. Hobbs, G. Janssen, R. Karuppusamy, D. R. Lorimer, A. G. Lyne, J. W. McKee, M. McLaughlin, L. E. Münch, B. B. P. Perera, N. Pol, A. Possenti, J. Sarkissian, B. W. Stappers, and G. Theureau. Strong-Field Gravity Tests with the Double Pulsar. *Physical Review X*, 11(4):041050, October 2021. doi: 10.1103/PhysRevX.11.041050.
- [65] S. Chandrasekhar. *The Mathematical Theory of Black Holes*. Oxford University Press, New York, 1983. The International Series of Monographs on Physics. Series editors: R. J. Elliot, J. A. Krumhuns, and D. H. Wilkinson.
- [66] Juri Poutanen. Accurate analytic formula for light bending in schwarzschild metric. *aap*, 640:A24, Aug 2020. ISSN 1432-0746. doi: 10.1051/0004-6361/202037471. URL <http://dx.doi.org/10.1051/0004-6361/202037471>.
- [67] Sergei M. Kopeikin and Gerhard Schäfer. Lorentz covariant theory of light propagation in gravitational fields of arbitrary-moving bodies. *Phys. Rev. D*, 60(12):124002, December 1999. doi: 10.1103/PhysRevD.60.124002.
- [68] Daniel Kapec and Alexandru Lupasca. Particle motion near high-spin black holes. *Class. Quant. Grav.*, 37(1):015006, 2020. doi: 10.1088/1361-6382/ab519e.

- [69] Samuel E. Gralla and Alexandru Lupsasca. Null geodesics of the kerr exterior. *Phys. Rev. D*, 101:044032, 02 2020. doi: 10.1103/PhysRevD.101.044032. URL <https://link.aps.org/doi/10.1103/PhysRevD.101.044032>.
- [70] Claudio F. Paganini, Blazej Ruba, and Marius A. Oancea. Characterization of Null Geodesics on Kerr Spacetimes. *arXiv e-prints*, art. arXiv:1611.06927, November 2016. doi: 10.48550/arXiv.1611.06927.
- [71] A. M. Ghez, S. Salim, S. D. Hornstein, A. Tanner, J. R. Lu, M. Morris, E. E. Becklin, and G. Duchêne. Stellar Orbits around the Galactic Center Black Hole. *ApJ*, 620 (2):744–757, February 2005. doi: 10.1086/427175.
- [72] S. Gillessen, F. Eisenhauer, S. Trippe, T. Alexander, R. Genzel, F. Martins, and T. Ott. Monitoring stellar orbits around the Massive Black Hole in the Galactic Center. *Astrophys. J.*, 692:1075–1109, 2009. doi: 10.1088/0004-637X/692/2/1075.
- [73] M. Grould, F. H. Vincent, T. Paumard, and G. Perrin. General relativistic effects on the orbit of the S2 star with GRAVITY. *A&A*, 608:A60, December 2017. doi: 10.1051/0004-6361/201731148.
- [74] GRAVITY Collaboration, R. Abuter, A. Amorim, N. Anugu, M. Bauböck, M. Benisty, J. P. Berger, N. Blind, H. Bonnet, W. Brandner, A. Buron, C. Collin, F. Chapron, Y. Clénet, V. Coudé Du Foresto, P. T. de Zeeuw, C. Deen, F. Delplancke-Ströbele, R. Dembet, J. Dexter, G. Duvert, A. Eckart, F. Eisenhauer, G. Finger, N. M. Förster Schreiber, P. Fédou, P. Garcia, R. Garcia Lopez, F. Gao, E. Gendron, R. Genzel, S. Gillessen, P. Gordo, M. Habibi, X. Haubois, M. Haug, F. Haußmann, Th. Henning, S. Hippler, M. Horrobin, Z. Hubert, N. Hubin, A. Jimenez Rosales, L. Jochum, K. Jocu, A. Kaufer, S. Kellner, S. Kendrew, P. Kervella, Y. Kok, M. Kulas, S. Lacour, V. Lapeyrère, B. Lazareff, J.-B. Le Bouquin, P. Léna, M. Lippa, R. Lenzen, A. Mérand, E. Müller, U. Neumann, T. Ott, L. Palanca, T. Paumard, L. Pasquini, K. Perraut, G. Perrin, O. Pfuhl, P. M. Plewa, S. Rabien, A. Ramírez,

- J. Ramos, C. Rau, G. Rodríguez-Coira, R.-R. Rohloff, G. Rousset, J. Sanchez-Bermudez, S. Scheithauer, M. Schöller, N. Schuler, J. Spyromilio, O. Straub, C. Straubmeier, E. Sturm, L. J. Tacconi, K. R. W. Tristram, F. Vincent, S. von Fellenberg, I. Wank, I. Waisberg, F. Widmann, E. Wieprecht, M. Wiest, E. Wieszorrek, J. Woillez, S. Yazici, D. Ziegler, and G. Zins. Detection of the gravitational redshift in the orbit of the star S2 near the Galactic centre massive black hole. *A&A*, 615: L15, July 2018. doi: 10.1051/0004-6361/201833718.
- [75] Dinesh Singh, Kinwah Wu, and Gordon E. Sarty. Fast spinning pulsars as probes of massive black holes’ gravity. *MNRAS*, 441(1):800–808, June 2014. doi: 10.1093/mnras/stu614.
- [76] Kaye Jiale Li, Kinwah Wu, and Dinesh Singh. Spin dynamics of a millisecond pulsar orbiting closely around a massive black hole. *MNRAS*, 485(1):1053–1066, May 2019. doi: 10.1093/mnras/stz389.
- [77] Kaye Jiale Li, Kinwah Wu, Po Kin Leung, and Dinesh Singh. Relativistic scattering of a fast spinning neutron star by a massive black hole. *MNRAS*, 511(3):3602–3617, April 2022. doi: 10.1093/mnras/stab2925.
- [78] János Majár. Spin-spin interaction in the spin-precession equations. *Phys. Rev. D*, 80(10):104028, November 2009. doi: 10.1103/PhysRevD.80.104028.
- [79] János Majár and Balázs Mikóczi. Second order spin effects in the spin precession of compact binaries. *Phys. Rev. D*, 86(6):064028, September 2012. doi: 10.1103/PhysRevD.86.064028.
- [80] H. Hu, M. Kramer, D. J. Champion, N. Wex, A. Parthasarathy, T. T. Pennucci, N. K. Porayko, W. van Straten, V. Venkatraman Krishnan, M. Burgay, P. C. C. Freire, R. N. Manchester, A. Possenti, I. H. Stairs, M. Bailes, S. Buchner, A. D. Cameron, F. Camilo, and M. Serylak. Gravitational signal propagation in the double pulsar

studied with the MeerKAT telescope. *A&A*, 667:A149, November 2022. doi: 10.1051/0004-6361/202244825.

- [81] Sergei Kopeikin and Bahram Mashhoon. Gravitomagnetic effects in the propagation of electromagnetic waves in variable gravitational fields of arbitrary-moving and spinning bodies. *Phys. Rev. D*, 65(6):064025, March 2002. doi: 10.1103/PhysRevD.65.064025.
- [82] Jayant V. Narlikar. *The Schwarzschild solution*, page 126–142. Cambridge University Press, 2010.
- [83] Claudio F. Paganini, Blazej Ruba, and Marius A. Oancea. Characterization of Null Geodesics on Kerr Spacetimes. *arXiv e-prints*, art. arXiv:1611.06927, November 2016. doi: 10.48550/arXiv.1611.06927.
- [84] Xiaolin Yang and Jiancheng Wang. YNOGK: A New Public Code for Calculating Null Geodesics in the Kerr Spacetime. *The Astrophysical Journal Supplement Series*, 207(1):6, 07 2013. doi: 10.1088/0067-0049/207/1/6.
- [85] Jyotijwal Debnath, Manjari Bagchi, Avishek Basu, and (Paper-I). A study of the light bending phenomenon under full general relativity for a pulsar in a binary with a Schwarzschild black hole. *MNRAS*, 524(4):5411–5431, October 2023. doi: 10.1093/mnras/stad2147.