Aspects of interplay between gravity and quantum information

By

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Abstract

In recent years, significant advancements have emerged at the intersection of quantum information theory and gravitational physics. These developments encompass a variety of topics, including but not limited to the AdS/CFT correspondence and the Hawking information paradox. Within this domain, this thesis presents work that focuses on research that has been inspired by recent insights related to the island formula, as well as applications stemming from algebraic quantum field theory.

The work comprises of two papers, the first of which addresses the notion of theory dependence in the reconstruction of black hole interiors utilizing Hawking radiation. This work is set in the context of a two-sided black hole that is coupled to a thermal bath, specifically in the framework of Jackiw-Teitelboim gravity. The question asked is of how theory dependence influences reconstruction efforts, framing it through the lens of the island formula. The results tell us that at extremely late times, reconstructing the interior of a black hole using Hawking radiation necessitates an understanding of the microscopic details of the theory, which can be interpreted as a dependence on the ultraviolet details of a quantum theory of gravity.

In the next work, I introduce an operator decomposition formula derived from the Zassenhaus decomposition applicable to unbounded operators within two-dimensional conformal field theory in Minkowski spacetime. This work uses half-sided translations — a construction that has been important in recent theoretical developments – as the prototypical example. The derivation of this result requires formulation of a regularization pro-

cedure aimed at transforming ill-defined operators, such as density matrices in quantum field theory, into well-defined entities. Furthermore, a 'centred' version of the Zassenhaus formula is derived to achieve the result. It is demonstrated how this decomposition can be applied to an infinite class of operators, and the governing differential equations that describe the processes are also derived.

Chapter 1

Introduction

1.1 Hawking's information paradox

Einstein's theory of general relativity (GR), the crown jewel of modern theoretical physics, revolutionized humanity's understanding of what we call space and time. Prior to Einstein, space and time were thought to provide a fixed, stationary, arena for physical entities to interact amongst each other, quite like how a stage functions in an act of drama. General relativity however taught us, contrary to the Newtonian understanding, that space and time are inseparable elements of a single entity called "spacetime", which is the supreme actor in the act of nature, that affects, and gets affected, by all the matter. A central lesson of the theory is that gravitation is better understood, not as a force, but as an effect of curvature of spacetime.

After a century of rigorous research in general relativity, we have acquired substantial knowledge about the universe. We now know that there exist stellar bodies called supermassive black holes, believed to be present at the centers of galaxies. We know of the phenomena of black hole mergers, in which gravitational waves – ripples in spacetime – get emitted, which have been experimentally detected on our planet. We also know that the thermodynamic entropy of a black hole, very surprisingly[1], only depends on its area, and that the only way for second law of thermodynamics to hold in presence of black holes is that they must evaporate.[2] We have mobile phones using satellite transmissions, which rely on time dilation effects, and it is why we have the internet. To make the point, we have learnt a lot from general relativity.

However, there are some questions concerning gravitation, a few as old as general relativity itself [3], that continue to mystify and baffle us. These chiefly concern the theory of gravitational interactions when matter densities are so large that general relativity predicts singularities in the associated spacetimes. This occurs most commonly inside black holes. Specifically, it can be observed in the Schwarzschild black hole through the use of the Kretschmann scalar, which diverges as one approaches the center of the black hole. Such high densities are also expected at the early stages of the universe when it was very small, and we do not fully understand the physics of that period. Further, a question that has persisted even after fifty years of its discovery is the apparent information loss observed by Hawking during black hole evaporation. Today, these problems are well-known in the high-energy physics community for their connection with quantum mechanics, and they are considered key signatures of the limitations of a classical theory of gravitation.

The Hawking information paradox in particular, in the recent years, has been the subject of many intense research programmes that have lead to significant advancements at the intersection of quantum information theory and gravitational physics. The paradox was discovered in mid 1970s, starting with Hawking's demonstration that black holes emit thermal radiation. This finding was particularly surprising because (i) it indicated that black holes are not eternal and eventually evaporate, and (ii) because thermal radiation does not retain detailed information about its source; the only extractable information

[1]Thermodynamic entropy is ordinarily an extensive quantity, which means that it scales with the volume of the system. However, for black holes, the scaling is with area.

[2]Usual black holes are colder than the ambient temperature of universe due to the cosmic microwave backgroud, hence they cannot radiate. However, hotter primordial black holes are being investigated for experimental detection of Hawking radiation.

[3]In the 1920s, it was already an interesting question to ask what effect spacetime would have on the predictions quantum mechanics makes for small objects.

from the radiation is the temperature of the emitting body.

Hawking's results led to a paradoxical situation within the framework of quantum mechanics because of the following reason. If a black hole evaporates completely, with the outgoing radiation not carrying any microscopic information about the matter that fell into the black hole, that information is lost forever. More specifically, the initial quantum state of the black hole when regarded as pure, the final state after evaporation consequently would be a mixed state. As quantum mechanics says an isolated system should evolve unitarily, this is a violation of unitarity. Whether this process is realized in nature or not is only one of the puzzles in the context of black hole evaporation process, and a detailed account of the paradox is given in [1]. However, the most recent progress on the problem, reviewed in [2], has emerged from studying toy models of black hole evaporations in AdS/CFT.

1.2 In the intersection of gravity and quantum information

The last two decades of research in AdS/CFT duality – the statement that any conformal field theory on $\mathbb{R} \times \mathbb{S}^{d-1}$ is equivalent to a quantum gravity theory (string theory) in a $d + 1$ dimensional asymptotically anti-de-Sitter space – have proven instrumental for our understanding of the interplay between quantum information and gravity. This can be significantly attributed to a few monumental works such as the Ryu-Takayanagi formula, bulk spacetime arising through entanglement in the boundary theory, and entanglement wedge reconstruction, among others. Further more, the developments have been able to provide a better of understanding of the Hawking's information paradox, through what is known as the 'island paradigm'. Below I will review a few key developments in this field that will be important for this thesis.

1.2.1 The Ryu-Takayanagi formula, quantum extremal surfaces, and the island paradigm

One of the most significant advancements in the AdS/CFT duality framework has been the discovery of the Ryu-Takayanagi formula [3]. The formula relates the entanglement entropy of a region in the holographic conformal field theory A, at leading order in G_N, to a geometric quantity in the bulk spacetime, specifically, the area of a minimal surface in the bulk spacetime that is anchored on the boundary of that region. Mathematically, it can be expressed as:

$$S_{\mathrm{CFT}}(A) = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \tag{1.2.1}$$

This result is applicable for time independent states, and was later generalized to time dependent states by Hubeny, Rangamani, and Takayanagi. Further, after including leading quantum corrections by Engelhardt, Faulkner, Lewkowcyz, Maldacena and Wall [4, 5, 6], it was upgraded to what is called the quantum extremal surface formula.

The quantum extremal surface formula is a generalization of the Ryu-Takayanagi formula that defines what is known as the entanglement wedge of a region A in the conformal field theory on the boundary to a bulk region W_A. The entanglement wedge plays a central role in understanding the bulk physics from the boundary theory, through the programme of 'bulk reconstruction' (see chapter 2 for details.) The quantum extremal surface formula can be expressed as:

$$S_{\mathrm{CFT}}(A) = \min_{\gamma_A}\left(\frac{\mathrm{Area}(\gamma_A)}{4G_N} + S_{\mathrm{bulk}}(W_A)\right) \tag{1.2.2}$$

This formula has proved to be central to the understanding of black hole information paradox[4], through what is known as the 'island paradigm' .

[4]While the AdS spacetime is much different from our own universe, it can contain black holes, and several features of black holes are universal, such as the presence of event horizons and Hawking radiation.

An island I, known to most prominently occur in 2d models of gravity in AdS spacetimes, is a special kind of entanglement wedge that is disconnected from boundary. As discussed above, to a boundary region A, there belongs an entanglement wedge W_A in the bulk that is anchored on A. One therefore would ask, to what 'system' does an island belong, if its not anchored on any part of the boundary? This can be answered fairly generally. Suppose that the spacetime, possibly containing black holes, is coupled to a system χ. Then the gravitational entropy of χ, $\mathbf{S}(\chi)$, in parallel with (1.2.2) is given by [7, 8, 9]

$$\mathbf{S}(\chi) = \min_I \left[\text{ext}_I \left(\frac{A(\partial I)}{4G_N} + S_{\text{bulk}}(\chi \cup I) \right) \right]. \tag{1.2.3}$$

Here I is a codimension-1 region in the bulk, called an island, S_{bulk} is the usual von Neumann entropy computed on curved spacetime, and $A(\partial I)$ is the area of its boundary. This island I then is the entanglement wedge for the system χ, and the island formula is a generalization of the Ryu-Takayanagi formula to include islands.

Now I can state the island paradigm in connection with the Hawking paradox. As I will discuss in chapter 2, even though the AdS spacetime are much different from our universe, with the help of a suitably coupled system χ[5], one can realize the Hawking's information paradox in 2d toy models of gravity and in fact can be resolved [10, 7, 11, 9]. I will now write the central statement of this island paradigm.

Island paradigm in the context of Hawking like paradoxes in AdS/CFT:

After the Page time of a Hawking evaporation process for a black hole, a region called 'island' emerges in the black hole interior, which is the entanglement wedge of the Hawking radiation, and in principle, bulk reconstruction can be done in this island using only the radiation degrees of freedom.

[5]This system χ is usually a kind of a energy reservoir, which can collect Hawking radiation, such a thermal bath $\mathcal{B}$ which I will describe below.

Apart from the island paradigm, in the last few years, another set of important developments has appeared in the intersection of gravity and quantum information. These are the application of ideas from algebraic quantum field theory (AQFT), and have given us clues to address the problem of defining operators in a theory with gravity.

1.2.2 Insights from algebraic quantum field theory and going beyond AdS/CFT

Firstly, from the work of Witten [12], in the recent years it has become a well known fact that algebra of operators involved in quantum field theory are of type III_1[6] and that entanglement entropy and operator algebras in quantum field theory are deeply interconnected. This proves relevant for holography as well, because a theory of gravity at low energies is a quantum field theory on a curved spacetime, and thus one can use AQFT tools to understand the holography in the limit $G_N \to 0$.

A critical study following the AQFT methods was done by Leutheusser and Liu [13], [14]. They showed emergence of sharp black hole horizons in AdS/CFT as a consequence of this type III nature, along with a boundary construction of bulk slices that can enter the black hole (infalling time evolution). These results were derived using the construction of the so-called half-sided translations, which are known to exist specifically only for type III_1 algebras, which means, for spacetimes of fixed curvature with $G_N \to 0$. Half-sided translations will be important for this thesis and I will return to them shortly.

Following LL, Witten in [15] showed that by considering perturbative G_N corrections in the canonical picture to the type III_1 algebra[7], or a little differently for a microcanonical picture with Chandrasekaran and Penington [16], the pertinent algebra becomes type II_∞.

[6] cf. chapter 4 for a defintion.

[7] To be precise, the algebra of single trace operators, which is dual to operators in the black hole exteriors via causal wedge reconstruction.

Crucially, in [16], the authors also derived the quantum extremal surface formula through algebraic methods, as an "emergent description of entanglement entropy for certain CFT states", in contrast to the earlier derivation of Faulkner Lewkowycz and Maldacena [6, 5] that uses Euclidean path integrals.

These works have had a cascading effect in the field into various directions, with one direction of going even beyond the AdS spacetime and into our universe. To that end, [17] describes an algebra of operators for the static patch in the de Sitter space to be of type II_1, and extends the arguments of type II_∞ algebras of operators, described in [15, 16], for the black hole exterior from asymptotically AdS spaces to asymptotically flat spaces. In fact, this was followed by an attempt to move beyond any fixed background and define algebras for dynamical gravity [18, 19], and it is an open question whether one could succeed in describing gravity this way. Thus we can see that algebraic quantum field theory methods, intimately connected to quantum entanglement, are becoming increasingly significant for understanding the nature of quantum gravity with or without holography, at least in the limit of $G_N \to 0$.

1.3 Thesis work

This work in this thesis is inspired by some of the developments listed above. The first project in this thesis focuses on understanding the black hole interior through the lens of quantum extremal surfaces and the island paradigm, and the third is inspired by the recent developments using the methods of algebraic quantum field theory. Below I will describe them and provide a brief summary of the results.

1.3.1 Understanding the black hole interior through radiation using partial information

This work, [20], builds on the work of Almheiri and Lin [9] who study the black hole interior in the Jackiw-Teitelboim gravity model (reviewed in chapter 2) with an unusual novelty – the model is given a set of random couplings. These random couplings are introduced in the model to emulate the unknown ultraviolet couplings of a quantum gravity theory. Their goal was of understanding the impact of the randomness on the reconstructability of the black hole interior, in the sense of bulk reconstruction mentioned above. With this motivation, their precise question was whether these random couplings are required to do entanglement wedge reconstruction inside the black hole. They find that while at early times the couplings are not important, at $t \sim O(e^{2S_{\rm BH}})$ they do become important, because a region emerges in the black hole interior, that is 'forbidden' from being reconstructed because of the random couplings. This region is the example of the 'island' mentioned above.

In our work, we adopt their model for unknown ultraviolet couplings and ask whether another (related) process requires the knowledge of the random couplings for completion: the interior reconstruction for a black hole attached to a thermal bath, using the bath radiation. This is promised to be doable by the island paradigm [11]. Our precise question in this work becomes whether a forbidden island due to random couplings appears again in the black hole interior, replacing the bath island.

We will see that this question is a Hawking-like paradox for 3 systems with unboundedly growing entanglement entropies. These three systems are the black hole, the bath $\mathcal{B}$, and a special agent $\mathcal{J}$ that has 'ownerships' of the forbidden island in [9], in the sense that island is the entanglement wedge of $\mathcal{J}$ (this will become clearer in chapter 3). The answer thus would come via asking, for what times $\mathcal{B}$ and $\mathcal{J}$ have entanglement wedges in the black hole interior, if any.

This question, when analyzed, required the following two steps: (i) computing the quantum extremal surfaces for all three systems, and (ii) selecting those that resolve all the paradoxes. The step (i) involves computations of boundary conformal field theory (BCFT)[8] correlation functions with changing boundary conditions. In step (ii) of selecting surfaces, we discovered a surprising result that the unique way to resolve the three paradoxes *does not coincide* with the choice of least entropy solutions – a result that goes against the intuition of quantum extremal surfaces. We show that the combined system undergoes two Page transitions, first at $t \sim O(1/G_N)$, when an island emerges that is owned by the thermal bath system $\mathcal{B}$ inside the black hole interior, and a second Page transition at $t \sim \exp(2S_{\rm BH})$, when similar to [9], a forbidden island due to random couplings appears, and in fact, replaces the island for the bath.

1.3.2 A stitching protocol for operators influencing different regions

In this work, I study the half-sided translations associated to Rindler wedge algebras for conformal field theories in 1+1 Minkowski spacetime, generated by an unbounded operator $\mathcal{G}$, in terms of quadratic forms G, G' made from entanglement Hamiltonians of the underlying algebras such that $\mathcal{G} = G + G'$. As I will explain in chapter 5, a quadratic form, is the mathematical object that can be used to study objects such as quantum fields, which are not operators on the Hilbert space, meaning their action on Hilbert space vectors is not a vector, however, their matrix elements are well defined. I will show that despite entanglement Hamiltonians being such ill-defined operators on Hilbert space, G, G' can be regularized using smooth bump functions to operators $\hat{G}, \hat{G}'$ with well-defined commutators. This regularization process is applicable in general to all kinds of smeared quadratic forms.

Further, I will use the regulated operators $\hat{G}, \hat{G}'$ to do a centered Zassenhaus expansion

[8] Most often, CFTs are defined on the complex plane, however a BCFT is defined on a manifold with a boundary. BCFT is not to be confused with the holographic CFT on the boundary.

of $\exp(i\mathcal{G}s)$ which respects causality. I will show that in fact half-sided translations is a special case in a large class of operators O for which a similar decomposition can be done by defining $O = O_L + O_R$ with O_L, O_R chosen approproiately.

Through this generalization, the results provide a way to combine operators defined in the two different algebras which have some overlap, such that together they can be used to define operators in a larger region. Each of these algebras can be regarded, through the timelike tube theorem [18] as the set of operations that can be performed by an observer in a fixed time interval, and therefore, the stitching protocol may be regarded as a form of communication between the observers in these regions.

This thesis is organized as follows. In the next chapter, I provide more details on the island formula and entanglement wedge reconstruction, and review the Jackiw Teitelboim model for 2d black holes. In chapter 3, I present the work on understanding the theory depedence of black hole interior reconstruction, when using radiation as the entity for reconstruction. In chapter 4, I review some of the key aspects of von Neumann algebras and some basic definitions in functional analysis that are relevant for understanding the work on studying the Zassenhaus decomposition of operators in 2d CFT, which is presented in chapter 5.

Chapter 2

On bulk reconstruction and JT gravity

AdS-CFT is a firmly established field with numerous excellent reviews available [21, 22]. While it is impossible to give a comprehensive summary of this subject in a few pages, I would like to return on some topics touched in the introduction, which we will need in this thesis later.

The primary topics to review below are the Ryu-Takayanagi or more generally the Quantum Extremal Surface (QES) formula, the notion of entanglement wedges, and entanglement wedge reconstruction. Then in the next section, I will reiterate the Hawking's information paradox, and discuss how it can be studied in 2 dimensional Jackiw Teitelboim gravity, briefly mentioning the consequent emergence of islands and replica wormholes. The reader who is familiar with these concepts can skip ahead to the next chapter.

2.1 The Ryu-Takayangi formula and Entanglement wedges

Given that AdS-CFT is a duality between a boundary theory and a bulk theory living in one higher dimension, one can ask the following question. Suppose a subgregion R on the boundary is of special interest to us, and we are interested in understanding the region in the bulk that is dual to R, alone. What is that region, and how does one find it? This region is called the entanglement wedge of R, which I will denote as $\mathcal{E}[R]$, and its found using the Ryu-Takayanagi formula, or more precisely the QES formula.

First let me recall what is a quantum extremal surface (QES). Let γ_R be a bulk surface anchored on R at the boundary. For it to be a QES, it should extremize the "generalized entropy" associated to γ_R

$$S_{gen}(\gamma_R; R) = \frac{A(\gamma_R)}{4G_N} + S_{\text{bulk}}(r) \tag{2.1.1}$$

where r is any spatial region[1] bounded by the surface γ_R and the boundary R, and $S_{bulk}(r)$ is the von Neumann entropy of quantum fields on this region, which is of fixed curvature.

A key result [3, 23, 4, 6, 5, 16] is that von Neumann entropy of quantum fields in region R on the boundary is given by

$$S(R) = \text{ext}_{\gamma_R} S_{gen}(\gamma_R; R) \tag{2.1.2}$$

and the domain of dependence[2] of the corresponding r, denoted as $\diamond_r$, is the entanglement wedge $\mathcal{E}[R]$. In other words, this $\diamond_r$ is the bulk region dual to the boundary region R, and now let me recall in what sense.

2.1.1 Entanglement wedge reconstruction

The programme of writing bulk operators in terms of boundary operators is called bulk reconstruction and is a central part of AdS-CFT. The programme that we will be interested in is the entanglement wedge reconstruction, which is a specific type of bulk reconstruc-

[1] All we need is γ_R to be homologous to R, and any spatial codimension 1 manifold anchord on $\gamma_R \cup R$ is allowed.

[2] Given a region $\mathcal{A}$, all points which have all timelike and lightlike curves passing through them and meeting $\mathcal{A}$, either in future or past, is the domain of dependence of $\mathcal{A}$. Strictly speaking entanglement wedges are codimension-0 regions in the bulk, but often when studying time independent geometries, we restrict our attention to a fixed time slice, and loosely call r itself as the entanglement wedge.

tion, which is based on the ideas of quantum error correction (QEC). [24, 25, 22]

The idea can be summarized through the following points.

- One considers a CFT state $\rho \in \mathcal{L}(\mathcal{H})$, and one takes a partial trace over it, to get a reduced density matrix ρ_R for the region R. This in QEC language can be thought of as a corruption of state by a noise channel to $\mathcal{N}(\rho)$.[3]

- In QEC, it is well known that while any ρ can corrupt, only some can be recovered. Suppose we can find a recovery channel $\mathcal{R}$ such that $\mathcal{R} \circ \mathcal{N}(\rho) \simeq \rho, \quad \forall \rho \in \mathcal{L}(\mathcal{H}_{code})$ for some Hilbert space $\mathcal{H}_{\text{code}}$. Such a recovery map can do quantum error correction for the noise channel $\mathcal{N}$ for states in $\mathcal{H}_{\text{code}}$.

- In the Heisenberg representation of QEC, operators get corrupted, and recovery is performed by $\mathcal{R}^\dagger$. This is the relevant framework for bulk reconstruction in AdS-CFT; usually $\mathcal{H}_{\text{code}}$ is the Hilbert space built from finite excitations of the vacuums state[4], and the recovery map $\mathcal{R}$ is obtained through the Petz map or the universal recovery map [26].

- Then one could formally write the reconstruction of a bulk operator $\phi(x)$ in terms of boundary operator Φ localized in R as

$$\Phi = \mathcal{R}^\dagger(\phi(x)), \quad x \in \mathcal{E}[R]$$

  where R is our original boundary region of interest, and $\mathcal{E}[R]$ is its entanglement wedge.

Apart from entanglement wedge reconstruction, the subject of bulk reconstruction has a lot more important and interesting areas, e.g. causal wedge reconstruction, and modular flow methods. In this thesis we will not dive deep into how to perform bulk reconstruction

[3]The noise channel to implement a partial trace can be realized through a swap operation.

[4]Or any holographic state, which can include states with black holes, e.g. the thermofield double state for two CFTs.

using any of these, and therefore we can restrict ourselves with these facts. However, the concept of entanglement wedges will be crucial for us, which we will return to, and discuss at length, in the next chapter.

2.2 Hawking's paradox, 2d Gravity, and Islands

In 1975, Hawking made a groundbreaking discovery that black holes emit radiation, which has a thermal character. This finding was unexpected, as it indicated that black holes are not eternal and will eventually evaporate. Moreover, it raised significant concerns since thermal radiation does not retain specific information about its source; the only extractable detail is the temperature of the body that emits it. Consequently, this leads to a paradox: if a black hole completely evaporates while continuously emitting thermal radiation, the detailed microscopic information about the matter that fell into it is seemingly lost forever. In more precise terms, while the initial quantum state of the black hole can be considered pure, the state that remains after evaporation appears mixed. This scenario contradicts the principle of unitarity in quantum mechanics, which states that an isolated system should evolve in a unitary manner.

While the 4 dimensional gravity has shown little sign to be amenable to study this evaporation process, in the recent years, a lot of progress has been possible in 2d models of gravity, where the paradox can be studied in detail. Below I will review a particular model of 2d gravity, called Jackiw-Teitelboim (JT) gravity, which has been used widely to study the Hawking's information paradox, and this model will be vital for us in the next chapter.

2.2.1 2d Jackiw-Teitelboim gravity

This subsection is primarily sourced from my paper [20] with Sitender Kashyap *and* Roji Pius.

In the truest sense, Einstein gravity is a four dimensional theory, and it is now well known that in 3 or lower dimensions, the theory has very limited dynamics. More specifically, the sourceless Einstein theory in 3d is trivial - all the solutions are flat, whereas in 2d, the Einstein tensor itself vanishes identically. The lower dimensional models of gravity therefore go beyond the Einstein-Hilbert action, use non-zero cosmological constants and matter couplings to create effective dynamics that mimics physical gravity.

One such model is the Jackiw-Teitelboim (JT) gravity, which is a 2d theory of gravity with a dilaton field. It is particularly relevant because the model can be realized as the near horizon limit of an extremal 4d Reissner-Nordström black hole [27]. One surprising aspect of this limit is that even though the 4d black hole is in asymptotically flat spacetime, the JT gravity is a theory in AdS_2. In fact, the AdS-JT gravity has been used widely [10, 11, 9, 7, 28] as a toy model for studying black holes. I will review some of it important aspects below, for more detailed reviews see [21, 29, 30, 31].

The action for JT gravity coupled to a CFT can be split into two parts $S = S_0 + S(\mathcal{M}, \partial\mathcal{M})$, where S_0 is topological and is given by

$$S_0 = \frac{\phi_0}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x\,\sqrt{-g}\,R + 2\int_{\partial\mathcal{M}} K\right]. \tag{2.2.1}$$

This term provides a large contribution $\frac{\phi_0}{4G_N}$ to the entropy of the black hole (in the limit $G_N \to 0$). The dynamical piece is given by

$$S(\mathcal{M}, \partial\mathcal{M}) = \frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x\,\sqrt{-g}\phi\left(R + \frac{2}{\ell_{AdS}^2}\right) + 2\int_{\partial\mathcal{M}} \phi_b K\right] + I_{\mathrm{CFT}}[g] \tag{2.2.2}$$

where R is the Ricci scalar of the spacetime $\mathcal{M}$, K is the trace of the extrinsic curvature of the boundary $\partial\mathcal{M}$, ϕ_b is the boundary value of the dilaton field and ℓ_{AdS} is the AdS radius which we will set to 1. The specific manner in which $\partial\mathcal{M}$ is carved out of pure AdS_2 will be called a *cutout*. This cutout is responsible for breaking the diffeomorphism invariance of the S_0 and is intimately connected to the AdS_2 bulk dynamics. Below we will show what cutout gives rise to a black hole solution. It is important to note that I_{CFT} does not couple to dilaton ϕ. Consequently its bulk equations of motion due to $\delta_\phi S$ variation give $R = -2$ and hence the geometry is locally that of AdS_2.

To describe the boundary, we use the Poincare patch shown in figure 2.1, whose metric is

$$ds^2 = \frac{-dt^2 + dz^2}{z^2} = \frac{-4dx^+dx^-}{(x^+ - x^-)^2}. \tag{2.2.3}$$

where $x^\pm = t \mp z$. The standard boundary conditions on the cut out are imposed to be

$$g_{uu}|_{bdy} = \frac{1}{\epsilon^2} = \frac{1}{z^2}\left(-\left(\frac{dt}{du}\right)^2 + \left(\frac{dz}{du}\right)^2\right)|_{bdy} \qquad \phi|_{bdy} = \phi_b = \frac{\phi_r}{\epsilon}. \tag{2.2.4}$$

Here u is the boundary time.

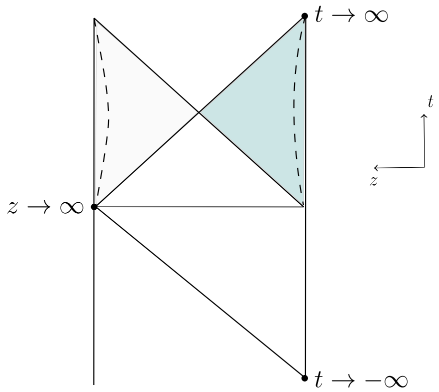

Figure 2.1: Poincare patch with the right black hole exterior highlighted in green. The left exterior in beige is outside the Poincare patch, which when using y coordinates covering the right exterior can be obtained from a $y \to y + i\beta/2$. The dashed lines represent cutouts. The infinite range of Poincare time t maps to a 2π range of the global time. Note that z increases to the left here, and accordingly we choose $x^\pm = t \mp z$. The inclined boundaries including the left tip are reached for $z \to \infty$.

In these coordinates, the boundary is given by $t = f(u), z = \epsilon f'(u)$, where the gluing function $f(u)$ is obtained through the energy balance as follows. The ADM mass of the gravitating region is

$$M(u) = -\frac{\phi_r}{8\pi G_N}\{f(u), u\}. \tag{2.2.5}$$

Energy conservation requires that the change in above ADM mass equal the net flux of energy across the boundary curve,

$$\frac{dM(u)}{du} = -\frac{d}{du}\left(\frac{\phi_r}{8\pi G_N}\{f(u), u\}\right) = T_{y^+y^+} - T_{y^-y^-} \tag{2.2.6}$$

which, given the stress tensor profile, can be solved for $f(u)$.

Next we see how the gluing map determines the dilaton profile $\phi(x^+, x^-)$. Varying action with respect to metric in the Poincare patch gives the equations for dilaton:

$$\begin{aligned} 2\partial_{x^+}\partial_{x^-}\phi + \frac{4}{(x^+ - x^-)^2}\phi &= 16\pi G_N \langle T_{x^+x^-}\rangle \\ -\frac{1}{(x^+ - x^-)^2}\partial_{x^+}\left((x^+ - x^-)^2\partial_{x^+}\phi\right) &= 8\pi G_N\langle T_{x^+x^+}\rangle \\ -\frac{1}{(x^+ - x^-)^2}\partial_{x^-}\left((x^+ - x^-)^2\partial_{x^-}\phi\right) &= 8\pi G_N\langle T_{x^-x^-}\rangle \end{aligned} \tag{2.2.7}$$

We further note that $T_{x^+x^-}$ can be absorbed into the definition of ϕ_0 and as such can be dropped from the first equation in (2.2.7). With such a choice a general solution to these equations, upto an $SL(2, R)$ transformation, can be written as

$$\begin{aligned} \phi(x^+, x^-) \;=\; & -\frac{2\pi\phi_r}{\beta}\frac{x^+ + x^-}{x^+ - x^-} - \frac{8\pi G_N}{x^+ - x^-}\int_0^{x^-} dt(x^+ - t)(x^- - t)T_{x^-x^-} \\ & + \frac{8\pi G_N}{x^+ - x^-}\int_0^{x^+} dt(x^+ - t)(x^- - t)T_{x^+x^+} \end{aligned} \tag{2.2.8}$$

and for $T_{x^-x^-} = 0$ along the cutout, the dilaton takes form [**?**, 28]

$$\phi(x^+, x^-) = -\phi_r\left(\frac{2f'(y^+)}{x^+ - x^-} - \frac{2f''(y^+)}{f'(y^+)}\right). \tag{2.2.9}$$

In the above we have to think of $x^+ = f(y^+)$ i.e. the Poincare coordinates are related to the Minkowski coordinate by the gluing map.

The field $\phi + \phi_0$ is related to the area of the transverse directions that are integrated out when the action S is viewed as a dimensional reduction from theories in higher dimensions. If for example the AdS_2 arises from the near horizon geometry of an extremal black hole then ϕ_0 corresponds to the area of the extremal black hole, while ϕ denotes deviations from this extremal value. Consequently the entropy formulae will have ϕ in the area term of the entropy formulae. As we move towards boundary, dilaton value increases to reflect surface area of a 2 sphere, as in higher dimensions In this spirit, specifying dilaton specifies the geometry of associated higher dimensional black hole spacetime.

Eternal black hole

One can interpret the interior of Rindler patches in AdS_2 as the exterior of a two sided black hole of temperature β^{-1}. Now we describe how to make use of the above equations to couple a black hole solution in the AdS_2 to the Minkowski bath. Because the black hole will be coupled to both on both sides, we use this left-right symmetry to focus on only one side. We demand that the black hole is in equilibrium with the bath, there is no net flux and hence $\frac{dT}{du} = 0$ and hence

$$\partial_u \{f(u), u\} = 0. \tag{2.2.10}$$

A solution that corresponds to a temperature $\frac{1}{\beta}$ is given by

$$f(u) = e^{\frac{2\pi u}{\beta}}. \tag{2.2.11}$$

Having solved for the gluing function, we can use it to extend the coordinates $y^{\pm}$ that were earlier defined in the bath region to the gravity region as well via

$$x^{\pm} = f(y^{\pm}) = \pm \exp\left(\pm \frac{2\pi}{\beta} y^{\pm}\right). \tag{2.2.12}$$

Figure 2.1 shows the right exteriro where $x^{\pm} = f(y^{\pm})$ is valid as green, and the left exterior, which can be reached by $y \to y + i\beta/2$, in beige. Given the map (2.2.12), the Poincare metric in (2.2.3) becomes

$$ds^2 = -\left(\frac{2\pi}{\beta}\right)^2 \frac{dy^+dy^-}{\sinh^2 \frac{\pi}{\beta}(y^- - y^+)}$$

and the dilaton profile using (2.2.9) is

$$\phi = \frac{2\pi\phi_r}{\beta}\frac{1}{\tanh \frac{\pi}{\beta}(y^- - y^+)}.$$

We will however be mostly working in Kruskal-Szekeres coordinates

$$w^\pm = \pm e^{\pm\frac{2\pi y_R^\pm}{\beta}} = \pm\left(x_R^\pm\right)^{\pm 1} \qquad\qquad w^\pm = \mp e^{\mp\frac{2\pi y_L^\pm}{\beta}} = \mp\left(x_L^\pm\right)^{\mp 1}. \tag{2.2.13}$$

In this coordinate the metric takes the form

$$ds^2 = \frac{4dw^-dw^+}{(1+w^-w^+)^2}. \tag{2.2.14}$$

The dilaton profile in the gravitating region in the w-coordinates takes the form

$$\phi = \phi_0 + \frac{2\pi\phi_r}{\beta}\frac{1-w^+w^-}{1+w^+w^-}. \tag{2.2.15}$$

Therefore the location of the singularity is given by $w^+w^- = \frac{1}{\theta}$, where $\theta = \frac{2\pi\phi_r - \beta\phi_0}{2\pi\phi_r + \beta\phi_0}$.

Further, the future horizon of the black hole is at $w^- = 0$ and past horizon is at $w^+ = 0$. Finally the location of the physical boundary of the black hole geometry that is being glued to the bath is given by $w^+w^- = -e^{\frac{2\pi\epsilon}{\beta}} \simeq -1$.

2.2.2 Islands and replica wormholes

Using the above models and models similar to it, the Hawking and Hawking like information paradoxes have been realized and solved in toy models of AdS-CFT [11, 9, 32, 28],

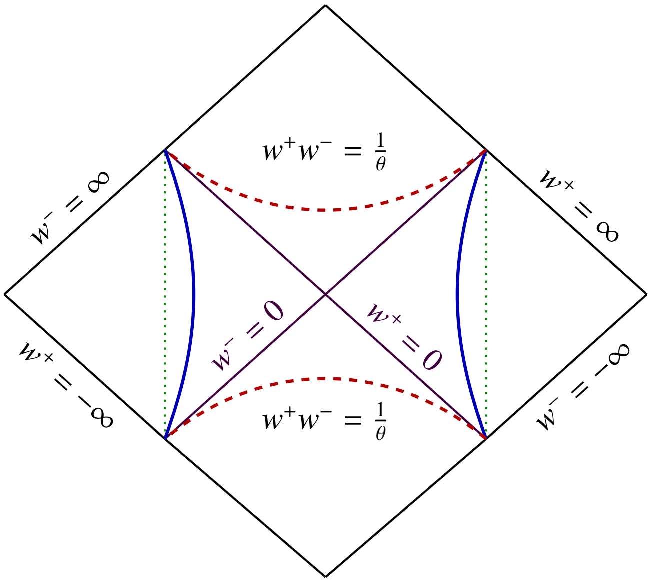

Figure 2.2: Kruskal-Szekeres, or the w coordinates, cover this entire diamond region, which includes the attached baths. The vertical lines are the AdS_2 boundaries and the dotted line is the singularity $w^+w^- = \theta^{-1}$.

which means that the Page curve for the Hawking radiation has been obtained; a curve that describes the entanglement entropy of the Hawking radiation as a function of time. The Page curve is a non-monotonic curve, which starts at zero, increases to a maximum value, and then decreases to zero as the black hole evaporates completely, implying that the information about the black hole is not lost, but rather encoded in the Hawking radiation. This is regarded as a major achievement by most people, and it has been achieved by using the so-called island formula (2.2.16).

For generality, assume that the black hole is coupled to a system χ, and let $\mathbf{S}(\chi)$ be the gravitational entropy of χ, then

$$\mathbf{S}(\chi) = \min_I \left[\text{ext}_I \left(\frac{A(\partial I)}{4G_N} + S(\chi \cup I) \right) \right] \tag{2.2.16}$$

where I is a codimension-1 region in the bulk, called an island, S is the usual von Neumann entropy computed on curved spacetime, and $A(\partial I)$ is the area of its boundary. In effect, I is the entanglement wedge for the system χ, and the island formula is a generalization of the Ryu-Takayanagi formula to include islands.

To summarize the motivation behind the island formula and how it provides a resolution to Hawking's paradox, in the last decade it has been motivated that Hawking's saddle point calculation for entropy did not take into account another saddle which becomes dominant at later stage of evaporation: the replica wormhole saddle [33, 7, 2, 8]. These wormholes emerge in the gravitational path integral for the computation of entanglement entropy of the Hawking radiation, done via the replica trick, and the "island" is a manifestation of the replica wormhole saddle.[5] The island formula (2.2.16) is thus a generalization of the Ryu-Takayanagi formula to include these islands.

In the next chapter, we will see that the island formula can be used to understand about "theory dependence" of black hole interior's reconstructibility when reconstruction is done using the Hawking radiation degrees of freedom.

[5]For more details, see [28, 7].

Chapter 3

Theory dependence of interior reconstruction

This chapter is primarily sourced from my paper [20] with Sitender Kashyap *and* Roji Pius.

3.1 Introduction

As I briefly mentioned in chapter 2, the resolution of Hawking information paradox [34] of an evaporating black hole in AdS_2 requires introducing a non-trivial entanglement wedge[1] that contains the black hole interior after the Page time for the Hawking radiation [33, 7, 8, 2]. This appearance of a non-vanishing entanglement wedge for Hawking radiation after the Page time is result of a non-perturbative gravitational effect, the replica wormhole. Usually, non-perturbative effects are highly theory dependent. The energy spectrum of a black hole [35, 36], the S-matrix of a black hole that determines its formation, and evaporation of the black hole [37] are some of the examples of such quantities.

[1]Need to add background on entanglement wedges, on Hawking paradox, on replica wormholes, maybe a few lines about general AdS spaces.

Since the ability of late Hawking radiation to reconstruct the black hole interior is a non-perturbative effect, it is natural to suspect that the interior reconstruction might depend on the details of the theory.

An eternal AdS_2 black hole in equilibrium with a finite temperature bath also comes with an information paradox very much like the Hawking information paradox. The paradox, the unbound growth of bath entropy, is due to the continuous exchange of the Hawking radiation and the radiation from the bath. The resolution of this paradox also requires introducing a non-trivial entanglement wedge for the bath radiation after the Page time [8]. Compared to an evaporating black hole, a black hole in equilibrium with a finite temperature bath is a more convenient setup for studying the theory dependence of the reconstruction from radiation due to the absence of any backreaction. For an eternal AdS black hole with no matter escaping the AdS boundary, it is already demonstrated in [9] that the bulk reconstruction of the interior of the black hole is highly theory dependent at late times. They achieved this by making the boundary conditions of the bulk matter fields random, and showing that the reference Hilbert space that encodes the information about this randomness possesses a non-trivial entanglement wedge that contains the black hole interior including the region near the singularity.

In this chapter, I will elaborate on my work with Sitender Kashyap and Roji Pius in [20], where following [9], we analyse the dependence of the interior reconstruction using the bath radiation on the boundary conditions of the bulk matter fields. For this we consider a JT gravity black hole in equilibrium with a non-gravitating bath at finite temperature and introduce a matter CFT coupled to gravity in the black hole region having reflecting boundary conditions for the fields in it. We assume that that these boundary conditions are drawn from a probability distribution. We denote the probability for the i^{th} field to have a boundary condition J_i as $P(J_i)$. [2] Then the black hole density matrix defined using the Euclidean path integral will depend on the boundary conditions

[2] In the SYK picture, $\mathbf{J} = \{J_1, \cdots, J_i, \cdots\}$ would correspond to the random couplings that appear in the SYK Hamiltonian.

$\mathbf{J} = \{J_1, \cdots, J_i, \cdots\}$ of the bulk CFT matter fields and the associated probability distribution $P(J) = \{P(J_1), \cdots, P(J_i), \cdots\}$. The purification of this density matrix requires the bath Hilbert space $\mathcal{H}_{\text{bath}}$ and also the introduction of an environment, an auxiliary reference Hilbert space $\mathcal{H}_{\text{journal}}$, which is referred as the 'journal' Hilbert space. The journal Hilbert space encodes the information about the boundary conditions of the bulk CFT matter fields. Therefore the dependence of the black hole interior reconstruction using the bath radiation on the the boundary conditions of the matter fields can be characterised by determining the entanglement wedge of the journal [24]. The goal of this paper is the determination of the entanglement wedge of the journal for this setup.

The physical significance of this problem was already discussed in [9] for an evaporating black hole which has two additional systems other than the black hole, the Hawking radiation and the journal. Our setup similarly has two additional systems other than the black hole, the bath radiation and the journal. However, the absence of backreaction makes our setup more convenient for analysing the same problem. After the black hole Page time the bath radiation and the journal strive for the ownership of the interior of the black hole. The winner is expected to be decided by the rate of the entropy growth of the two systems. As it is computed in this paper, initially the entropy of bath radiation grows linearly and the journal entropy grows logarithmically. Therefore, at first the bath radiation is expected to capture the black hole interior. The non-triviality is in figuring out whether the ownership of the interior is ever transferred to the journal. If the bath radiation retains the interior forever, then it means that the interior reconstruction is insensitive to details of the bulk theory, which suggests that the unknown details of the bulk theory can be determined by making measurements on the bath radiation. If the ownership is transferred to the journal at a late time, then it implies that interior reconstruction is theory dependent. The main result of this work is that the bath radiation transfers ownership of the black hole interior to the journal at a later time.

Let me briefly delineate how we will proceed. We determine the entanglement wedge of the subsystems, black hole, bath and journal, by demanding that the entropies of these subsystems satisfy all the constraints imposed by unitarity. There are two such constraints, first is that the von Neumann entropy of any subsystem must be less than its thermal entropy and the second is that the entropies of the subsystems must satisfy the extended strong subadditivity (eSSA) due to Carlen and Lieb [38]. The first constraint demands that the von Neumann entropy $S(\rho_{\text{BH}})$ of the eternal black hole density matrix ρ_{BH} must be less than $2S^0_{\text{BH}}$, where S^0_{BH} is the Bekenstein-Hawking entropy of the one side of the eternal black hole. Since black hole has only a finite number of degrees of freedom, S^0_{BH} is finite. This demands that the entanglement entropy of the black hole must not be an ever-growing function of boundary time. However, the black hole entropy obtained by the replica computation without including any Euclidean wormhole contribution becomes more than $2S^0_{\text{BH}}$ after the black hole Page time. This violation of thermal entropy bound can be cured by removing the black hole interior from the combined entanglement wedge of the boundary CFTs dual to the eternal black hole. The resulting generalised entropy of the black hole after the Page time saturates the thermal entropy bound $2S^0_{\text{BH}}$. On the contrary, the entropy $S(\rho_{\text{bath}})$ of the bath density matrix ρ_{bath} and the entropy $S(\rho_{\text{journal}})$ of the journal density matrix ρ_{journal} can have unbounded growth due to the infinite number of degrees of freedom they possess. Hence one naively expects that the interior of the black hole after the Page time may be co-owned by the bath and the journal.[3] However, using the second constraint we argue below that this is not true.

If the state of the combined system of the black hole, the bath and the journal is pure, then the eSSA after the black hole Page time states that

$$S(\rho_{\text{bath}}) + S(\rho_{\text{journal}}) - 2S^0_{\text{BH}} \geq 2\, max\left\{S(\rho_{\text{journal}}) - 2S^0_{\text{BH}}, S(\rho_{\text{bath}}) - 2S^0_{\text{BH}}, 0\right\}. \tag{3.1.1}$$

Using the replica trick we can compute $S(\rho_{\text{bath}})$ and $S(\rho_{\text{journal}})$. In the absence of any non-trivial islands, $S(\rho_{\text{bath}})$ and $S(\rho_{\text{journal}})$ grow linearly and logarithmically respectively with

[3] By co-ownership, we mean that neither the bath nor the journal individually owns the interior, only the combined system does.

respect to the boundary time. Due to the larger growth of bath entropy, the eSSA takes the following form right after the black hole Page time

$$S(\rho_{\text{bath}}) - S(\rho_{\text{journal}}) \leq 2S^0_{\text{BH}}. \tag{3.1.2}$$

This inequality can be satisfied only if the bath owns an island that contains the black hole interior after black hole Page time. After including such an island, $S(\rho_{\text{bath}})$ becomes $2S^0_{\text{BH}}$. However, at a later time the logarithmic growth of $S(\rho_{\text{journal}})$ makes it larger than $2S^0_{\text{BH}}$. At this stage the eSSA takes the following form

$$S(\rho_{\text{journal}}) - S(\rho_{\text{bath}}) \leq 2S^0_{\text{BH}}. \tag{3.1.3}$$

Clearly, this inequality is satisfied until $S(\rho_{\text{journal}})$ becomes $4S^0_{\text{BH}}$. Subsequently, in order to satisfy the eSSA, the bath must transfer the ownership of the black hole interior to the journal. Introducing such a non-trivial entanglement wedge that contains the interior of the black hole makes the rate of the entropy growth of the journal same as that of the bath and saturates the eSSA, thus restoring unitarity. This implies that the reconstruction of the black hole interior using the bath radiation at late times requires the complete description of the theory which includes specifying the boundary conditions of all the fields at the AdS boundary.

This will be carried out in the following subsections. Chapter 2 has already introduced the AdS_2 black hole in equilibrium with a finite temperature non-gravitating bath, and the setup described below will build on it. In this case, there will be two kinds of conformal matter, one having transparent boundary conditions along the boundary of the gravitational region, and another having random reflecting boundary conditions drawn from a distribution. In section 3.3, we will determine the entanglement wedge of the random boundary conditions to characterise the theory dependence of the black hole interior reconstruction using the bath radiation.

3.2 The setup

Consider a black hole solution of JT gravity with inverse temperature β coupled to a bath having the same temperature. We assume that gravity is absent in the bath. We introduce two CFTs into this spacetime. They will be referred as CFT_1 and CFT_2. The CFT_1 has central charge c_1 and CFT_2 has central charge c_2. The CFT_2 is restricted to the gravitating AdS_2 region, while the CFT_1 lives in the full spacetime, which is the AdS_2 and the bath region together. This is done by setting transparent boundary conditions for fields in CFT_1 and reflecting boundary conditions for fields in CFT_2. Both the CFTs are coupled to the metric in the gravitational region. However, they are not coupled to the dilaton field. This makes the black hole spacetime locally AdS_2, even though gravity is dynamical in the black hole region. We also assume that CFT_1 and CFT_2 do not directly interact with each other. An additional feature of CFT_2 is that the boundary conditions of the fields in this theory are drawn from a distribution. In the dual holographic side this arises from “unknown couplings" whose information is present in the aforementioned system called journal [9, 39, 40].

3.2.1 Random boundary conditions and the journal

Let me elaborate on the idea and the use of the system journal. We have the second CFT, the CFT_2, in the AdS_2 region with reflecting boundary conditions for the matter field along its boundary. Since there is no additional net flow across the interface between bath and the AdS_2 region due to CFT_2, introduction of CFT_2 does not change the geometry of the spacetime. For computational tractability we have chosen a CFT_2 that does not interact with CFT_1. Assume that the probability for the i^{th} field in CFT_2 to have a boundary condition J_i is $P(J_i)$. As mentioned in the introduction, the information of the boundary conditions of CFT_2 matter fields $J = \{J_1, \cdots, J_i, \cdots\}$ and the associated probability distribution $P(J)$ are encoded in the density matrix of the black hole. This black hole density matrix cannot be purified only by the bath Hilbert space $\mathcal{H}_{\text{bath}}$, it also requires introducing an auxiliary reference Hilbert space $\mathcal{H}_J$ which is referred as the 'journal' Hilbert space. Let $\{|J_i\rangle_{\text{journal}}\}$, $\{|\psi_k, J_i\rangle_{\text{BH}}\}$ and $\{|\gamma_{k'}\rangle_{\text{bath}}\}$ be basis for $\mathcal{H}_{\text{journal}}$, $\mathcal{H}_{\text{BH}}$ and $\mathcal{H}_{\text{bath}}$ respectively. We choose $\{|J_i\rangle_{\text{journal}}\}$ to be orthonormal. Then the purified state can be expressed as

$$|\Psi\rangle = \sum_i \sqrt{P(J_i)} \left(\sum_{k,k'} A_{k,k'} |\psi_k, J_i\rangle_{\text{BH}} |\gamma_{k'}\rangle_{\text{bath}} \right) |J_i\rangle_{\text{journal}}. \tag{3.2.1}$$

Each $|J_i\rangle_{\text{journal}}$ corresponds to a choice of boundary condition for CFT_2 at the physical boundary of the eternal black hole spacetime.

We take the CFT_2 to be a free theory of c_2 non-compact bosons $X_1, \cdots, X_{c_2}$ with action

$$S = \sum_{i=1}^{c_2} \frac{1}{2\pi} \int d^2 w \partial X_i \bar{\partial} X_i. \tag{3.2.2}$$

For this theory the boundary condition J_i corresponds to the boundary value of the boson X_i at the AdS boundary. We also assume that the boundary conditions are drawn from a Gaussian distribution having standard deviation $1/\delta$

$$P(J) = \frac{\delta}{\sqrt{2\pi}} e^{-\frac{\delta^2}{2} J^2}. \tag{3.2.3}$$

It was shown in [9] that for an eternal black hole the boundary time evolution produces entanglement growth between the black hole and the journal. This leads to an unbounded logarithmic growth of the journal entropy, producing a unitarity paradox. This information paradox was resolved by introducing an island for the journal which includes the interior of the black hole.

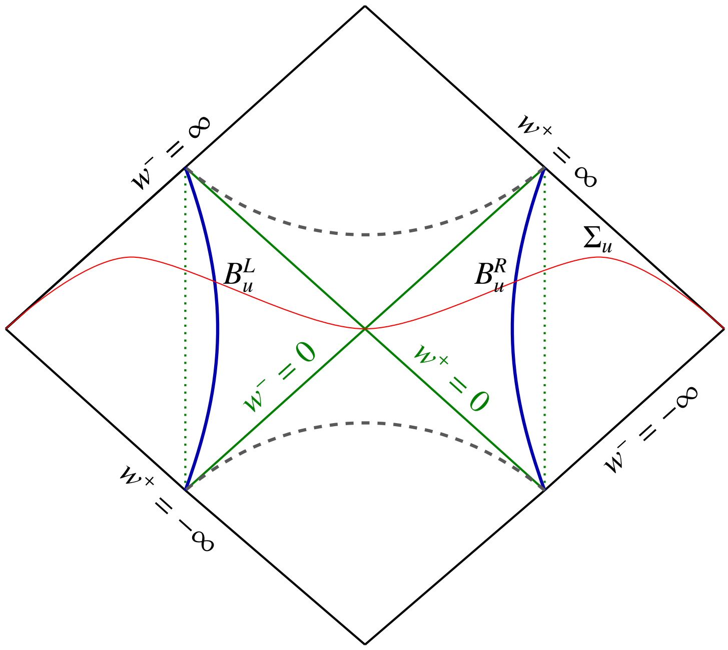

Figure 3.1: The equal time slice Σ_u in the glued spacetime intersects the right AdS boundary at B_u^R and the left AdS boundary at B_u^L.

3.3 The entanglement wedge of journal

In this section, we shall determine the entanglement wedge of the journal at late times, which is the main goal of this paper.

3.3.1 Black hole quantum extremal surfaces

Consider the setup described in section 3.2. The early time von Neumann entropy of the black hole density matrix can be computed using the replica trick and is given by

$$\mathbf{S}_{\mathrm{BH}}(u) = S^1_{\mathrm{BH}}(u) + S^2_{\mathrm{BH}}(u), \tag{3.3.1}$$

where $S^1_{\mathrm{BH}}(u)$ and $S^2_{\mathrm{BH}}(u)$ are the CFT$_1$ and CFT$_2$ contributions to the black hole entropy. The CFT$_1$ contribution can be obtained from the twist operator correlation function as follows

$$S^1_{\mathrm{BH}}(u) = -\lim_{n\to 1}\frac{1}{1-n}\ln\langle\sigma_1(B^L_u)\sigma_1(B^R_u)\rangle. \tag{3.3.2}$$

Here σ_1 denotes the twist fields in the orbifold version of CFT_1 having scaling dimension of $\Delta_n = \frac{c_1}{12}\left(n - \frac{1}{n}\right)$. The points B_u^L and B_u^R are points on the left and right boundary where the equal time slice Σ_u corresponding to the boundary time u intersects the left and right black hole boundaries, see figure 3.1. As u increases, it is assumed that the point B_u^R moves along the positive time direction of the right boundary and the point B_u^L is moving along the negative time direction of the left boundary. The correlation function $\langle\sigma_1(B_u^L)\sigma_1(B_u^R)\rangle$ is evaluated on a complex plane with the w-coordinates described in the previous section with metric $ds^2 = \frac{4dw^- dw^+}{(1+w^-w^+)^2}$. It can be evaluated by Weyl transforming it into a correlation function on the w-plane with flat metric. The resulting correlation function is

$$\langle\sigma_1(B_u^L)\rangle\sigma_1(B_u^R)\rangle = \left(\frac{\left(1 + w^-_{B_u^L} w^+_{B_u^L}\right)\left(1 + w^-_{B_u^R} w^+_{B_u^R}\right)}{4\left(w^+_{B_u^R} - w^+_{B_u^L}\right)\left(w^+_{B_u^R} - w^+_{B_u^L}\right)}\right)^{\Delta_n}. \tag{3.3.3}$$

Substituting the w-coordinates (w^+, w^-) of the point B_u^L and B_u^R given by

$$\begin{aligned}(w^+_{B_u^L}, w^-_{B_u^L}) &= \left(-e^{-\frac{2\pi(u-\epsilon)}{\beta}}, e^{\frac{2\pi(u+\epsilon)}{\beta}}\right),\\ (w^+_{B_u^R}, w^-_{B_u^R}) &= \left(e^{\frac{2\pi(u-\epsilon)}{\beta}}, -e^{\frac{-2\pi(u+\epsilon)}{\beta}}\right)\end{aligned} \tag{3.3.4}$$

gives the CFT_1 contribution to the black hole entropy as

$$S^1_{\mathrm{BH}}(u) = \frac{c_1}{3}\ln\left(\frac{\beta}{\pi\epsilon}\cosh\left(\frac{2\pi u}{\beta}\right)\right). \tag{3.3.5}$$

The CFT_2 contribution to the black hole entropy is obtained from the correlation function of the boundary condition changing operators averaged over the distribution $P(\mathbf{J})$ as follows

$$S^2_{\mathrm{BH}}(u) = -\lim_{n\to 1}\frac{1}{1-n}\ln\left(\int_{\mathbf{J}^1,\cdots,\mathbf{J}^n}\prod_{i=1,k=1}^{c_2,n} dJ_i^k P(J_i^k)\langle O_{\mathbb{J}}(B_u^L) O_{\mathbb{J}}(B_u^R)\rangle\right). \tag{3.3.6}$$

The operator $O_{\mathbb{J}}$ denotes the boundary condition changing operator that changes the boundary conditions of the CFT_2 fields. It changes the boundary conditions of the fields $\{X_1, \cdots, X_{c_2}\}$ from $\mathbf{J}^k = \{J_1^k, \cdots, J_{c_2}^k\}$ to $\mathbf{J}^{k+1} = \{J_1^{k+1}, \cdots, J_{c_2}^{k+1}\}$ as we go from the k-th sheet to the $(k+1)$-th sheet of the replica manifold for $k = 1, \cdots, n$. The scaling dimension of $O_{\mathbb{J}}$ is given by

$$\Delta_{\mathbb{J}} = \sum_{i=1,k}^{c_2,n} \left(\frac{J_i^{k+1} - J_i^k}{2\pi} \right)^2 . \tag{3.3.7}$$

The cut out of AdS_2 region is a region in the disk with w-coordinates. The CFT_2 correlation function $\langle O_{\mathbb{J}}(B_u^L) O_{\mathbb{J}}(B_u^R) \rangle$ is calculated on the w-disk and can be obtained with the help of the doubling trick. Finally, the integration over the boundary conditions $\mathbf{J}$ can be performed by using the concept of circularly invariant matrices [41]. The final result is given by

$$S_{\mathrm{BH}}^2(u) \approx \frac{c_2}{2} \ln\left(\frac{u}{\beta\delta^2} \right). \tag{3.3.8}$$

Therefore, the black hole entropy at late times has unbounded growth as given below

$$S_{\mathrm{BH}}^1(u) \approx \frac{2\pi c_1}{3\beta} u + \frac{c_2}{2} \ln\left(\frac{u}{\beta\delta^2} \right). \tag{3.3.9}$$

Clearly, such a growth will lead us to information paradox at late times. The resolution of this information paradox requires determining the quantum extremal surface (QES) associated with the black hole. This is done by minimising the generalised entropy of the black hole after removing an interval $A_u^L A_u^R$ from the restriction of the equal time slice Σ_u to the AdS_2 region. The generalised entropy of the black hole for the interval $B_u^L A_u^L \cup B_u^R A_u^R$ is given by

$$\mathbf{S}_{\mathrm{BH}}^{gen}(u) = \frac{\phi\left(A_u^L\right) + \phi\left(A_u^R\right)}{4G_N} + S_{\mathrm{BH}}^{gen,1}(u) + S_{\mathrm{BH}}^{gen,2}(u), \tag{3.3.10}$$

where the first term is the area term. The area term is equal to the sum of the value of dilaton field given in (2.2.15) at points A_u^L and A_u^R. $S_{\mathrm{BH}}^{gen,1}(u)$ and $S_{\mathrm{BH}}^{gen,2}(u)$ denote the CFT_1

and CFT_2 contributions to the generalised black hole entropy. The CFT_1 contribution is

$$\begin{aligned}
S_{\mathrm{BH}}^{gen,1}(u) &= -\lim_{n\to 1}\frac{1}{1-n}\ln\langle\sigma_1\left(B_u^L\right)\sigma_1\left(A_u^L\right)\sigma_1\left(A_u^R\right)\sigma_1\left(B_u^R\right)\rangle \\
&\approx -\lim_{n\to 1}\frac{1}{1-n}\ln\langle\sigma_1\left(B_u^L\right)\sigma_1\left(A_u^L\right)\rangle\langle\sigma_1\left(A_u^R\right)\sigma_1\left(B_u^R\right)\rangle \\
&\approx \frac{c_1}{6}\ln\left(\left(\frac{\beta}{\pi\epsilon}\right)^2\frac{\left(e^{-\frac{2\pi u}{\beta}}+w^+_{A_u^L}\right)\left(e^{\frac{2\pi u}{\beta}}-w^-_{A_u^L}\right)}{\left(1+w^-_{A_u^L}w^+_{A_u^L}\right)}\right)\left(\frac{\left(e^{-\frac{2\pi u}{\beta}}+w^-_{A_u^R}\right)\left(e^{\frac{2\pi u}{\beta}}-w^+_{A_u^R}\right)}{\left(1+w^-_{A_u^R}w^+_{A_u^R}\right)}\right).
\end{aligned} \tag{3.3.11}$$

In the second step we made the approximation by assuming that the points A_u^L and A_u^R are well separated.

The CFT_2 contribution is given by

$$\begin{aligned}
S_{\mathrm{BH}}^{gen,2}(u) &= -\lim_{n\to 1}\frac{1}{1-n}\ln\int_{\mathbf{J}^1,\cdots,\mathbf{J}^n}\prod_{i=1,k=1}^{c_2,n}dJ_i^kP(J_i^k)\langle O_{\mathbb{J}}\left(B_u^L\right)\sigma_2\left(A_u^L\right)\sigma_2\left(A_u^R\right)O_{\mathbb{J}}\left(B_u^R\right)\rangle \\
&\approx -\lim_{n\to 1}\frac{1}{1-n}\ln\int_{\mathbf{J}^1,\cdots,\mathbf{J}^n}\prod_{i=1,k=1}^{c_2,n}dJ_i^kP(J_i^k)\langle O_{\mathbb{J}}\left(B_u^L\right)\sigma_2\left(A_u^L\right)\rangle\langle\sigma_2\left(A_u^R\right)O_{\mathbb{J}}\left(B_u^R\right)\rangle,
\end{aligned} \tag{3.3.12}$$

where σ_2 denotes the CFT_2 twist operators. The correlators $\langle O_{\mathbb{J}}\left(B_u^L\right)\sigma_2\left(A_u^L\right)\rangle$ and $\langle\sigma_2\left(A_u^R\right)O_{\mathbb{J}}\left(B_u^R\right)\rangle$ are evaluated on the w-plane where CFT_2 is defined. Since in the Euclidean version this region is a cutout of disk, these correlators can be calculated by using the doubling trick. The correlator $\left\langle O_{\mathbb{J}}\left(w^+_{B_u^L},w^-_{B_u^L}\right)\sigma_2\left(w^+_{A_u^L},w^-_{A_u^L}\right)\right\rangle$ evaluated on the Euclidean AdS_2 is given by

$$\left\langle O_{\mathbb{J}}\left(w^+_{B_u^L},w^-_{B_u^L}\right)\sigma_2\left(w^+_{A_u^L},w^-_{A_u^L}\right)\right\rangle = G_n\left(\mathbb{J}\right)\left(\left(\frac{\pi\epsilon}{\beta}\right)\frac{\left(1+w^-_{A_u^L}w^+_{A_u^L}\right)}{\left(1-e^{-\frac{2\pi u}{\beta}}w^-_{A_u^L}\right)\left(1+e^{\frac{2\pi u}{\beta}}w^+_{A_u^L}\right)}\right)^{\Delta_{\mathbb{J}}}. \tag{3.3.13}$$

The coefficient $G_n\left(\mathbb{J}\right)$ is related to the n-point function of boundary condition changing operators kept on a disk

$$G_n(\mathbb{J}) = \prod_{k\neq l, i=1}^{c_2} \left| e^{\frac{2\pi i(k-1)}{n}} - e^{\frac{2\pi i(l-1)}{n}} \right|^{\mu^i_{kl}}, \tag{3.3.14}$$

where $\mu^i_{kl} = \frac{(J^{k+1}_i - J^k_i)(J^{l+1}_i - J^l_i)}{2\pi^2}$. After performing the averaging over the Gaussian distribution again by using the integration method based on circulant matrix we obtain the generalised bath entropy as follows

$$S^{gen,2}_{\text{BH}}(u) \approx \frac{c_2}{2}\ln\left(\ln\left(\left(\frac{\beta}{\pi\epsilon}\right)^2 \frac{\left(1 - e^{-\frac{2\pi u}{\beta}} w^-_{A^L_u}\right)\left(1 + e^{\frac{2\pi u}{\beta}} w^+_{A^L_u}\right)\left(1 + e^{\frac{2\pi u}{\beta}} w^-_{A^R_u}\right)\left(1 - e^{-\frac{2\pi u}{\beta}} w^+_{A^R_u}\right)}{\left(1 + w^-_{A^L_u} w^+_{A^L_u}\right)\left(1 + w^-_{A^R_u} w^+_{A^R_u}\right)}\right)\right). \tag{3.3.15}$$

Extremising the generalised bath entropy with respect to $w^+_{A^L_u}$ and $w^-_{A^L_u}$ gives the following QES equations

$$\begin{aligned}
-\frac{\pi\phi_r}{G_N\beta}\frac{w^-_{A^L_u}}{\left(1 + w^+_{A^L_u} w^-_{A^L_u}\right)^2} + \left(\frac{c_1}{6} + \frac{c_2}{2\ln\left(\frac{\beta}{\pi\epsilon}\right)}\right)\left(\frac{1}{e^{-\frac{2\pi u}{\beta}} + w^+_{A^L_u}} - \frac{w^-_{A^L_u}}{\left(1 + w^+_{A^L_u} w^-_{A^L_u}\right)}\right) &= 0 \\
-\frac{\pi\phi_r}{G_N\beta}\frac{w^+_{A^L_u}}{\left(1 + w^+_{A^L_u} w^-_{A^L_u}\right)^2} - \left(\frac{c_1}{6} + \frac{c_2}{2\ln\left(\frac{\beta}{\pi\epsilon}\right)}\right)\left(\frac{1}{e^{\frac{2\pi u}{\beta}} - w^-_{A^L_u}} + \frac{w^+_{A^L_u}}{\left(1 + w^+_{A^L_u} w^-_{A^L_u}\right)}\right) &= 0.
\end{aligned} \tag{3.3.16}$$

There exists a solution for this coupled equations at late times near the left future horizon of the black hole where $w^+_{A^L_u} w^-_{A^L_u} \approx 0$. The solution is given by

$$w^\pm_{A^L_u} = \mp\frac{G_N\beta}{6\pi\phi_r}\left(c_1 + \frac{3c_2}{\ln\left(\frac{\beta}{\pi\epsilon}\right)}\right)e^{\mp\frac{2\pi u}{\beta}}. \tag{3.3.17}$$

By repeating the same analysis we can find the QES in the right side of the black hole. It is given by

$$w^\pm_{A^R_u} = \pm\frac{G_N\beta}{6\pi\phi_r}\left(c_1 + \frac{3c_2}{\ln\left(\frac{\beta}{\pi\epsilon}\right)}\right)e^{\pm\frac{2\pi u}{\beta}}. \tag{3.3.18}$$

Substituting the QES solutions back to the generalised black hole entropy expression shows that at late time the black hole entropy becomes a constant equal to twice the area of black hole horizon. Therefore, this QES after Page time u_{Page}, tame the non-unitary growth of the black hole entropy.

3.3.2 Entanglement wedge of bath and the extended strong subadditivity

The QES computation in the previous subsection suggests that after Page time $u_{Page} \approx \frac{3S^0_{\text{BH}}\beta}{\pi c_1}$, where S^0_{BH} is area of the bifurcation horizon of the black hole, the combined system of the bath and the journal possesses a non-trivial entanglement wedge that contains the interior of the black hole. The entanglement wedge of the journal at late time must belong to the entanglement wedge of the combined system. Hence, we should search for a journal island satisfying the constraints of the extended strong subadditivity [**?**] inside the interval bounded by the black hole quantum extremal surfaces. The eSSA is an inequality satisfied by the von Neumann entropies of three subsystems of a larger quantum system which we explain below.

Consider a quantum system having Hilbert space $\mathcal{H}$ formed by taking the tensor product of the Hilbert spaces of three subsystems $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, i.e. $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. We denote the state of the larger quantum system by ρ^{123}, the state of the combined system having Hilbert space $\mathcal{H}^{ij} = \mathcal{H}^i \otimes \mathcal{H}^j$ by ρ^{ij}, and the state of the i^{th} subsystem having Hilbert space $\mathcal{H}^i$ by ρ^i. Then the eSSA inequality states that

$$S\left(\rho^{12}\right) + S\left(\rho^{23}\right) - S\left(\rho^{123}\right) - S\left(\rho^{2}\right) \geq 2\, max\left\{S\left(\rho^{1}\right) - S\left(\rho^{13}\right), S\left(\rho^{3}\right) - S\left(\rho^{13}\right), 0\right\}; \tag{3.3.19}$$

for the usual strong subadditivity inequality the right hand side is simply zero.

Now we make use of the above inequality as follows. Take subsystem 1 to be the journal, subsystem 2 to be the black hole and the subsystem 3 to be the bath. Then the eSSA satisfied by the entropies of the subsystems after Page time reads as follows

$$S_{\text{bath}}(u) + S_{\text{journal}}(u) - 2S^0_{\text{BH}} \geq 2\, max\left\{S_{\text{journal}}(u) - 2S^0_{\text{BH}}, S_{\text{bath}}(u) - 2S^0_{\text{BH}}, 0\right\}. \quad (3.3.20)$$

Using the replica method the entropies of bath and journal can be calculated. The bath entropy at large times is given by

$$S_{\text{bath}}(u) = -\lim_{n\to 1} \frac{1}{1-n} \ln\langle\sigma_1(B^L_u)\sigma_1(B^R_u)\rangle \approx \frac{2\pi c_1}{3\beta} u, \quad (3.3.21)$$

and the journal entropy is given by

$$S_{\text{journal}}(u) = -\lim_{n\to 1} \frac{1}{1-n} \ln\left(\int_{\mathbf{J}^1,\cdots,\mathbf{J}^n} \prod_{i=1,k=1}^{c_2,n} dJ^k_i P(J^k_i)\langle O_{\mathbb{J}}(B^L_u) O_{\mathbb{J}}(B^R_u)\rangle\right) \approx \frac{c_2}{2} \ln\left(\frac{u}{\beta\delta^2}\right). \quad (3.3.22)$$

After Page time, since the bath entropy is significantly greater than the entropy of the journal, the eSSA takes the following form

$$S_{\text{bath}}(u) - S_{\text{journal}}(u) \leq 2S^0_{\text{BH}}. \quad (3.3.23)$$

It is clear from these expressions that the entropy of the bath and the journal violates the eSSA (3.3.23) after time $u = u_B > u_{Page}$, where u_B is the time at which the difference in $S_{\text{bath}}(u)$ and $S_{\text{journal}}(u)$ becomes $2S^0_{\text{BH}}$. The root cause of this violation is the linear growth of entanglement entropy of the bath while the journal only has logarithmic growth of entanglement entropy. Therefore, this violation of eSSA can be described as the bath information paradox.

An island for bath that is inside the interval enclosed by the quantum extremal surfaces of black hole might resolve this paradox. With this hope, let us search for a bath island by minimising the generalised entropy of bath associated with an arbitrary interval at $C^L_u C^R_u$

with respect to the points C_u^L and C_u^R. The generalised entropy of the bath is given by

$$S_{\text{bath}}^{gen} = \frac{\phi\left(C_u^L\right) + \phi\left(C_u^R\right)}{4G_N} + S_{\text{bath}}^{gen,1}(u) + S_{\text{bath}}^{gen,2}(u), \tag{3.3.24}$$

where $S_{\text{bath}}^{gen,1}(u)$ is the CFT_1 contribution to the bath generalised entropy

$$\begin{aligned} S_{\text{bath}}^{gen,1}(u) &= -\lim_{n\to 1}\frac{1}{1-n}\ln\langle\sigma_1(B_u^L)\sigma_1(C_u^L)\sigma_1(C_u^R)\sigma_1(B_u^R)\rangle \\ &\approx -\lim_{n\to 1}\frac{1}{1-n}\ln\langle\sigma_1(B_u^L)\sigma_1(C_u^L)\rangle\langle\sigma_1(C_u^R)\sigma_1(B_u^R)\rangle \\ &\approx \frac{c_1}{6}\ln\left(\left(\frac{\beta}{\pi\epsilon}\right)^2\frac{\left(e^{-\frac{2\pi u}{\beta}} + w^+_{C_u^L}\right)\left(e^{\frac{2\pi u}{\beta}} - w^-_{C_u^L}\right)}{\left(1 + w^-_{C_u^L}w^+_{C_u^L}\right)}\right)\left(\frac{\left(e^{-\frac{2\pi u}{\beta}} + w^-_{C_u^R}\right)\left(e^{\frac{2\pi u}{\beta}} - w^+_{C_u^R}\right)}{\left(1 + w^-_{C_u^R}w^+_{C_u^R}\right)}\right), \end{aligned} \tag{3.3.25}$$

and $S_{\text{bath}}^{gen,2}(u)$ is the CFT_2 contribution to the bath generalised entropy

$$\begin{aligned} S_{\text{bath}}^{gen,2}(u) &= -\lim_{n\to 1}\frac{1}{1-n}\ln\int_{\mathbf{J}^1,\cdots,\mathbf{J}^n}\prod_{i=1,k=1}^{c_2,n} dJ_i^k P(J_i^k)\langle\sigma_2(C_u^L)\sigma_2(C_u^R)\rangle \\ &\approx -\lim_{n\to 1}\frac{1}{1-n}\ln\int_{\mathbf{J}^1,\cdots,\mathbf{J}^n}\prod_{i=1,k=1}^{c_2,n} dJ_i^k P(J_i^k)\langle\sigma_2(C_u^L)\rangle\langle\sigma_2(C_u^R)\rangle \approx 0. \end{aligned} \tag{3.3.26}$$

Here we used the fact that any one point function of a primary field in AdS_2 disk is identity. The solution for the associated coupled QES equations at late time is given by

$$w^{\pm}_{C_u^L} = \mp\frac{G_N\beta}{6\pi\phi_r}c_1 e^{\mp\frac{2\pi u}{\beta}}, \qquad w^{\pm}_{C_u^R} = \pm\frac{G_N\beta}{6\pi\phi_r}c_1 e^{\pm\frac{2\pi u}{\beta}}. \tag{3.3.27}$$

Substituting the QES solutions back into the expression for the generalised bath entropy (3.3.24) gives a constant equal to twice the area of black hole horizon, i.e.

$$S_{\text{bath}}^{gen}(u) = 2S_{\text{BH}}^0, \qquad u > u_B. \tag{3.3.28}$$

We must check whether the island with boundaries that matches with the above quantum extremal surfaces resolves the bath information paradox that appeared soon after black hole Page time u_{Page}. The eSSA inequality (3.3.23) is satisfied after the black hole Page time if we replace with $S_{\text{bath}}(u)$ with $S^{gen}_{\text{bath}}(u)$, as long as $S_{\text{journal}}(u) \leq S^{gen}_{\text{bath}}(u) \sim 2S^{0}_{\text{BH}}$. Until $u = u_I$ at which $S_{\text{journal}}(u_I) = 2S^0_{\text{BH}}$, the eSSA reduces to the demand that $S_{\text{journal}}(u) \geq 0$. Thus, the inclusion of the bath island enables all the three subsystems to satisfy the eSSA inequality (3.3.23) at least till $u = u_I$.

3.3.3 Transfer of the ownership of the black hole interior from bath to journal

After time $u = u_I$, while the bath owns a non-trivial island that contains the black hole interior, the eSSA (3.3.20) is given by

$$S_{\text{journal}}(u) - S^{gen}_{\text{bath}}(u) \leq 2S^0_{\text{BH}}, \qquad u > u_I. \tag{3.3.29}$$

It is straightforward to see that this inequality will be violated after time $u = u_J$, at which $S_{\text{journal}}(u_J) = 4S^0_{\text{BH}}$. At late times, for $u > u_J$, this leads to another unitarity violation or information paradox. In order to resolve this paradox, we shall search for an island for the journal by minimising the generalised entropy of journal associated with an arbitrary interval at $D^L_u D^R_u$. The generalised entropy of the journal associated with this interval is given by

$$S^{gen}_{\text{bath}} = \frac{\phi\left(D^L_u\right) + \phi\left(D^R_u\right)}{4G_N} + S^{gen,1}_{\text{journal}}(u) + S^{gen,2}_{\text{journal}}(u). \tag{3.3.30}$$

Here $S^{gen,1}_{\text{journal}}(u)$ is the CFT_1 contribution to the bath generalised entropy

$$S^{gen,1}_{\text{journal}}(u) = -\lim_{n\to 1}\frac{1}{1-n}\ln\langle\sigma_1(D^L_u)\sigma_1(D^R_u)\rangle \approx \frac{c_1}{6}\ln\left(\frac{\left(w^+_{D^R_u} - w^+_{D^L_u}\right)\left(w^-_{D^R_u} - w^-_{D^L_u}\right)}{\left(1 + w^-_{D^L_u}w^+_{D^L_u}\right)\left(1 + w^-_{D^R_u}w^+_{D^R_u}\right)}\right) \tag{3.3.31}$$

and $S^{gen,2}_{\text{journal}}(u)$ is the CFT_2 contribution to the bath generalised entropy.

$$\begin{aligned}
S^{gen,2}_{\text{journal}}(u) &= -\lim_{n\to 1}\frac{1}{1-n}\ln\int_{\mathbf{J}^1,\cdots\mathbf{J}^n}\prod_{i=1,k=1}^{c_2,n} dJ_i^k P(J_i^k)\langle O_{\mathbb{J}}(B_u^L)\sigma_2(D_u^L)\sigma_2(D_u^R)O_{\mathbb{J}}(B_u^R)\rangle \\
&\approx -\lim_{n\to 1}\frac{1}{1-n}\ln\int_{\mathbf{J}^1,\cdots\mathbf{J}^n}\prod_{i=1,k=1}^{c_2,n} dJ_i^k P(J_i^k)\langle O_{\mathbb{J}}(B_u^L)\sigma_2(D_u^L)\rangle\langle\sigma_2(D_u^R)O_{\mathbb{J}}(B_u^R)\rangle \\
&\approx \frac{c_2}{2}\ln\left[\ln\left(\left(\frac{\beta}{\pi\epsilon}\right)^2\frac{\left(1-e^{-\frac{2\pi u}{\beta}}w^-_{D_u^L}\right)\left(1+e^{\frac{2\pi u}{\beta}}w^+_{D_u^L}\right)\left(1+e^{\frac{2\pi u}{\beta}}w^-_{D_u^R}\right)\left(1-e^{-\frac{2\pi u}{\beta}}w^+_{D_u^R}\right)}{\left(1+w^-_{D_u^L}w^+_{D_u^L}\right)\left(1+w^-_{D_u^R}w^+_{D_u^R}\right)}\right)\right].
\end{aligned} \tag{3.3.32}$$

The generalised entropy minimisation with respect to $w^+_{D_u^L}$ and $w^-_{D_u^L}$ gives the following equations

$$\begin{aligned}
-\frac{\pi\phi_r}{G_N\beta}\frac{w^-_{D_u^L}}{\left(1+w^+_{D_u^L}w^-_{D_u^L}\right)^2}+\frac{c_2}{2\ln\left(\frac{\beta}{2\pi\epsilon}\right)}\left(\frac{1}{e^{-\frac{\pi u}{\beta}}+w^+_{D_u^L}}-\frac{w^-_{D_u^L}}{\left(1+w^+_{D_u^L}w^-_{D_u^L}\right)}\right)&=0 \\
-\frac{\pi\phi_r}{G_N\beta}\frac{w^+_{D_u^L}}{\left(1+w^+_{D_u^L}w^-_{D_u^L}\right)^2}-\frac{c_2}{2\ln\left(\frac{\beta}{2\pi\epsilon}\right)}\left(\frac{1}{e^{\frac{\pi u}{\beta}}-w^-_{D_u^L}}+\frac{w^+_{D_u^L}}{\left(1+w^+_{D_u^L}w^-_{D_u^L}\right)}\right)&=0.
\end{aligned} \tag{3.3.33}$$

The solution near horizon at late time is given by

$$\begin{aligned}
w^\pm_{D_u^L} &= \mp\frac{G_N\beta c_2}{2\pi\phi_r\ln\left(\frac{\beta}{2\pi\epsilon}\right)}e^{\mp\frac{2\pi u}{\beta}} \\
w^\pm_{D_u^R} &= \pm\frac{G_N\beta c_2}{2\pi\phi_r\ln\left(\frac{\beta}{2\pi\epsilon}\right)}e^{\pm\frac{2\pi u}{\beta}}.
\end{aligned} \tag{3.3.34}$$

Substituting this back to the expression (3.3.30) gives the generalised entropy of the subsystem journal as follows

$$\begin{aligned}
S^{gen}_{\text{journal}}(u) &= 2S^0_{\text{BH}}+\frac{c_1}{3}\ln\left[\frac{\beta}{\pi\epsilon}\cosh\left(\frac{2\pi u}{\beta}\right)\right] \\
&= 2S^0_{\text{BH}}+S_{\text{bath}}(u).
\end{aligned} \tag{3.3.35}$$

Let us allow the journal to own the island that contains the black hole interior instead of the bath after time $u = u_J$. Then the eSSA relation for $u > u_J$ is given by

$$S^{gen}_{\text{journal}}(u) - S_{\text{bath}}(u) \leq 2S^0_{\text{BH}}, \qquad u > u_J. \tag{3.3.36}$$

Using (3.3.35) we can verify that this eSSA relation gets saturated after time $u = u_J$. Therefore, transferring the ownership of the black hole interior after time $u = u_J$ from the bath to the journal restores unitarity. Consequently, the reconstruction of the black hole interior from radiation at late times requires complete knowledge of the theory.

Chapter 4

On operator theory and von Neumann algebras

4.1 Definitions in operator theory

Below, I will list a few definitions and results that will be relevant for later discussions; some of these will be relevant for unbounded operators on infinite dimensional Hilbert spaces, which present various interesting subtleties that are absent for bounded operators. The primary reference for this section is [42].

Definition 4.1.1. *(Unboundedness and the operator norm.) Let A be an operator acting on a Hilbert space $\mathcal{H}$, then we can define its operator norm as*

$$||A|| = \sup_{\psi \in \mathcal{H}} \frac{||A\psi||}{||\psi||} \in [0, \infty]. \tag{4.1.1}$$

Let me remark that is well known that position and momentum in quantum mechanics are realized as unbounded operators acting on infinite dimensional Hilbert spaces such as $L^2(\mathbb{R}^d)$. If one wants to study an unbounded operator A, one must therefore restrict to a domain D_A, where it may be well defined.

Definition 4.1.2. *(Adjoint.) The adjoint of an operator A is defined through the inner product, via the equation*

$$\langle \psi, A\phi \rangle = \left\langle A^{\dagger}\psi, \phi \right\rangle, \quad \phi \in D_A, \psi \in D_{A^{\dagger}}. \tag{4.1.2}$$

For bounded operators, one has $\phi, \psi \in \mathcal{H}$. However, for an unbounded operator A, one has to be very careful because the domains D_A and $D_{A^{\dagger}}$ are generally not equal. An important fact is that the adjoint of an operator A is unique if it is defined on a dense set of the Hilbert space, which I will define next.

Definition 4.1.3. *(Dense set.) A set D is said to be dense in a set $\mathcal{H}$ if for every $x \in \mathcal{H}$, there exists a sequence $\{a_n\}$ entirely in D such that $a_n \to x$. An operator $A : D \to \mathcal{H}$ for a dense set $D \subset H$ is said to be densely defined, and has a unique adjoint. We will always work with densely defined unbounded operators.*

Definition 4.1.4. *(Symmetric operator.) Let $A : D_A \to \mathcal{H}$ be a densely defined unbounded operator. If*

$$\forall \psi, \phi \in D_A, \quad \langle \psi, A\phi \rangle = \langle A\psi, \phi \rangle \tag{4.1.3}$$

then A is called symmetric.

Definition 4.1.5. *(Self-adjoint operator.) Let A be a symmetric operator. If $D_A = D_{A^{\dagger}}$, then $A = A^{\dagger}$ is a self adjoint operator.*

Definition 4.1.6. *(Quadratic form.) A quadratic form is a map $q : Q(q) \times Q(q) \to \mathbb{C}$, where $Q(q)$ is a dense subset of $\mathcal{H}$ called the form domain, such that $q(\cdot, \psi)$ is anti-linear and $q(\phi, \cdot)$ is linear on $\phi, \psi \in Q(q)$. A quadratic form is called symmetric if $q(\phi, \psi) = q^*(\psi, \phi)$.*

We will always be concerned with symmetric quadratic forms. The quantum fields in QFT are known to be operator valued distributions, or alternatively, (unbounded) symmetric quadratic forms. It is appropriate to switch to physics notation to bring this point out; we can think

$$q(\phi, \psi) = \langle \phi | Q\psi \rangle, \quad \psi, \phi \in D_q(q).$$

The theory of quadratic forms tells us that Q is guaranteed to exist as a bounded operator if q is a bounded form. However, for unbounded forms, as the ones in quantum field theory, the association is with unbounded operators, and is not always guaranteed. Despite this, one may insist on writing the right side of (**??**) for some formally defined Q, which is the common practice in QFT. To wit, if we denote our field as $Q(x)$, then in the quadratic form language it means there is a q_x such that $q_x(\phi, \psi) = q_x(\psi, \phi)^* = \langle \phi | Q(x)\psi \rangle$, which is a non-singular complex valued function of variable x for $\phi, \psi \in D_q$ [42]. In the more common language, we usually say that while $Q(x)$ may not be an operator, it can have well defined matrix elements $\langle \phi | Q | \psi \rangle$.

Next, let us look at some aspects of bounded operator theory, particularly, the von Neumann algebras.

4.2 von Neumann algebras

In the mathematical literature there are many different kinds of algebras, and it can often be confusing to keep track of them. In this section, let us slowly build the construction known as a von-Neumann algebra piece by piece, which will be a central object for further discussions. A natural start is to take a Hilbert space $\mathcal{H}$, and study the set of bounded operators $B(\mathcal{H})$ on it.[1] In order to do that, let A, B be two bounded operators on $\mathcal{H}$ and

[1]This is a Banach space, which means it has a norm with respect to which the space is complete. This norm for $a \in B(\mathcal{H})$ is given by the operator norm (4.1.1).

$\alpha \in \mathbb{C}$. Then we can see that

$$||\alpha A|| = |\alpha| \cdot ||A|| \tag{4.2.1}$$

$$||A + B|| \leq ||A|| + ||B|| \tag{4.2.2}$$

$$||AB|| \leq ||A|| \cdot ||B|| \tag{4.2.3}$$

which means that multiplication by a scalar, addition among bounded operators and multiplication of bounded operators, all result in bounded operators. While the first one is trivial, we can show the second and third line as follows. Note,

$$|(A + B)\psi| = |A\psi + B\psi| \leq |A\psi| + |B\psi| \tag{4.2.4}$$

by triangle inequality, which every norm satisfies. Take the supremum over ψ and we get (4.2.2). Next use the Cauchy Schwarz inequality for the inner product on $\mathcal{H}$, for two appropriate vectors:

$$|\langle A\psi|B\psi\rangle| \leq |A\psi| \cdot |B\psi| \tag{4.2.5}$$

But what is the left side? It is $\left\langle\psi|A^\dagger B|\psi\right\rangle = ||A^\dagger B||$. Now if we can show $A^\dagger$ and A both have the same norm, then we are done. To do that, note

$$|Ax| = \sup_{|y|=1} |\langle Ax|y\rangle| \tag{4.2.6}$$

$$= \sup_{|y|=1} |\left\langle x|A^\dagger y\right\rangle| \leq |x| \cdot ||A^\dagger|| \tag{4.2.7}$$

which establishes $||A|| \leq ||A^\dagger||$. We can do the reverse as well; that will give $||A|| = ||A^\dagger||$. Here we have used the identity $||u|| = \sup_{|v|=1} |\langle u|v\rangle|$, which is correct because the supremum is obtained for $v = \frac{u}{|u|}$. Also note that here we used the fact that $A^\dagger$ can act on any $y \in \mathcal{H}$, which is part of the definition of the adjoint for bounded operators [42].

Preceding discussions show that $\mathcal{B}(\mathcal{H})$ is an **algebra**. That is, simply, a vector space equipped with a bilinear multiplication operation under which vector space is closed. The equations (4.2.1), (4.2.2) establish the vector space structure of $\mathcal{B}(\mathcal{H})$, and (4.2.3) establishes that it is an algebra.

Now, let us proceed to a mathematical point. A vector space is a group for vector addition, therefore it contains in particular a zero vector, which is the identity element for the addition group. When dealing with an algebra, along with addition, we are dealing with operator multiplications through the bilinear product, and we'd like to have an identity operator that acts by multiplication. We further would like an an algebra that is closed under taking adjoints. This gives us a $*$-unital algebra, which is often called just a $*$-algebra by physicists. Now we can give a definition.

Definition 4.2.1. *A unital $*$-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if $\mathcal{A} = \mathcal{A}''$.*

Here $\mathcal{A}'$ is the commmutant of the algebra $\mathcal{A}$, and likewise $\mathcal{A}''$ is the double commutant of $\mathcal{A}$. We will be interested in von Neumann algebras which are factors because using these we can build more general algebras. A factor is defined as follows.

Definition 4.2.2. *A von Neumann algebra $\mathcal{A}$ is a called a factor if $\mathcal{A} \cap \mathcal{A}' = 1 \cdot \mathbb{C}$*

Now we can turn to briefly discuss the type classification of von Neumann algebras.

4.2.1 Types of von Neumann algebras

The von Neumann algebras have an intricate theory in which they are classified into three basic types, and then further classified according to more details. The three basic types are called type I, type II, and type III. The study of quantum mechanics for instance involves the use of type I von Neumann algebras, whereas it has been known for long now that type III algebras play a central role in quantum field theory. Over the recent years, interestingly, evidence is emerging that type II algebras may be involved in a description of quantum gravity. In this section, following [43], I will give a basic classification of these types, and then focus our attention to a particular construction known as the half-sided translations.

In order to classify the algebras, we need a few definitions.

Definition 4.2.3. *(Murray - von Neumann equivalence) Let $P, Q \in \mathcal{A}$ be two projectors in a von Neumann algebra $\mathcal{A}$. They are said to be equivalent if there exists a $V \in \mathcal{A}$ such that $V^*V = P$ and $VV^* = Q$.*

Definition 4.2.4. *A projection $P \in \mathcal{A}$ is said to be a finite projection (with respect to $\mathcal{A}$) if there is no projection $Q < P$ that is Murray - von Neumann equivalent to P.*

When we say $Q < P$, it means that $QP = PQ = Q$, but $Q \neq P$. Now, we can classify the algebras as

Definition 4.2.5. *A finite projector $P \in \mathcal{A}$ is said to be minimal if there is no other projection Q with $0 < Q < P$.*

Now, we can give the type classification

Definition 4.2.6. *A factor $\mathcal{A}$ is called Type I if it contains a non-zero minimal projection.*

Definition 4.2.7. *A factor $\mathcal{A}$ is called Type II if it contains non-zero finite projections (with respect to $\mathcal{A}$) but no non-zero minimal projections.*

Definition 4.2.8. *A factor $\mathcal{A}$ is called type III if it contains no non-zero finite projections (with respect to $\mathcal{A}$).*

This is the Murray - von Neumann classification of the algebras. A more deeper classification can be done using hyperfinite algebras e.g. the Infinite Tensor Product of Finite Factors of type I (IFTPI) construction, of Araki and Woods [44]. A modern account on it is available in [12] on how to do such constructions, and notably, it discusses the algebras of type III_1, which are the most relevant for quantum field theory. A central result which is now widely known is that type II and type III algebras when studied in quantum field theories leave a signature in the form of infinite entanglement entropy between subregions in the spacetime, which is associated to the non-factorizability of Hilbert spaces that could have acted on the separate regions in the spacetime.[2] I will have more to say about these points in the next chapter. For now, let us note, that it is only for these type III_1 algebras for which the construction we will study next is known to exist, which is called the half-sided translations.

4.3 Modular theory and review of half-sided translations

This subsection is primarily sourced from my paper [45].

Let me recall a few basic facts from algebraic quantum field theory before we begin. In quantum field theory, one can associate to any open region a von Neumann algebra of operators, which is type III_1 in nature. The theory of type III_1 algebras developed by Tomita and Takesaki, often called the modular theory, is the starting point in the construction of half-sided translations.

Consider a system of quantum fields in flat spacetime, in a state $|\Omega\rangle$ that is cyclic and separating for a von Neumann algebra $\mathcal{M}$ of operators. Cyclicity of the state means that $\mathcal{M}$ when acting on $|\Omega\rangle$ produces a dense subset of the full Hilbert space, and separability means that no operator in the algebra $\mathcal{M}$, except the trivial operator, annihilates $|\Omega\rangle$. These conditions are necessary and sufficient to define the Tomita operator S whose polar decomposition is

$$S = J_{\mathcal{M}} \Delta_{\mathcal{M}}^{1/2} \tag{4.3.1}$$

[2]Appropriately, these algebras are sometimes called ‘wild’ in contrast with ‘tame’ type I algebras.

where $\Delta_{\mathcal{M}} = \exp(-K_{\mathcal{M}})$ implements the automorphisms of the algebra $\mathcal{M}$,

$$e^{iK_{\mathcal{M}}t}\mathcal{M}e^{-iK_{\mathcal{M}}t} = \mathcal{M}, t \in \mathbb{R}, \tag{4.3.2}$$

with $J_{\mathcal{M}}$ an anti-unitary operator satisfying $J_{\mathcal{M}}\mathcal{M}J_{\mathcal{M}} = \mathcal{M}'$. Through the action of $J_{\mathcal{M}}$, one can see that the S operator knows about $\mathcal{M}'$; in fact $\Delta_{\mathcal{M}}$ also generates the automorphisms of $\mathcal{M}'$. The operator $K_{\mathcal{M}}$ is unbounded in general, and the equation (4.3.2) describes the *modular flow* of $\mathcal{M}$, and the operator $K_{\mathcal{M}}$ is the *modular* Hamiltonian for $\mathcal{M}$.

The set of half-sided translations is an intimately connected construction to the modular flow. Let me list the conditions of existence of half-sided translations below. We need two von Neumann algebra $\mathcal{M}, \mathcal{N}$ with $\mathcal{N} \subset \mathcal{M}$ that satisfy:

- The state $|\Omega_0\rangle$ is cyclic and separating for $\mathcal{M}$ and $\mathcal{N}$.

- The half-sided modular flow of $\mathcal{N}$ with $\Delta_{\mathcal{M}}$ lies within $\mathcal{N}$.

$$\Delta_{\mathcal{M}}^{-it}\mathcal{N}\Delta_{\mathcal{M}}^{it} \subset \mathcal{N}, \quad t \leq 0 \text{ (or } t \geq 0). \tag{4.3.3}$$

If satisfied, [46, 47, 48, 49, 50, 51] guarantee that

- Half-sided translations exist and are described by a unitary group $U(s) = e^{-i\mathcal{G}s}, s \in \mathbb{R}$ generated by a positive generator $\mathcal{G}$.

- $U(s)|\Omega_0\rangle = |\Omega_0\rangle, s \in \mathbb{R}$, or equivalently $\mathcal{G}|\Omega_0\rangle = 0$.

- $U^\dagger(s)\mathcal{M}U(s) \subset \mathcal{M}, \quad s \leq 0$

- $\mathcal{N} = U^\dagger(-1)\mathcal{M}U(-1)$.

- $[K_{\mathcal{M}}, K_{\mathcal{N}}] = -2\pi i(K_{\mathcal{M}} - K_{\mathcal{N}})$ and $K_{\mathcal{M}} - K_{\mathcal{N}} = 2\pi\mathcal{G}$.

- Given an $\mathcal{M}, \mathcal{N}$ and $|\Omega_0\rangle$ defined this way, $U(s)$ is unique.

- $\mathcal{M}$ needs to be a type III_1 von Neumann for $\mathcal{G}$ to be non-trivial.

Chapter 5

Spatial decomposition of operators in QFT

This chapter is primarily sourced from my paper [45].

In the algebraic quantum field theory (AQFT), an important group of unitary transformations that can translate fields through the spacetime is given by the half-sided translations [46, 50]. As suggested by their name, these transformations are unlike the ordinary spacetime translations, for their definition requires specification of *two* von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ satisfying $\mathcal{N} \subset \mathcal{M}$, which together characterize them. It is with respect to these algebras, that the word "translations" is used to describe the group, for when they exist, half of their group elements preserve the algebra $\mathcal{M}$, while the other half do not. Introduced in 1992 by Borchers in [46], and studied at length thereafter, the half-sided translations have enabled important advancements in AQFT [47, 48, 49, 50, 51, 52], and in the recent years, following the work of Leutheusser and Liu [14, 13], they have served as powerful tools in AdS/CFT for studying quantum field theories and quantum gravity [53, 54, 55, 56, 15, 57].

To what do the half-sided translations owe such a success in helping us understand quantum field theories, and how does one construct them, are two questions with the same answer – the properties of modular Hamiltonians. The generator $\mathcal{G}$ of half-sided translations for the given pair $\mathcal{M}$ and $\mathcal{N}$ is a function of their automorphism generators[1], known as the modular Hamiltonians, commonly denoted $K_{\mathcal{M}}, K_{\mathcal{N}}$ respectively [58]. To wit,

$$\mathcal{G} = \frac{1}{2\pi}(K_{\mathcal{M}} - K_{\mathcal{N}}). \tag{5.0.1}$$

In the past decade, the role of modular Hamiltonians has been studied at length in AdS/CFT and in particular they have been shown to be of primary importance for bulk reconstruction [5, 59, 60, 61, 62]. However, there is a qualification needed in this description that we must provide immediately, which will also allow us to introduce another key object for this work.

The high energy community often refers to the modular Hamiltonian as the “full modular Hamiltonian", in contrast to the quantities called “half-modular Hamiltonians" or often simply modular Hamiltonians, which are obtained by the logarithm of appropriate density operators. It is the latter, which have received most of the attention, and in condensed matter community these operators have been studied for many decades by the name entanglement Hamiltonians [63, 64]. While this observation may seem frivolous, the point is that there is an indisputable difference between the notions of modular Hamiltonians and entanglement Hamiltonians, which is the following. $K_{\mathcal{M}}$, the modular Hamiltonian for the algebra $\mathcal{M}$, generates the automorphisms for $\mathcal{M}$ and its commutant $\mathcal{M}'$ both. In contrast, writing $K_{\mathcal{M}} = H_{\mathcal{M}} - H_{\mathcal{M}'}$ in terms of entanglement Hamiltonians, the operators $H_{\mathcal{M}}$ and $H_{\mathcal{M}'}$ generate automorphisms of $\mathcal{M}$ and $\mathcal{M}'$ separately. When focusing on properties such as the subregion duality, the natural object of study is the entanglement Hamiltonian, which is the reason why it has been preferred in problems concerning subregions of spacetime. The point is also that the half-sided translations, modular Hamiltonians, and the entanglement Hamiltonians are intimately connected, and in fact, in terms of the

[1] A brief review is in section 2.

entanglement Hamiltonians, one can split $\mathcal{G}$ as

$$\mathcal{G} = G + G'; \quad G = \frac{1}{2\pi}(H_{\mathcal{M}} - H_{\mathcal{N}}), \quad G' = \frac{1}{2\pi}(H_{\mathcal{N}'} - H_{\mathcal{M}'}). \tag{5.0.2}$$

Having introduced the key objects of interest, let us now come to the premise of this work. The half-sided translations are able to translate a local operator $\phi(x,t)$ through unbounded regions in spacetime. This is because the involved modular Hamiltonians $K_{\mathcal{M}}, K_{\mathcal{N}}$ have a non-trivial effect on elements of both the algebras $\mathcal{M}, \mathcal{N}$ and their commutants. In contrast, the action of an entanglement Hamiltonian $H_{\mathcal{M}}$ on a local operator ϕ placed in the causal diamond $\diamond_{\mathcal{M}}$ associated to $\mathcal{M}$ cannot take it outside $\diamond_{\mathcal{M}}$, simply because $H_{\mathcal{M}}$ generates the automorphism of $\mathcal{M}$. The operators G, G' similarly cannot be expected to transport $\phi(x,t)$ through unbounded regions in spacetime either. One may then ask, how is it that G and G' combine together to create the effect of $\mathcal{G}$. We can make this language more precise as follows. Since $\mathcal{G}$ is the generator of half-sided translations, the group elements are given by $\exp(i\mathcal{G}s)$ for $s \in \mathbb{R}$. One may ask whether it is possible to understand this group element by "stitching" the effect of e^{iGs} and $e^{iG's}$ in some way.

The natural intuition from experience in quantum mechanics is to consider the Zassenhaus expansion of $e^{i\mathcal{G}s}$ [65, 66], using the nested commutators of G and G'. One may therefore hope that an expansion of the form

$$e^{is\mathcal{G}} = e^{isG'} e^{isG} \cdot \prod_{k=2}^{\infty} e^{(is)^k C_k} \quad (?), \tag{5.0.3}$$

with every C_k made from nested commutators of G, G', will provide the answer. However, even if one ignores the difficulty of computing the infinite set $\{C_k\}$ for the moment, there is an issue that $\mathcal{G}$ is generally an unbounded operator. Usually Zassenhaus decompositions involve exponentials of bounded operators on both sides, which can always be defined on the full Hilbert space. Fortunately $\mathcal{G}$ is self-adjoint, thus $e^{is\mathcal{G}}$ is unitary and therefore defined on the full Hilbert space $\mathcal{H}$. However, the status of the right side of (5.0.3) is questionable, in whether the decomposition works on any subspace of the Hilbert space. [2]

Importantly, the suggestion (5.0.3) faces a major objection which is the fact that entanglement Hamiltonians $\{H_{\mathcal{M}}, H_{\mathcal{N}}, H_{\mathcal{M}'}, H_{\mathcal{N}'}\}$ and therefore G, G' along with $\{C_k\}$ are ill-defined operators because of the continuum nature of quantum field theory [12]. This phenomenon is related to the fact that entanglement Hamiltonians cannot be regarded as logarithms of density matrices defined on separate Hilbert spaces in a strict sense because type III_1 algebras do not admit a trace [43]. An ill-defined operator here means that one whose matrix elements for suitable states may be well-defined, but as a map between states with a unique adjoint, it cannot be defined on any dense set,[3] that is, the mapped vector always has infinite norm. Consequently, even if one can compute the commutators of G and G', the right side of (5.0.3) will be an ill-defined operator which can be meaningful in its matrix elements for states in some dense subset of the Hilbert space at its best.

In this work, we will show this difficulty is surmountable by a regularization procedure. We will see that is possible to trade G, G' for operators $\hat{G}, \hat{G}'$ obeying $\hat{G} + \hat{G}' = \mathcal{G}$ and mimicking the former, such that all the nested commutators of $\hat{G}, \hat{G}'$ are well-defined operators. Consequently we will show that $e^{i\mathcal{G}s}$ can be decomposed using $\hat{G}, \hat{G}'$, not in a right-sided, but a centered Zassenhaus expansion of the form:

$$e^{is\mathcal{G}} = e^{is\hat{G}'} \cdot \prod_{k=2}^{\infty} e^{(is)^k D_k} \cdot e^{is\hat{G}} \tag{5.0.4}$$

where $\{D_k\}$ are made from nested commutators of $\hat{G}, \hat{G}'$, with $D_k = \lambda_k D(\epsilon) + \Sigma_k(\epsilon)$; λ_k is a c-number, ϵ is a regularization parameter, $\Sigma_k(\epsilon)$ are contributions arising from the associ-

[2] This expansion can indeed fail; cf. 5.4.

[3] Dense definition is required for a unique adjoint [42]. Put precisely, they are not operators, but are studied using unbounded bilinear forms, or operator valued distributions, and quantum fields fall in the same class. See section 5.2 for more details.

ated regularization, and importantly, $D(\epsilon)$, the primary component of D_k, is independent of k. We will in fact show that half-sided translations can be a special case in a large class of operations that permit such a Zassenhaus decomposition.

More specifically, we will be working with a general 2d conformal field theory in Minkowski spacetime, taking half-sided translations for $\mathcal{M}$ corresponding to the Rindler wedge as our prototypical example, and obtain the following results:

- We will show that norms of states $H_{\mathcal{M}}|\chi\rangle$ for generic CFT states $|\chi\rangle$ have a universal ultraviolet divergence coming from the boundary of the associated integral as suggested in [12], but also a universal infrared divergence. Contrarily, the states $G|\chi\rangle$ will only have a universal ultraviolet divergence that is marginal, and is easy to regularize. This will allow us to trade G, G' for $\hat{G}, \hat{G}'$ by a boundary modification of the integral, such that $\hat{G}|\chi\rangle, \hat{G}'|\chi\rangle$ have a finite norm. (Section 5.2).

- The operators $\exp(is\hat{G})$ and $\exp(is\hat{G}')$ will be shown to be equivalent to $\exp(is\mathcal{G})$ in regions of spacetime causally disconnected to $\hat{G}'$ and $\hat{G}$ respectively. However $\exp(is\hat{G}), \exp(is\hat{G}')$ will turn out to be non-unitary, and therefore $\hat{G}, \hat{G}'$ will not be self-adjoint. (Section 5.2.3).

- We will consider the most general integrated stress tensor on a connected interval in 1+1 dimension:

$$O = \int_{-b_L}^{b_R} dx \cdot [f(x)T(x) + \overline{f}(x)\overline{T}(x)] \tag{5.0.5}$$

  and split it into two parts, $O = O_L + O_R$, where

$$O_L = \int_{-b_L}^{-a_L} dx \cdot [f(x)T(x) + \overline{f}(x)\overline{T}(x)] + \int_{-a_L}^{a_R} dx \cdot [L(x)T(x) + \overline{L}(x)\overline{T}(x)] \tag{5.0.6}$$

and

$$O_R = \int_{a_R}^{b_R} dx \cdot [f(x)T(x) + \overline{f}(x)\overline{T}(x)] + \int_{-a_L}^{a_R} dx \cdot [R(x)T(x) + \overline{R}(x)\overline{T}(x)] \quad (5.0.7)$$

both involving degrees of freedom from a common region $(-a_L, a_R)$. See figure 5.1. Requiring the conditions on the functions

$$R(x) + L(x) = f(x), \quad \overline{R}(x) + \overline{L}(x) = \overline{f}(x), \quad x \in (-a_L, a_R) \quad (5.0.8)$$

$$R(-a_L) = \overline{R}(-a_L) = L(a_R) = \overline{L}(a_R) = 0, \quad (5.0.9)$$

we will show that if R satisfies the non-linear differential equation (5.3.26) and $\overline{R}$ the analogous one for $\alpha \to -\alpha$, then using the $i\epsilon$ prescription of [67] of comuputing commutators from the stress tensor OPE, all the nested commutators of O_R, O_L are proportional to $[O_R, O_L]$, permitting a formally defined closed form Zassenhaus expansion of the type

$$e^{iOt} = e^{\lambda(t)} \cdot e^{iO_L t} \cdot e^{iO_R t} \cdot \exp(\Omega(t)[O_R, O_L]) \quad (5.0.10)$$

for functions $\Omega(t)$ and $\lambda(t)$. We will see that half-sided translations corresponding to $\mathcal{M}$ as a Rindler wedge is a special case in this class, and thus we can obtain a formal expression

$$e^{i\mathcal{G}s} = e^{iG's} e^{iGs} \exp(f(s)[G, G']). \quad (5.0.11)$$

of the type (5.0.3). However the involved terms won't be operators unless regularized. (Section 5.3).

- We will show that regularizing commutators like $[G, G']$ is more tricky than regularizing G and G', especially when one wishes to compute *all* of their nested commutators. This will require use of non-analytic bump functions [68] for regularization of G, G'; these functions on the real line are infinitely differentiable smooth functions

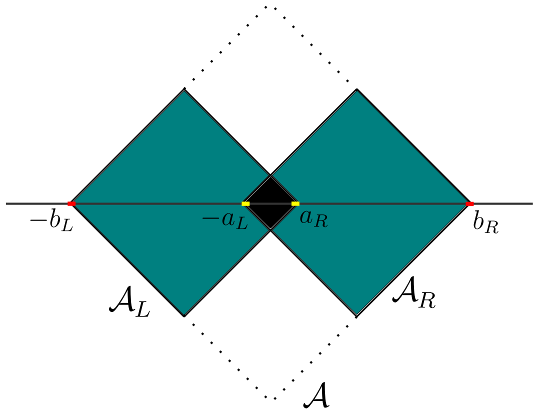

Figure 5.1: The setup for general treatment concerning the split of O in (5.0.5) into O_L, O_R. If e^{iOt} is an operator in the algebra $\mathcal{A}$, the results can be understood as that diamond region shown in black can enable the stitching of the operators in $\mathcal{A}_L$ and $\mathcal{A}_R$ to give e^{iOt} in $\mathcal{A}$ through a Zassenhaus decomposition. However, one may need to regularize them over small intervals near the boundaries, which are shown in yellow and red.

and will enable all nested commutators of $\hat{G}, \hat{G}'$ to be operators. (Section 5.3.4).

- We will show that even after regularization of G, G', the right-sided expansion of form (5.0.3) fails when excitations spacelike to the overlap region (black diamond in figure 5.1) are involved. The resolution of this will need a centered Zassenhaus formula, which we will derive following the methods of [69]. Consequently, we will be able to write

$$\exp(is\mathcal{G}) = e^{is\hat{G}'} \prod_{k=2}^{\infty} \exp\left[(is)^k \left(\frac{i^{k-2}}{k(k-2)!} D(\epsilon) + \Sigma_k(\epsilon)\right)\right] \cdot e^{is\hat{G}} \tag{5.0.12}$$

$$\text{where } D(\epsilon) = 2ai \int_{\epsilon}^{2a-\epsilon} T(x) dx \tag{5.0.13}$$

for $2a = a_R, a_L = 0$ and $b_L, b_R \to \infty$ when comparing to figure 5.1. $\Sigma_k(\epsilon)$ are stress tensor integrals over the intervals $(0, \epsilon)$ and $(2a - \epsilon, 2a)$ obtained by nested Wronskians of bump functions (5.3.37). In figure 5.1, these intervals are shown in yellow. Further, for conjugations $e^{is\mathcal{G}}\phi(x)e^{-is\mathcal{G}}$ with x confined to the overlap region and away from the yellow intervals, the Σ_k in (5.0.12) can be ignored, and

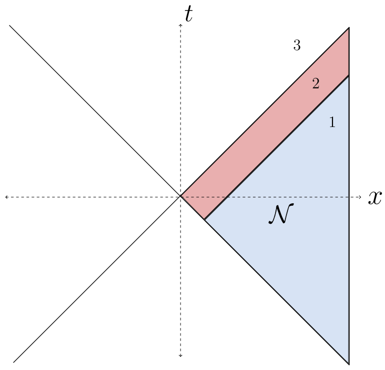

Figure 5.2: Rindler wedges and regions 1,2,3; the condition (4.3.3) is satisfied by algebras of the regions 1 and 2.

the infinite product simplifies to the closed form

$$\exp(is\mathcal{G}) \overset{*}{=} e^{is\hat{G}'} \cdot \exp\left[(e^{-s}(s+1)-1)D(\epsilon)\right] \cdot e^{is\hat{G}} \tag{5.0.14}$$

where $*$ denotes this conditional equality. (Section 5.4).

In the next section, we will give a brief review of half-sided translations, and the method of [67] for computing commutators of stress tensors. In sections 3, 4, and 5, we will obtain the mentioned results, and will conclude in section 6 with some future directions.

5.1 Background and methods

5.1.1 Half-sided translations for Rindler wedge

Consider that our conformal system is in the vacuum state $|0\rangle$ corresponding to time coordinate t, and let the algebras $\mathcal{M}, \mathcal{N}$ correspond to the causal diamond regions

$$\diamond_M = \{x^+ > t_0 + x_0, \quad x^- < t_0 - x_0\}, \quad \diamond_N = \{x^+ > t_0 + x_0, x^- < t_0 - x_0 - 2a\}. \tag{5.1.1}$$

These are Rindler wedges, with $M = (x_0, \infty)$ at $t = t_0$, and $N = (x_0 + a, \infty)$ at $t = t_0 - a$. The vacuum state $|0\rangle$ is cyclic and separating for the algebras $\mathcal{M}, \mathcal{N}$, which obey $\mathcal{N} \subset \mathcal{M}$ and satisfy the property (4.3.3). Thus half-sided translations exist and are generated by $\mathcal{G} = \frac{1}{2\pi}(K_{\mathcal{M}} - K_{\mathcal{N}})$, where $K_{\mathcal{M}}$ and $K_{\mathcal{N}}$ are 2π times the boost operators preserving $(t, x) = (t_0, x_0)$ and $(t, x) = (t_0 - a, x_0 + a)$ respectively [12, 64, 61]. This result is in fact independent of (x_0, t_0), and is given by

$$\mathcal{G} = 2aP^+ = a(H + P) = 2a \int_{-\infty}^{\infty} T(x)dx. \tag{5.1.2}$$

It is derived using a Poincaré generator and a stress tensor method in 6.0.2 and the result allows us to set $x_0 = 0, t_0 = 0$. See figure 5.2, where the Minkowski space is divded into three regions, where $R_1 = \diamond_{\mathcal{N}}, R_2 \cup R_1 = \diamond_{\mathcal{M}}$ and complement of $R_1 \cup R_2$ is R_3. We will refer to these regions later.

Each K decomposes in terms of entanglement Hamiltonians $H_{\mathcal{M}}, H_{\mathcal{N}}, H_{\mathcal{M}'}$, and $\mathcal{H}_{\mathcal{N}'}$ as discussed in the introduction, for

$$H_{\mathcal{M}} = 2\pi \int_{0}^{\infty} dx\ x \cdot T_{00}(x, 0), \tag{5.1.3}$$

$$H_{\mathcal{M}'} = -2\pi \int_{-\infty}^{0} dx\ x \cdot T_{00}(x, 0) \tag{5.1.4}$$

$$H_{\mathcal{N}} = 2\pi \int_{a}^{\infty} dx\ (x - a)T_{00}(x, -a), \tag{5.1.5}$$

$$H_{\mathcal{N}'} = -2\pi \int_{-\infty}^{a} dx\ (x - a)T_{00}(x, -a). \tag{5.1.6}$$

Let us now take the first step to compute the commutators of these quantities, by studying commutators of stress tensors in 2d CFT.

5.1.2 Commutators from the OPE

We will now address how to compute commutators of the stress tensor from their OPE. This can be done via the $i\epsilon$ prescription [67]. Using the Wick rotation $\tau = it$ or $t = -i\tau$, we will apply the rule

$$[O_i(t,x), O_j(0)] = \lim_{\epsilon\to 0}[O_i(t-i\epsilon,x)O_j(0) - O_i(t+i\epsilon,x)O_j]. \tag{5.1.7}$$

Note that stress tensor OPEs differ from the usual CFT definition by factors of -2π, which is explained in [70]. The $T(z)T(w)$ OPE is then

$$T(z)T(w) = \frac{c}{8\pi^2(z-w)^4} - \frac{T(w)}{\pi(z-w)^2} - \frac{\partial T(w)}{2\pi(z-w)}. \tag{5.1.8}$$

To get a useful answer from (5.1.7), we need to integrate both sides with respect to test functions. But first we must write (5.1.8) in terms of t. We will be using $z = x - t$ as our variable, therefore,

$$[T(z), T(w)] = \lim_{\epsilon\to 0}[T(x-(t-i\epsilon))T(w) - T(x-(t+i\epsilon))T(w)] \tag{5.1.9}$$

which after a contour integral against a test function (either in upper half or the lower half plane) gives

$$[T(z), T(w)] = -i(T(z)+T(w))\cdot\partial_z\delta(z-w) + \frac{ic}{24\pi}\partial_z^3\delta(z-w). \tag{5.1.10}$$

Note the very important minus sign compared to [67] that arises from our convention of writing $z = x + i\tau = x - t$. A similar calculation can be done to obtain $[T(z), \phi(w,\overline{w})]$ for primary field $\phi(w,\overline{w})$ of conformal weight $(h,\overline{h})$ that gives

$$[T(z), \phi(w,\overline{w})] = -i\left(\phi(z,\overline{w}) + (h-1)\phi(w,\overline{w})\right)\cdot\delta'(z-w). \tag{5.1.11}$$

Note that when dealing with anti-holomorphic components a minus sign will be picked because $\bar{z} = x+t$ and (5.1.9) is sensitive to the sign of t. Further note that because $T(x)\overline{T}(y)$ OPE does not have any singular terms, their commutator is zero. An explicit calculation of unequal time commutator relations can be done easily for the free massless scalar field to verify these statements. We will use these observations in section 5.3.

5.2 On G and G' as operators

Quantum fields are postulated to be operator valued distributions, which are symmetric bilinear forms over a dense domain $D \subset \mathcal{H}$ [71]. This means a field $A(x)$ has well-defined matrix elements $\langle\Psi|A(x)|\Phi\rangle$ for $\Psi, \Phi \in D$ but $A(x)$ is not an operator itself. The products of fields, or their derivatives, are similarly operator valued distributions [58], although one may need to regularize them first. For instance, the product of two fields e.g. $A(x)A(y)$ is divergent as $x \to y$. This is well known in CFT, where one can do a normal ordering when defining composite fields such as the stress tensor [72]. Consequently, composite fields can be defined to have finite matrix elements. However, bilinear forms are generally not operators.

Producing operators from fields requires smearing against smooth functions [71, 12]. These functions are usually required to have fast decay at infinity to get sensible operators, but this isn't always the case. For example, the boost $K_{\mathcal{M}}$ associated to the Rindler wedge $\diamond_{\mathcal{M}}$ (5.1.1)

$$K_{\mathcal{M}} = \int_{-\infty}^{\infty} x\, T_{00}(x)\, dx \tag{5.2.1}$$

is a well-defined operator even though the smearing function x is unbounded at infinity. As we have been discussing, this property is not enjoyed by the entanglement Hamiltonians $H_{\mathcal{M}}$ and $H_{\mathcal{M}'}$ given in (5.1.3), even though $K_{\mathcal{M}} = H_{\mathcal{M}} - H_{\mathcal{M}'}$. While $H_{\mathcal{M}}, H_{\mathcal{M}'}$ are well-defined bilinear forms, they fail to be operators [12]. Below we will show this and study the reason for this failure, which will be important later for discussion of G and G'.

5.2.1 Entanglement Hamiltonians as operators?

Let us show that $H_{\mathcal{M}}$ and $H_{\mathcal{M}'}$ are not operators by a contradiction proof. We will use the fact that K is a well-defined operator with the vacuum $|0\rangle$ in its domain, and the OPE of stress tensors to arrive at our result.

Let us start by assuming that $H_{\mathcal{M}}$ and $H_{\mathcal{M}'}$ are densely defined operators on domains $D_{\mathcal{M}}$ and $D_{\mathcal{M}'}$ and hence have unique adjoints. As $H_{\mathcal{M}}, H_{\mathcal{M}'}$ are integrals of $T_{00}(x)$, they are symmetric operators. Now, note that the domain of $K = H_{\mathcal{M}} - H_{\mathcal{M}'}$ contains $|0\rangle$. In fact $K|0\rangle = 0$, as boost is a symmetry. Crucially, $D_K \subset D_{\mathcal{M}} \cap D_{\mathcal{M}'}$, and therefore,

$$|0\rangle \in D_K \implies |0\rangle \in D_{\mathcal{M}}, D_{\mathcal{M}'}. \tag{5.2.2}$$

In other words, $H_{\mathcal{M}}|0\rangle = H_{\mathcal{M}'}|0\rangle$ is a state of finite norm. This we will show is false, by computing the norm using the OPE. Because $H_{\mathcal{M}}$ is symmetric and $|0\rangle \in D_{\mathcal{M}}$, note that norm squared of $H_{\mathcal{M}}|0\rangle$ is $\langle 0|H_{\mathcal{M}}^2|0\rangle$ which is[4]

$$||H_{\mathcal{M}}|0\rangle||^2 = \int_0^\infty dx \int_0^\infty dy \cdot xy \cdot \langle 0|T_{00}(x+i\epsilon)T_{00}(y)|0\rangle \tag{5.2.3}$$

where we have put in an $i\epsilon$ to appropriately define the product $H_{\mathcal{M}}^2$; we will it take to zero at the end of calculation. Writing $T_{00}(x,0) = T(x) + \overline{T}(x)$, and using the OPE, the central charge term gives

$$\frac{c}{4\pi^2}\int_0^\infty dx \int_0^\infty dy \cdot xy \cdot \frac{1}{(x-y+i\epsilon)^4} = \frac{c}{24\pi^2}\int_0^\infty \frac{xdx}{(x+i\epsilon)^2} \tag{5.2.4}$$

which is a divergent integral, both because of $x = 0$ contributions and the asymptotic

[4]As the form is unbounded, it follows that $|\chi\rangle, H_{\mathcal{M}}|\chi\rangle \in D_{\mathcal{M}}$, then $\langle H_{\mathcal{M}}\chi|H_{\mathcal{M}}\chi\rangle = \langle\chi|H_M^2|\chi\rangle$. Last step is due to the fact that a symmetric operator A coincides with its adjoint $A^\dagger$ on the domain of A.

contributions. Therefore $|0\rangle \notin D_{\mathcal{M}}$. We could have similarly done it for $H'_{\mathcal{M}}$. Thus we have a contradiction, and therefore $H_{\mathcal{M}}, H_{\mathcal{M}'}$ are not operators. □

Extension using Reeh-Schlieder theorem

Concerning the norm of states $H_M|\chi\rangle$, we can actually derive a stronger result. It was noted in [12] that norm of $H_{\mathcal{M}}|\chi\rangle, H_{\mathcal{M}'}|\chi\rangle$ has a universal ultraviolet divergence. Using the Reeh-Schlieder theorem and the OPE of the stress tensor we can explicitly show that $H_{\mathcal{M}}|\chi\rangle, H_{\mathcal{M}}|\chi\rangle$ norms are infinite for χ in an arbitrarily large dense set of the vacuum sector of the Hilbert space, with both ultraviolet and infrared divergences.

The key point in the derivation concerns the radius of convergence of the stress tensor OPE, which is determined by the distance to the nearest operator not included in the expansion [70]. If we identify some states χ such that in computation of $||H_{\mathcal{M}}|\chi\rangle||^2$ we can draw a circle around $T(x)$ and $T(y)$ containing no other insertion as $x, y \to 0$ and $x, y \to \infty$, then their norm squares are of the type (5.2.4).

The Reeh-Schlieder theorem tells us that such states χ can approximate any state in the vacuum sector of the Hilbert space $\mathcal{H}_0$. More precisely, let $\mathcal{A}_{\mathcal{U}}$ be the polynomial algebra associated to an open region $\mathcal{U}$ in spacetime, that is, algebra generated using operators of the form $\int_{\mathcal{U}} f(x)\phi(x)dx$. The theorem states that $\mathcal{A}_{\mathcal{U}}|0\rangle$ is a dense subset of $\mathcal{H}_0$ for any $\mathcal{U}$ [58, 12].

Now, for us, provided that $\mathcal{U}$ is a bounded region and does not contain $(0,0)$, we can always find an $\alpha_{\mathcal{U}}, \beta_{\mathcal{U}}$ such that for $0 \leq x, y < \alpha_{\mathcal{U}}$ and $x, y \geq \beta_{\mathcal{U}}$, the OPE for $T(x)T(y)$ converges. For any such $\mathcal{U}$, we can take any $\chi \in \mathcal{A}_{\mathcal{U}}|0\rangle$, and we will find that $||H_{\mathcal{M}}|\chi\rangle||^2$ diverges because of the χ independent central charge term. For instance, the asymptotic contribution is

$$\int_{\beta_{\mathcal{U}}}^{\infty} dx \int_{\beta_{\mathcal{U}}}^{\infty} dy \cdot xy \cdot \frac{1}{(x-y+i\epsilon)^4} = \int_{\beta_{\mathcal{U}}}^{\infty} dx\, x \frac{3\beta_{\mathcal{U}} + i\epsilon - x}{6(\beta_{\mathcal{U}} + i\epsilon - x)^3} \sim \int_{\beta_{\mathcal{U}}}^{\infty} \frac{dx}{x}. \tag{5.2.5}$$

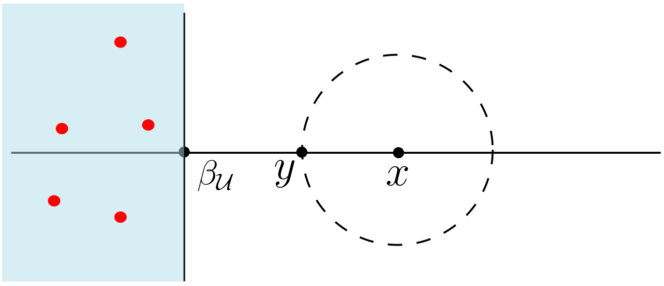

Figure 5.3: The OPE of two stress tensor insertions $T(x)T(y)$ converges for $x, y > \beta_{\mathcal{U}}$ where $\beta_{\mathcal{U}}$ is chosen such that all excitations to produce χ lie before $x < \beta_{\mathcal{U}}$.

See figure 5.2 for an illustration. Similarly there is a contribution from near $x = 0$.

Let us note two key points in the above discussion which will be important below. The first is that the integrals (5.2.4), (5.2.5) are divergent not only because of $x = 0$ contributions, but also due to infrared contributions coming from the slow decay of $1/x$. The second point is the observation that these integrals are marginally divergent. That is, if the integrand had been changed from $x^{-1} \to x^{-1+\kappa}$ near the origin, and $x^{-1} \to x^{-1+\kappa}$ asymptotically, for any $\kappa > 0$, the integrals would have converged. We will next see that $||G|\chi\rangle||^2$ and $||G'|\chi\rangle||^2$ are not infrared divergent, and their UV divergence can be regularized conveniently because of its marginal nature.

5.2.2 Forms G, G' to operators $\hat{G}, \hat{G}'$

Let us rewrite $G = H_{\mathcal{M}} - H_{\mathcal{N}}, G' = H_{\mathcal{N}'} - H_{\mathcal{M}'}$ obtained from (5.1.3) using properties of the stress tensor in 2d CFT as follows. From the definition,

$$G = \frac{H_{\mathcal{M}} - H_{\mathcal{N}}}{2\pi} = \int_0^\infty dx \cdot x \cdot T_{00}(x, 0) - \int_a^\infty dx \cdot (x - a) \cdot T_{00}(x, -a) \tag{5.2.6}$$

$$= \int_0^\infty dx \cdot x \cdot T_{00}(x, 0) - \int_0^\infty dx \cdot x \cdot T_{00}(x + a, -a). \tag{5.2.7}$$

Shifting the integration variable, and using the transformation rule $T_{00}(x,t) = T(x-t) + \overline{T}(x+t)$,

$$G = \int_0^\infty dx \cdot x \cdot [T(x) + \overline{T}(x)] - \int_0^\infty dx \cdot x \cdot [T(x+2a) + \overline{T}(x)] \tag{5.2.8}$$
$$= \int_0^\infty dx \cdot x \cdot [T(x) - T(x+2a)] \tag{5.2.9}$$
$$= \int_0^{2a} dx \cdot x \cdot T(x) + 2a \int_{2a}^\infty dx \cdot T(x). \tag{5.2.10}$$

Likewise,

$$G' = \int_0^{2a} dx \cdot (2a - x) \cdot T(x) + 2a \int_{-\infty}^0 dx \cdot T(x) \tag{5.2.11}$$

These mimic $2aP^+$ as $x \to \pm\infty$, which is a well-defined operator, and therefore the norm of $G|\chi\rangle, G'|\chi\rangle$ cannot have a universal infrared divergences. Near $x = 0$ however, the behaviour is same as $H_{\mathcal{M}}$, and therefore the norms of the states $G|\chi\rangle, G|\chi'\rangle$ show a universal UV divergence. However, as we saw in the last subsection, this divergence is marginal and can be fixed conveniently.

Let us replace G, G' by suitable $\hat{G}, \hat{G}'$. It is important to recognize that a change in G must be balanced with a change in G' to maintain $\mathcal{G} = G + G' = \hat{G} + \hat{G}'$, and because there are two points requiring regulation, i.e. $x = 0$ and $x = 2a$, we will need two functions μ and ν. Let

$$\hat{G} = 2a \int_{2a}^\infty dx \cdot T(x) + \int_0^\epsilon \mu(x) T(x) + \int_\epsilon^{2a-\epsilon} dx \cdot x \cdot T(x) + \int_{2a-\epsilon}^{2a} \nu(x) T(x) \tag{5.2.12}$$
$$\hat{G}' = 2a \int_{-\infty}^0 dx \cdot T(x) + \int_0^\epsilon [2a - \mu(x)] T(x) + \int_\epsilon^{2a-\epsilon} dx \cdot (2a - x) \cdot T(x) + \int_{2a-\epsilon}^{2a} [2a - \nu(x)] T(x) \tag{5.2.13}$$

for $\epsilon \ll 1$. One possible choice for $\mu(x)$ and $\nu(x)$ is the minimal regularization

$$\mu(x) = x\left(\frac{x}{\epsilon}\right)^{\epsilon}, \quad \nu(x) = 2a - \frac{1}{\epsilon^{\epsilon}}(2a-x)^{1+\epsilon} \tag{5.2.14}$$

which is sufficient to ensure that norms of $\hat{G}|\chi\rangle, \hat{G}'|\chi\rangle$ do not display universal UV divergences. This prescription satisfies continuity at $x = \epsilon, 2a - \epsilon$, which is an obvious requirement. Note that in the limit $\epsilon \to 0$, $\mu(x) = \nu(x) = x$, which suggests G, G' by themselves possibly fall just marginally short of being operators. If the divergence was stronger, requiring a different regularization like $\mu(x) \sim x^2$, continuity would demand $\frac{1}{\epsilon}$ factors in the denominator which would diverge in the limit $\epsilon \to 0$.

Let us note two more points in this subsection. First is that the contributions from the additional boundaries introduced in the various integral splits we considered cancel against each other, therefore we don't need to regularize integrands there. Second point, which we will return to in section 5.3.4, is that any analytic function used for regularization will have a derivative problem at the points $x = \epsilon, x = 2a - \epsilon$. For example, $\mu(x) = x^{1+\epsilon}\epsilon^{-\epsilon}$ at $x = \epsilon$ is ϵ, but its derivative at $x = \epsilon$ is not zero. We can regularize using higher degree polynomials which would have N derivatives continuous at $x = \epsilon$, but N cannot be infinite. This will be an important point when studying commutators later, which will force us to use non-analytic bump functions for regulation.

5.2.3 The conjugations $e^{iGs}\phi e^{-iGs}$ and $e^{iG's}\phi e^{-iG's}$

What is the nature of the formal operators G, G', and what dynamics do they generate? Is it consistent with causality? To understand this, let us study the behaviour of e^{iGs} and $e^{iG's}$. At the end, we will see that our discussions will allow conclusions for $e^{is\hat{G}}, e^{is\hat{G}'}$. Let us start with the quantity

$$\Phi(s; x, t) = e^{iGs}\phi(x, t)e^{-iGs} \tag{5.2.15}$$

for a primary field ϕ. This will be a bilinear form again, because G is known to have problems being an operator as discussed above. However we can do this computation anyway by the Hadamard lemma,

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{(is)^n}{n!} \mathrm{ad}_G^n \phi(x,t) \tag{5.2.16}$$

where the notation is $\mathrm{ad}_G\phi = [G,\phi], \mathrm{ad}_G^k\phi = [G, \mathrm{ad}_G^{k-1}\phi]$, and $\mathrm{ad}_G^0\phi = \phi$. Now $[G,\phi]$ is given by[5]

$$[G,\phi(x,t)] = \int_0^{2a} dz \cdot z[T(z),\phi(x,t)] + 2a \int_{2a}^{\infty} dz \cdot [T(z),\phi(x,t)]. \tag{5.2.17}$$

This requires us to obtain $[T(z),\phi(x,t)]$ which can be done using $i\epsilon$ prescription starting from the OPE. We will find (5.1.11):

$$[T(z),\phi(w,\overline{w})] = -i\left(\phi(z,\overline{w}) + (h-1)\phi(w,\overline{w})\right) \cdot \delta'(z-w)$$

where $\overline{w} = x+t, w = x-t$ and h is the holomorphic conformal weight for ϕ. Let us at first place (x,t) in region $R_1 = \diamond_N$ of figure 5.2 and define for convenience

$$I_1 = 2a\int_{2a}^{\infty} dx \cdot T(x), \quad I_2 = \int_0^{2a} dx \cdot x \cdot T(x) \tag{5.2.18}$$

where the notation references the same figure. Then

$$\begin{aligned}
[G,\phi(x,t)] &= 2a\int_{2a}^{\infty} dz \cdot [T(z),\phi(x,t)] && (5.2.19)\\
&= -i\,2a\int_{2a}^{\infty} dz \cdot (\phi(z) + (h-1)\phi(w,\overline{w})) \cdot \delta'(z-w) && (5.2.20)\\
&= 2ai\int_{2a}^{\infty} dz \cdot \partial_z\,(\phi(z) + (h-1)\phi(w,\overline{w})) \cdot \delta(z-w) && (5.2.21)
\end{aligned}$$

[5]More precisely these integrals should be understood as integrals over matrix elements of the involved operator valued distributions.

provided we are far away from boundary $x-t = 2a$. This gives us $i\partial_w\phi(w)$, but $w = x-t = -x^-$, and hence

$$[G, \phi(x,t)] = -2ai\partial_-\phi(x,t) \quad \text{or} \quad \frac{1}{i}[\phi(x,t), G] = 2a \cdot \partial_-\phi(x,t) \tag{5.2.22}$$

which is the standard Heisenberg equation of motion for ϕ under the operator G. Due to the constant velocity, $\Phi(s)$ becomes a standard Taylor series, describing the flow of ϕ along the null line x^-. This establishes that G is acting like $\mathcal{G}$ provided we are in region 1. When we approach the boundary, we have to be more careful to include the boundary term. Then,

$$[I_1, \phi] = -2ai[\phi(z, \overline{w}) + (h-1)\phi(w, \overline{w})]\delta(z-w)\Big|_{2a}^{\infty} + 2ai\int_{2a}^{\infty} dz\ \partial_z\phi(z)\delta(z-w) \tag{5.2.23}$$

which at the boundary $w = 2a$ becomes

$$ai \cdot \partial\phi(2a, \overline{w}) + 2ai\frac{1}{2\epsilon} \cdot [\phi(2a, \overline{w}) + (h-1)\phi(2a, \overline{w})] \tag{5.2.24}$$

by approximating δ function as $\frac{1}{2\epsilon}$ on a width of 2ϵ. This is important because integration by parts requires the boundary term to be regular. On the other hand, for $[I_2, \phi]$

$$[I_2, \phi] = \int_{\epsilon}^{2a} dz \cdot z \cdot [T(x), \phi(y)] \tag{5.2.25}$$

$$= -i\int_0^{2a} dz \cdot z \cdot (\phi(z, \overline{w}) + (h-1)\phi(w, \overline{w})) \cdot \delta'(z-w) \tag{5.2.26}$$

$$= -iz[\phi(z, \overline{w}) + (h-1)\phi(w, \overline{w})] \cdot \delta(z-w)\Big|_0^{2a} + i\int_0^{2a} dz \cdot [z\partial\phi(z, \overline{w}) + h\phi(w, \overline{w})]\delta(z-w). \tag{5.2.27}$$

At the boundary $w = 2a$, this term also contributes, and we get

$$-2ai[\phi(2a,\overline{w}) + (h-1)\phi(2a,\overline{w})] \cdot \frac{1}{2\epsilon} + \frac{i}{2}\left(2a\partial\phi(2a,\overline{w}) + h\phi(2a,w)\right). \quad (5.2.28)$$

Hence when I_1, I_2 are combined, the singularity cancels for $w = 2a$, but note there still is a discontinuity because of $\frac{h}{2}\phi(2a,\overline{w})$. This is due to the fact that writing $G = \int_0^\infty dx \cdot f(x)T(x)$, the function f has a discontinuous derivative at $x = 2a$. For $w < 2a$, only I_2' s integral contributes,

$$[I_2, \phi(w,\overline{w})] = i \cdot w\partial_w\phi(w,\overline{w}) + ih\phi(w,\overline{w}) \quad (5.2.29)$$

where the first term produces dilation, and the second produces a multiplicative effect. We can see this better in $\Phi(s)$. Take the double commutator,

$$[I_2, [I_2, \phi(w,\overline{w})]] = i \cdot w[I_2, \partial\phi(w,\overline{w})] + ih \cdot [I_2, \phi(w,\overline{w})]. \quad (5.2.30)$$

We can take the derivative action on w outside the commutator, to write

$$[I_2, [I_2, \phi]] = i(h + w\partial)[I_2, \phi(w,\overline{w})] \quad (5.2.31)$$

$$= i(h + w\partial)[i(h + w\partial)\phi(w,\overline{w})]. \quad (5.2.32)$$

We therefore can see,

$$\mathrm{ad}_{I_2}^k\phi(w,\overline{w}) = i^k(h + w\partial)^k\phi(w,\overline{w}) \quad (5.2.33)$$

and thus, if $(w, \overline{w})$ was in region 2,

$$\Phi(s) = \sum_{j=0}^{\infty} \frac{(is)^j}{j!} \mathrm{ad}_{I_2}^j \phi(w, \overline{w}) \tag{5.2.34}$$

$$= \sum_{j=0}^{\infty} \frac{(-s)^j}{j!} (h + w\partial)^j \phi(w, \overline{w}) \tag{5.2.35}$$

$$= e^{-s(h+w\partial)} \phi(w, \overline{w}) = e^{-sh} \phi(e^{-s} w, \overline{w}). \tag{5.2.36}$$

Thus we see, that operator is moving towards $w = 0$ (through a dilation), the boundary of region 2 and 3, and its also diminishing due to the e^{-sh} factor.

Calculations for G'

Similar calculations can be done for $e^{iG's}$ by symmetry. We will find

$$[G', \phi(w, \overline{w})] = i(2a - w)\partial_w \phi(w, \overline{w}) - ih\phi(w, \overline{w}) \tag{5.2.37}$$

for $(w, \overline{w})$ in region 2, which implies

$$\Phi(s) = e^{-s(-h+(2a-w)\partial)} \phi(w, \overline{w}) = \exp[(sh + s(w - 2a)\partial)]\phi(w, \overline{w}). \tag{5.2.38}$$

as long as $(w(s), \overline{w})$ remains in region 2. By a change of variable, $u = \ln(w - 2a)$, the exponential operator can become a translation operator which simplifies to

$$\Phi(s) = e^{sh} \phi(-e^s(2a - w) + 2a, \overline{w}) \tag{5.2.39}$$

and we can find the time till this expression is valid by setting $e^s(2a - w) + 2a = 0$, corresponding to $w = 0$ boundary of region 2, which gives

$$s_1 = \ln \frac{2a}{2a - w} \tag{5.2.40}$$

If $(w, \overline{w})$ was in region 3, then G' would act like $\mathcal{G}$, just like how G did in (5.2.22).

Now what if we replace G, G' by $\hat{G}, \hat{G}'$? The only changes will be near the boundaries. However, the factors e^{-sh}, e^{sh} will persist. Conjugation by $e^{i\hat{G}s}$ or $e^{i\hat{G}'s}$ therefore will change the operator norm, and thus these operators cannot be unitary. In other words, while $\hat{G}, \hat{G}'$ may be symmetric operators, they cannot be self-adjoint.[6]

5.3 Commutator computations

We will now compute the commutators for a general class of integrated stress tensors, and later specialize to the half-sided translations. The commutator of stress tensor is obtained by taking products of fields without regularizing them, and therefore $\delta(x)$ singularities appear naturally in them. As we will be integrating over these singularities, we will need to be careful especially in integration by parts. As mentioned above, for using the formula of integration by parts, we will need the boundary terms to be regular, and this will require us to express delta function as a $1/2\epsilon$ on interval of width 2ϵ, which we will take to zero at the end.

5.3.1 Commutators of integrals of stress tensor

Let us consider the operator

$$\mathcal{O} = \int_{-b_L}^{b_R} dx \cdot [f(x)T(x) + \overline{f}(x)\overline{T}(x)]. \tag{5.3.1}$$

[6]In the theory of bilinear forms, this corresponds to the case that $\hat{G}, \hat{G}'$ as forms are not closable [42].

We will split it into two parts, $O = O_L + O_R$, where

$$O_L = \int_{-b_L}^{-a_L} dx \cdot [f(x)T(x) + \overline{f}(x)\overline{T}(x)] + \int_{-a_L}^{a_R} dx \cdot [L(x)T(x) + \overline{L}(x)\overline{T}(x)] \tag{5.3.2}$$

and

$$O_R = \int_{a_R}^{b_R} dx \cdot [f(x)T(x) + \overline{f}(x)\overline{T}(x)] + \int_{-a_L}^{a_R} dx \cdot [R(x)T(x) + \overline{R}(x)\overline{T}(x)]. \tag{5.3.3}$$

The condition $O_L + O_R = O$ constrains the functions $L, R, \overline{L}$ and $\overline{R}$:

$$R(x) + L(x) = f(x), \quad \overline{R}(x) + \overline{L}(x) = \overline{f}(x), \quad x \in (-a_L, a_R). \tag{5.3.4}$$

Further, we would like to require continuity conditions, that

$$R(-a_L) = \overline{R}(-a_L) = L(a_R) = \overline{L}(a_R) = 0. \tag{5.3.5}$$

Using causality, these conditions allow us to write $[O_R, O_L]$ as integrals over just $(-a_L, a_R)$:

$$[O_R, O_L] = \int_{-a_L}^{a_R} dx \int_{-a_L}^{a_R} dy \left(R(x)L(y)[T(x), T(y)] + \overline{R}(x)\overline{L}(y)[\overline{T}(x), \overline{T}(y)]\right). \tag{5.3.6}$$

Below at first we will set $\overline{f}(x) = \overline{R}(x) = \overline{L}(x) = 0$ which will be sufficient to discuss the special case of half-sided translations for Rindler wedge. As the OPE of T and $\overline{T}$ does not have any singular terms, $[T(x), \overline{T}(y)] = 0$, therefore the extension to the more general case can be done in a straightforward manner at the end. Also, we will set $I = (-a_L, a_R)$ for brevity. Now we can write using (5.1.8)

$$[O_R, O_L] = \int_I dy \int_I dx \cdot L(y)R(x) \left(-i[T(x) + T(y)] \cdot \delta'(x - y) + \frac{ic}{24\pi} \cdot \delta'''(x - y)\right). \tag{5.3.7}$$

Thus,

$$[O_R, O_L] = i\int_I dy \int_I dx\, L(y)\left(\partial_x(R(x)\cdot[T(x)+T(y)]) - \frac{c}{24\pi}R'''(x)\right)\delta(x-y) + \Sigma_1 \tag{5.3.8}$$

where

$$\Sigma_1 = \int_I dy\left(-iL(y)R(x)\cdot[T(x)+T(y)]\delta(x-y)\Big|_{-a_L}^{a_R} + \frac{icL(y)R(x)}{24\pi}\cdot\delta''(x-y)\Big|_{-a_L}^{a_R}\right). \tag{5.3.9}$$

Because of the boundary condition $R(a_{-L}) = 0$, this becomes

$$\Sigma_1 = \int_I dy\left(-iL(y)R(a_R)\cdot[T(a_R)+T(y)]\delta(a_R-y) + \frac{icL(y)R(a_R)}{24\pi}\cdot\delta''(a_R-y)\right). \tag{5.3.10}$$

Because the integral is over the interval $(-a_L, a_R)$ and there is a δ function at the boundary, such an evaluation would need a lot of care, but in this case since $L(a_R) = 0$, this integral is easily zero. Therefore

$$[O_R, O_L] = i\int_I dy \int_I dx\cdot L(y)\left(R'(x)\cdot[T(x)+T(y)] + R(x)\cdot T'(x) - \frac{c}{24\pi}\cdot R'''(x)\right)\delta(x-y) \tag{5.3.11}$$

$$= i\int_I dy\cdot L(y)\left(2R'(y)\cdot T(y) + R(y)\cdot T'(y) - \frac{c}{24\pi}\cdot R'''(y)\right). \tag{5.3.12}$$

The non-trivial operator content comes from first integral, which is

$$i\int_I dy\; L(y)\cdot[2R'(y)\cdot T(y) + R(y)\cdot T'(y)]. \tag{5.3.13}$$

Integration by parts of the second term gives

$$i\left(L(y)R(y)\cdot T(y)\Big|_{-a_L}^{a_R} - \int_I dy\cdot T(y)\cdot[R'(y)L(y) + L'(y)R(y)]\right). \tag{5.3.14}$$

This new boundary term here is easily seen to be zero. Thus

$$[O_R, O_L] = -\frac{ic}{24\pi}\int_I dy \cdot L(y)R'''(y) + i\int_I dy \cdot [L(y)R'(y) - L'(y)R(y)]\, T(y). \tag{5.3.15}$$

If f is a constant, $L'(y) = -R'(y)$, and the non-trivial bulk term becomes $if\int_I dy \cdot R'(y)T(y)$. At this stage let us note the appearance of Wronskian of functions L, R defined by $W(L, R) = LR' - L'R$. We will see that these appear each time we take a nested commutator. Let us define $\omega(x) = L(x)R'(x) - L'(x)R(x)$ below.

5.3.2 Nested commutators

Next, let us evaluate $[O_R, [O_R, O_L]]$. Because the central charge term is proportional to identity operator,

$$[O_R, [O_R, O_L]] = i\int_I dy\int_I dx \cdot [R(x)T(x), \quad \omega(y)T(y)] \tag{5.3.16}$$

$$= i\int_I dy\int_I dx \cdot \omega(y) \cdot R(x) \cdot \Big(-i[T(x) + T(y)]\delta'(x - y) + \frac{ic}{24\pi}\delta'''(x - y)\Big). \tag{5.3.17}$$

This is the same form as $[O_R, O_L]$ with $L \to \omega$. Therefore

$$[O_R, [O_R, O_L]] = -\frac{i^2 c}{24\pi}\int_I dy \cdot \omega(y)R'''(y) + i^2\int_I dy \cdot [\omega(y)R'(y) - \omega'(y)R(y)]T(y) + \Sigma \tag{5.3.18}$$

but this time Σ is not zero trivially because $\omega(a_R) = -L'(a_R)R(a_R)$. Its evaluation will require a careful treatment of the integrals. For this we can approximate δ function as $\frac{1}{2\epsilon}$

over the interval $(a_R - \epsilon, a_R + \epsilon)$ for $\epsilon \to 0$. Thus

$$\Sigma = -iR(a_R)^2 L'(a_R)T(a_R) + R(a_R) \int_{a_R-\epsilon}^{a_R} dy\ \omega(y) \left(-i \cdot [T(a_R) + T(y)] \frac{1}{2\epsilon} + \frac{ic\delta''(a_R - y)}{24\pi} \right) \tag{5.3.19}$$

$$= -iR(a_R)^2 L'(a_R)T(a_R) - \frac{i}{2} R(a_R)\ \omega(a_R) \cdot 2T(a_R) + \int_{a_R-\epsilon}^{a_R} dy \cdot \omega(y) \cdot \frac{ic\delta''(a_R - y)}{24\pi} \tag{5.3.20}$$

$$= \int_{a_R-\epsilon}^{a_R} dy \cdot \omega(y) \cdot \frac{ic\delta''(a_R - y)}{24\pi}. \tag{5.3.21}$$

This is just a c-number, and in fact for linear ω this will be zero. Similarly the $[O_L, [O_R, O_L]]$ computation can be done and the boundary term will at most give a c-number.

5.3.3 Proportionality to $[O_R, O_L]$

Now, let us ask, when is $[O_R, [O_R, O_L]]$ proportional to $[O_R, O_L]$ (apart from additive c-numbers). Demanding this will ensure that Zassenhaus expansion of e^{iOt} in terms of O_L, O_R becomes very simple. This will give us a differential equation:

$$\omega = \alpha W(\omega, R) \implies fR' - Rf' = \alpha\left(\omega R' - \omega' R\right). \tag{5.3.22}$$

for proportionality constant α. Similarly in the $[O_L, [O_R, O_L]]$ computation,

$$i^2 \int_I dy \cdot [\omega(y)L'(y) - \omega'(y)L(y)]T(y) \tag{5.3.23}$$

is the non-trivial part, which we want to be proportional to $[O_R, O_L]$. Therefore, we demand

$$\omega = \beta W(\omega, L) = \beta(\omega L' - \omega' L) \tag{5.3.24}$$

for another proportionality constant β. Using the $R(x), L(x)$ obtained by solving these equations together will ensure that all nested commutators of O_L, O_R are proportional to $[O_R, O_L]$ for the case of $\overline{f} = \overline{L} = \overline{R} = 0$, and will factorize the expansion into just three terms: $e^{iO_L t}, e^{iO_R t}$, and $e^{\Omega(t)[O_R,O_L]}$ for some function $\Omega(t)$. Further, the central charge terms will all collect into a c-number of the form $e^{\lambda(t)}$ for some function $\lambda(t)$.

If we had chosen $f = L = R = 0$ instead of $\overline{f} = \overline{R} = \overline{L} = 0$, it is easy to see the change would only be the commutator relation of stress tensor components; compared to (5.1.10), there would be an overall minus sign. Since $[T(x), \overline{T}(y)] = 0$, we can say with both f and $\overline{f}$ non-zero that if $\overline{\omega}(x) = \overline{L}(y)\overline{R}'(y) - \overline{L}'(y)\overline{R}(y)$,

$$\overline{\omega} = -\alpha W(\overline{\omega}, \overline{R}) = -\beta W(\overline{\omega}, \overline{L}) \tag{5.3.25}$$

are the additional equations to solve to obtain $\overline{R}, \overline{L}$ to ensure all nested commutators are proportional to $[O_R, O_L]$ where α, β are the same as above.

Constant function f

By using $\omega = LR' - RL' = fR' - Rf'$ these become very complicated differential equations, e.g. $\omega = \alpha W(\omega, R)$ becomes

$$\alpha R(x)^2 f''(x) + R'(x)\left[-\alpha R(x) f'(x) - f(x)\right] - \alpha f(x) R(x) R''(x) + \alpha f(x) R'(x)^2 + R(x) f'(x) = 0 \tag{5.3.26}$$

which we will not discuss for general functions f in this work. However for a constant f, we can get the solutions easily. Using (5.3.22) and (5.3.24), we can write

$$\omega = \beta W(\omega, L) = \beta W(\omega, f) - \frac{\beta}{\alpha}\omega \tag{5.3.27}$$

Thus,

$$f' - \frac{\omega'}{\omega} f = \left(\frac{1}{\beta} + \frac{1}{\alpha} \right) \tag{5.3.28}$$

For a constant f, we know $\omega = fR'$, giving us

$$\frac{R'(x)}{R(x)} = \frac{d}{dx} \log R(x) = -\frac{1}{f} \left(\frac{\beta + \alpha}{\alpha\beta} \right) \implies R(x) = A \exp\left(-\frac{x}{f} \left(\frac{\alpha + \beta}{\alpha\beta} \right) \right) \tag{5.3.29}$$

but this is not the only solution. Using (5.3.22) directly we can see that

$$R(x) = \frac{x}{\alpha} + C \tag{5.3.30}$$

is another solution. In fact, if one requires that $[O_R + O_L, [O_R, O_L]] = 0, \alpha = -\beta$, so the solution (5.3.29) becomes trivial, leaving (5.3.30). The condition $[O_R + O_L, [O_R, O_L]] = 0$ holds for half-sided translations.

5.3.4 Special case of half-sided translations

Let us now set $f(x) = 2a, R(x) = x, a_L = 0$, and $a_R = 2a$ in (5.3.1) for $b_L, b_R \to \infty$. Comparing with (5.2.10) and (5.2.11), it makes $O_R = G$ and $O_L = G'$, and gives

$$\omega(x) = fR'(x) = 2a \tag{5.3.31}$$

Recall $\omega(x) = fR'(x)$ for constant f, making $\omega(x) = 2a$. Further, note that $R(x)$ is linear and ω is a constant, therefore there are no boundary terms, nor any terms involving central charge c. Further, we can compute $W(\omega, L) = W(2a, 2a - x) = -2a$, which is $-\omega(x)$ meaning that $\beta = -\alpha$, so that (5.3.29) is not a useful solution.

Taking care of the i factors, it is straightforward to write all the nested commutators:

$$\mathrm{ad}^k_G G' = i^{k-1}[G,G'], \quad \mathrm{ad}^k_{G'} G' = (-i)^{k-1}[G,G'], \quad \mathrm{ad}^j_{\mathcal{G}} G' = \delta_{j,1}\cdot[G,G'], \; j \geq 1 \quad (5.3.32)$$

with

$$[G,G'] = i\int_0^{2a} dx \cdot \omega(x)T(x) = 2ai\int_0^{2a} dx \cdot T(x). \quad (5.3.33)$$

Regularization

Following our discussions in section 5.2, we see that equation (5.3.33) clearly has problems coming from integral's boundaries $x = 0$ and $x = 2a$. Using the same logic as before, we can regularize it at the boundaries with functions $\mu(x) = (x/\epsilon)^{1+\epsilon}$ on $(0,\epsilon)$ and $\nu(x) = 2a - [(2a-x)/\epsilon]^{1+\epsilon}$ on $(2a-\epsilon, 2a)$ such that UV divergence vanishes, where this time $\epsilon \to 0$ makes μ,ν diverge. However there is a problem with such a regularization; fixing $[G,G']$ does not fix any of the nested commutators such as $[G,[G,G']]$.

This can be seen from the general results of this section. For any $R(x)$, we can compute $\omega(x) = fR'(x)$, and each time we nest commutators, we will have to differentiate $R(x)$. For any analytic function used for regularizing, there will be an N such that $R^{(N+1)}(\epsilon)$ is discontinuous. This follows from a simple Taylor expansion; demanding all derivatives of $R^{(n)}(x)$ vanish at $x = \epsilon$ for $n \geq 2$ will force $R(x)$ to be linear. This non-differentiability of $R(x)$ gives rise to singularities, because second derivative of a non-differentiable function has an infinite discontinuity. Thus, we see, that for defining commutators, we must ensure $R(x)$ post regulation is smooth, forcing us to go beyond analytic functions.

In the subject of smooth manifolds [68], such functions do exist, and they often rely on the function $\exp(-1/x)$, which has an essential singularity at $x = 0$ when x is treated as a complex variable. For our purposes,[7] we can define the function

$$\psi(t) = \frac{1}{C}\int_0^t ds\, \exp\left(-\frac{1}{s(1-s)}\right), \quad C = \int_0^1 ds\, \exp\left(-\frac{1}{s(1-s)}\right). \tag{5.3.34}$$

If we consider now a piecewise function

$$P(x) = \begin{cases} 0, & x \leq 0 \\ x\psi(\frac{x}{\epsilon}), & 0 \leq x \leq \epsilon \\ x, & x \geq \epsilon \end{cases} \tag{5.3.35}$$

it follows that $P^{(n)}(\epsilon) = 0$ for all $n \geq 2$, and $P^{(n)}(0) = 0$ for all $n \geq 0$. We can show this by noting that

$$\psi'(t) = \frac{1}{C}\exp\left(-\frac{1}{t(1-t)}\right). \tag{5.3.36}$$

Thus instead of taking the minimal regularization of (5.2.14), we will instead require

$$\mu(x) = x\psi\left(\frac{x}{\epsilon}\right), \quad \nu(x) = 2a - (2a-x)\psi\left(\frac{2a-x}{\epsilon}\right) \tag{5.3.37}$$

in the expressions (5.2.12). This modification for G, G' will regularize all the nested commutators in one go. However, note that any regularization, including this one, will break down the proportionality of the nested commutators (5.3.32). This will prevent a simple factorization of $e^{is\mathcal{G}}$ near the boundaries, but that will not be a major problem because we will be able to use (5.3.32) away from the boundaries.

[7] It is worth mentioning that there is a well known and simpler way to construct a smooth ramp function that tapers off to constant value using $f(x) = \{e^{-1/x}, x \geq 0;\ 0, x \leq 0\}$ by considering a ratio like $f(2-x)/[f(2-x)+f(x-1)]$, but the this ramp does not have a uniform slope, and therefore does not work for our problem.

5.4 Zassenhaus expansion for half-sided translations

The Zassenhaus expansion [65] is ubiquitous in quantum mechanics for writing operators $e^{\lambda(X+Y)}$ in terms of exponentials of X, Y and their commutators in a right sided expansion:

$$e^{\lambda(X+Y)} = e^{\lambda X} e^{\lambda Y} \cdot \prod_{i=2}^{\infty} e^{\lambda^n C_n} \tag{5.4.1}$$

for some C_n which can be obtained recursively by differentiating the expression repeatedly. In general this is a difficult task, however, there exists an efficient recursive method for their computation due to Casas et al [69]. Their result is

$$C_n = \begin{cases} \frac{1}{n} f_{1,n-1}, & n = 2,3,4 \\ \frac{1}{n} f\left(\left[\frac{n-1}{2}\right], n-1\right), & n \geq 5 \end{cases} \tag{5.4.2}$$

where

$$f_{1,k} = \sum_{j=1}^{k} \frac{(-1)^k}{j!(k-j)!} \mathrm{ad}_Y^{k-j} \mathrm{ad}_X^j Y, \tag{5.4.3}$$

$$f_{n,k} = \sum_{j=0}^{[k/n]-1} \frac{(-1)^j}{j!} \mathrm{ad}_{C_n}^j f_{n-1,k-nj}, \quad k \geq n. \tag{5.4.4}$$

where $\mathrm{ad}_X Y = [X, Y]$, $\mathrm{ad}_X^k Y = [X, \mathrm{ad}_X^{k-1} Y]$, and $\mathrm{ad}_X^0 Y = Y$.

5.4.1 Issue with the right sided expansion

We can demand a right sided expansion for half-sided translations, with $X = \hat{G}', Y = \hat{G}$ and $\lambda = is$,

$$e^{is\mathcal{G}} = e^{is\hat{G}'} e^{is\hat{G}} \prod_{n=2}^{\infty} e^{(is)^n C_n} \tag{5.4.5}$$

where C_n can be obtained using (5.4.22) and turn out to be integrals of stress tensor over $(0, 2a)$ that are closely related to (5.3.33). Can this expression describe half-sided translations? Let us ask this question by considering the conjugation for $\phi(x)$, for x spacelike to the interval $(0, 2a)$.

$$A(s, x) = \exp(is\mathcal{G})\phi(x)\exp(-is\mathcal{G}) \tag{5.4.6}$$

Since $e^{is\mathcal{G}}$ is unitary, and x is spacelike to $(0, 2a)$, by micro causality $\phi(x)$ commutes with all C_n, giving,

$$A(s, x) = e^{is\hat{G}'}e^{is\hat{G}} \cdot \phi(x) \cdot e^{-is\hat{G}}e^{-is\hat{G}'} \tag{5.4.7}$$

but from the discussions of section 5.2.3 conjugation by $e^{is\hat{G}'}$ and $e^{is\hat{G}}$ can transport $\phi(x)$ to regions causally connected with $(0, 2a)$. Without the contributions from C_n, $\phi(x)$ cannot cross the region 2 of figure 5.2 in a way $e^{i\mathcal{G}s}\phi(x)e^{-i\mathcal{G}s}$ does. Therefore, (5.4.5) cannot represent half-sided translations when excitations outside region 2 are involved.

5.4.2 Derivation of centered Zassenhaus formula

The key to avoid this problem is to transport $\phi(x)$ into the overlap region before the commutator terms act on it. This can be done using a centered expansion which will ensure that commutator terms always make their contribution. We therefore should look for the decomposition

$$e^{\lambda(X+Y)} = e^{\lambda X} \cdot \prod_{n=2}^{\infty} e^{\lambda^n D_n} \cdot e^{\lambda Y}. \tag{5.4.8}$$

Unfortunately the centered version of Zassenhaus expansion does not seem to be available in the literature and therefore we will have to derive it. We can take inspiration from [69], of proposing an expansion and taking derivatives to get $\{D_n\}$. We demand $e^{\lambda(X+Y)}$ has an expansion (5.4.8). Then let us define the functions

$$R_1(\lambda) = e^{-\lambda X} e^{\lambda(X+Y)} e^{-\lambda Y}, \quad F_1 = \frac{d}{d\lambda} R_1 \cdot R_1^{-1}. \tag{5.4.9}$$

Note that $R_1(\lambda)$ is equal to the infinite product $\prod_{n=2}^{\infty} e^{\lambda^n D_n}$. Using this, one can define

$$R_n(\lambda) = e^{-\lambda^n D_n} R_{n-1}(\lambda), \quad F_n = \frac{d}{d\lambda} R_n \cdot R_n^{-1}. \tag{5.4.10}$$

These functions like in [69] obey the recursive relation

$$F_n(\lambda) = e^{-\lambda^n \mathrm{ad}_{D_n}} (F_{n-1}(\lambda) - n D_n \lambda^{n-1}) \tag{5.4.11}$$

which enable us to find D_n' s recursively, starting from F_1. Therefore, let us begin the task.

$$\frac{\partial R_1}{\partial \lambda} = -X \cdot R_1 - R_1 \cdot Y + e^{-\lambda X}(X+Y) e^{\lambda(X+Y)} e^{-\lambda Y} \tag{5.4.12}$$

and now, we use

$$R_1'(\lambda) \cdot R_1^{-1} = -X - R_1 Y R_1^{-1} + e^{-\lambda X}(X+Y) e^{\lambda X} \tag{5.4.13}$$

$$= -X - e^{-\lambda X} e^{\lambda(X+Y)} Y e^{-\lambda(X+Y)} e^{\lambda X} + e^{-\lambda X}(X+Y) e^{\lambda X} \tag{5.4.14}$$

$$= -X + e^{-\lambda X} \left[-e^{\lambda(X+Y)} Y e^{-\lambda(X+Y)} + X + Y \right] e^{\lambda X} \tag{5.4.15}$$

$$= e^{-\lambda X} \left[-e^{\lambda(X+Y)} Y e^{-\lambda(X+Y)} + Y \right] e^{\lambda X} \tag{5.4.16}$$

$$= -e^{-\lambda \mathrm{ad}_X} (e^{\lambda \mathrm{ad}_{X+Y}} - I) Y \tag{5.4.17}$$

$$= -\sum_{j=0}^{\infty} \sum_{p=1}^{\infty} \frac{(-\lambda)^j \lambda^p}{j! p!} \mathrm{ad}_X^j \mathrm{ad}_{X+Y}^p Y \tag{5.4.18}$$

Like in [69], let us define

$$F_n(\lambda) = \sum_{k=n}^{\infty} f_{n,k}\lambda^k. \tag{5.4.19}$$

From this, we can extract $f_{1,k}$ by a change of index; λ^{j+p} must be set to λ^k, and the sum is over $k = 1$ to infinity. Thus

$$f_{1,k} = \sum_{j=0}^{k-1} \frac{(-1)^{j+1}}{j!(k-j)!}\mathrm{ad}_X^j \mathrm{ad}_{X+Y}^{k-j} Y. \tag{5.4.20}$$

Using the recursive relation (5.4.11)

$$f_{n,k} = \sum_{j=0}^{[k/n]-1} \frac{(-1)^{j+1}}{j!}\mathrm{ad}_{D_n}^j f_{n-1,k-nj} \tag{5.4.21}$$

One now finally obtains

$$D_n = \begin{cases} \frac{1}{n} f_{1,n-1}, & n = 2,3,4 \\ \frac{1}{n} f\left(\left[\frac{n-1}{2}\right], n-1\right), & n \geq 5 \end{cases}. \tag{5.4.22}$$

Compared to the right sided expansion in [69], only thing that has changed is the form of $f_{1,k}$.

Centered Zassenhaus expansion

First let us do the calculation for the forms G, G', then we will look at operators $\hat{G}, \hat{G}'$. we will set $Y = G, X = G'$, and $\lambda = is$.

$$f_{1,k} = \sum_{j=0}^{k-1} \frac{(-1)^{j+1}}{j!(k-j)!}\mathrm{ad}_{G'}^j \mathrm{ad}_{\mathcal{G}}^{k-j} G \tag{5.4.23}$$

$$= \frac{(-1)^k}{(k-1)!}\mathrm{ad}_{G'}^{k-1} \cdot [G', G] = \frac{(-i)^{k-1}(-1)^{k-1}}{(k-1)!}[G, G']. \tag{5.4.24}$$

It is easy to see that $f_{n,k} = f_{1,k}$ because of this simplified structure, and in fact for all $n \geq 2$:

$$D_n = \frac{1}{n} f_{1,n-1} = [G, G'] \cdot \frac{i^{n-2}}{n(n-2)!} \tag{5.4.25}$$

Therefore

$$\sum_{n=2}^{\infty} \lambda^n D_n = \frac{1}{i}[G, G'] \cdot \sum_{n=2}^{\infty} \frac{\lambda^n \cdot i^{n-1}}{n(n-2)!} = \frac{1}{i}[e^{i\lambda}(\lambda + i) - i][G, G'] \tag{5.4.26}$$

$$= [e^{-s}(s+1) - 1] \cdot [G, G']. \tag{5.4.27}$$

Therefore, we have the result

$$\exp(is\mathcal{G}) = e^{isG'} \cdot \exp([e^{-s}(s+1) - 1] \cdot [G, G']) \cdot e^{isG}. \tag{5.4.28}$$

As a comment, this closed form answer resulted from all nested commutators being proportional to the most basic one. Such a closed form solution for a Zassenhaus expansion was shown to hold for a class of bounded operators X, Y whose commutator was $[X, Y] = uX + vY + c1$ in [73]. For half-sided translations, the bilinear forms G, G' show some similarity to this prescription if we set $Y = G, X = G'$ with $u = v = 1$. That would mean $[G, G'] = \mathcal{G} + 1c$. This is not true in our case for any c, but it is true that $[\mathcal{G}, [G, G']] = 0$.

Upgrading to operators

Unfortunately, for $\hat{G}, \hat{G}'$, such a simple form will not appear. One can however show that because of smoothness of bump regulators, the changes will only be near the boundary, giving

$$\exp(is\mathcal{G}) = e^{is\hat{G}'} \prod_{n=2}^{\infty} \exp\left[(is)^n \left(\frac{i^{n-2}}{n(n-2)!} D(\epsilon) + \Sigma_n(\epsilon)\right)\right] \cdot e^{is\hat{G}} \tag{5.4.29}$$

$$D(\epsilon) = 2ai \int_{\epsilon}^{2a-\epsilon} T(x)\, dx \tag{5.4.30}$$

where the boundary terms $\Sigma_n(\epsilon)$ involve nested Wronskians of $\mu(x), \nu(x)$ with $R(x)$, and do not commute with the integral on $(\epsilon, 2a - \epsilon)$. To see this, note that (5.4.23) now has contributions from $k - j > 1$, all of which give operators that commute with integral on $(\epsilon, 2a - \epsilon)$. However, the $k - j = 1$ contribution itself is such that the its boundary terms do not commute with the part on $(\epsilon, 2a - \epsilon)$, which is $D(\epsilon)$ for any j.

Note that the boundary modifications can produce central charge factors, which can be estimated by numerical integrations of functions involving $\psi(t)$ in (5.3.34) over the intervals $(0, \epsilon)$ and $(2a - \epsilon, 2a)$. However, because of the $e^{-1/t}$ decay of $\psi(t)$ near $t = 0$, they can be assumed to be negligible. Further, the explicit form of $\Sigma_n(\epsilon)$ can be worked out by taking Wronskians repeatedly but this task is neither straightforward nor very illuminating for our present study.

The above result means that as long as x remains is in $(\epsilon, 2a-\epsilon)$, (5.4.29) takes the form of (5.4.28) for conjugation $e^{is\mathcal{G}}\phi(x)e^{-is\mathcal{G}}$. As a remark, in the above discussion we assumed s was positive. However s can be negative, and that case requires an interchange of $\hat{G}, \hat{G}'$, or alternatively, taking the adjoint of the equation (5.4.29).

5.4.3 Testing the result

We will now verify that (5.4.29) reproduces the effect of half-sided translations for fields that stay confined in region 2 and spacelike from $x = \epsilon, 2a - \epsilon$. In this case, we can ignore the $\Sigma_n(\epsilon)$ in each conjugation, and the product (5.4.29) is effectively given by

$$\exp(is\mathcal{G}) \stackrel{*}{=} e^{is\hat{G}'} \cdot \exp(g(s)D(\epsilon)) \cdot e^{is\hat{G}} \tag{5.4.31}$$

where $g(s)$ is the function from (5.4.28), and the $*$ is to denote that this holds only in our chosen case. Let's ask what effect $e^{g(s)D(\epsilon)}$ has on a $\phi(w, \overline{w})$, a primary. This will be similar to our calculations in section 5.2.3. We will need the $[T(z), \phi(x)]$ commutator, and first we are looking for

$$e^{g(s)D(\epsilon)}\phi(w, \overline{w})e^{-g(s)D(\epsilon)} \tag{5.4.32}$$

when w is in $(\epsilon, 2a - \epsilon)$. Using Hadamard lemma, this is same as

$$\sum_{n=1}^{\infty} \frac{g^n}{n!} \mathrm{ad}^n_{D(\epsilon)}\phi. \tag{5.4.33}$$

As we are away from the boundary, the boundary terms drop.

$$[D(\epsilon), \phi] = -2ai(-i)\int dx \cdot \partial_x\phi(x, \overline{w})\delta(x - w) = -2a \cdot \partial_w\phi(w, \overline{w}). \tag{5.4.34}$$

Thus,

$$\sum_{n=0}^{\infty} \frac{g^n}{n!}(-2a)^n \partial^n_w \phi(w, \overline{w}) = \phi(w - 2ag(s), \overline{w}). \tag{5.4.35}$$

Thus, if initial location was $w = w_0$, after the conjugation by $e^{g(s)D(\epsilon)}$ the location is $w = w_0 - 2ag(s)$. We can now ask, what if we were to apply $e^{i\hat{G}'s}$ on this ϕ. From (5.2.37), we already know the final answer would be

$$\phi(w(s), \overline{w}) = \exp[sh + s(w - 2a)\partial_w] \cdot \phi(w, \overline{w}) \tag{5.4.36}$$

where the derivative is to be evaluated at $w_0 - 2ag(s)$. Like we did in section 5.2.3, using the change of variable $u = \ln(w - 2a)$, this simplifies to a translation operation, giving

$$\phi(w(s), \overline{w}) = e^{sh}\phi(2a - e^s[2a - (w_0 - 2ag(s))], \overline{w}). \tag{5.4.37}$$

This is the result of composing $e^{g(s)D(\epsilon)}$ and $e^{i\hat{G}' s}$ in correct sequence. To include the effect of $e^{i\hat{G}s}$ at the beginning, following (5.2.36), we can do a rescaling $w_0 \to e^{-s}w_0$ and cancel e^{-sh}. Composing all three therefore gives

$$\phi(w(s), \overline{w}) = e^{-sh+sh}\phi(w_0 + 2a(1 - e^s) - 2ag(s)e^s). \tag{5.4.38}$$

Now finally if we set $g(s) = e^{-s}(s + 1) - 1$, the answer is

$$\phi(w(s), \overline{w}) = \phi(w_0 - 2as, \overline{w}) \tag{5.4.39}$$

which is exactly what we will get from $e^{i\mathcal{G}s}$, because $w = x - t$ and it decreases as t increases.

As a final remark, let us note that we were able to cancel the e^{-sh}, e^{sh} factors because we confined $\phi(w(s), \overline{w})$ for all s entirely to the region 2. This would not have happened otherwise, and this is one of the reasons why on its own, (5.4.28) is not correct, and the boundary terms in (5.4.29) are important. The figures 5.4 and 5.5 are polynomial approximations of the derivatives of $R(x)$ for half-sided translations with and without regularization with $\mu(x)$ and $\nu(x)$ of (5.3.37), which show the non-trivial edge effects that arise near the boundary.

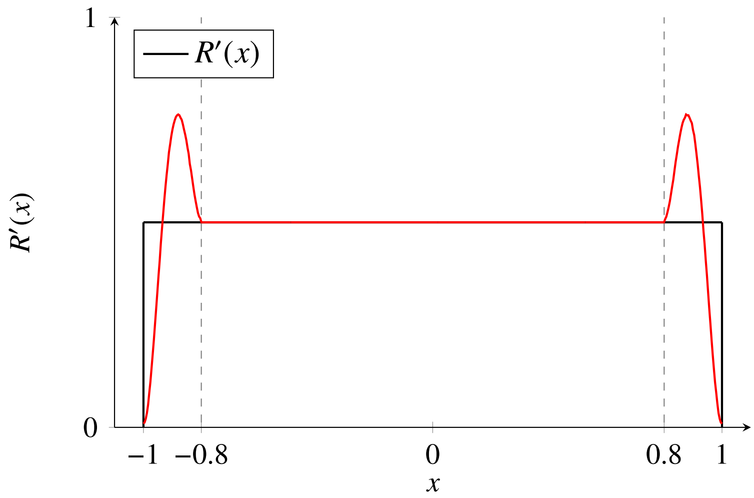

Figure 5.4: First derivative of $R(x)$ as a smoothed function with a 5th-degree polynomial on $(-1, 1)$ with $\epsilon = 0.2$ shown in red; the original $R(x) \propto x$ having a constant derivative is shown in black.

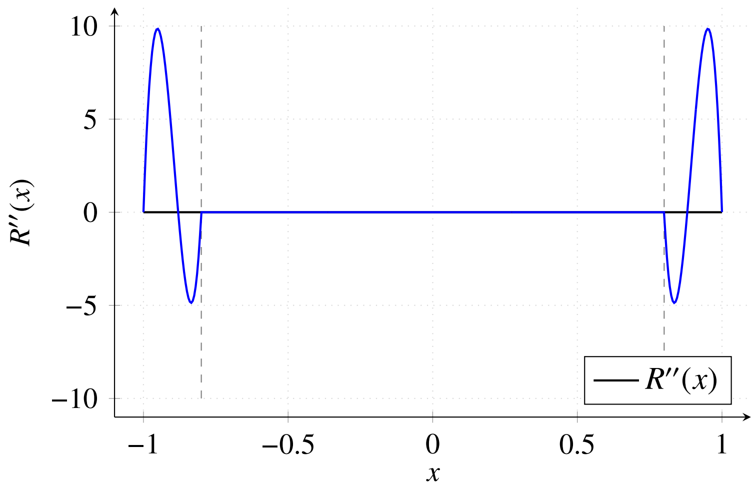

Figure 5.5: Second derivative of $R(x)$ as a smoothed function with a 5th-degree polynomial on $(-1, 1)$ with $\epsilon = 0.2$.

Chapter 6

Summary and conclusions

The research presented in this thesis has drawn inspiration from recent advancements from the intersection of the fields of quantum information and quantum gravity. The work presented builds on the studies on quantum extremal surfaces and islands for understanding the black hole interior through Hawking radiation, as well as providing development within the framework of quantum field theory using ideas of algebraic quantum field theory.

In the first study, we explored how the reconstructibility of an AdS_2 eternal black hole in thermal equilibrium with a finite temperature bath, using the Hawking radiation, is affected by the underlying theoretical framework. This was accomplished by employing a conformal field theory (CFT) that includes matter fields with random reflecting boundary conditions at the edges of the AdS_2 space. By applying the island formula and the extended strong subadditivity principle introduced by Carlen and Lieb, we showed that, at later times, the reference Hilbert space, which encodes information about the random boundary conditions, has an entanglement wedge that contains the interior of the black hole, including regions near the singularity. This result suggests that a comprehensive understanding of the theory is essential for accurately reconstructing the region surrounding the black hole's singularity from the emitted radiation at late enough times.

An intriguing observation in this work was that the combined state of the black hole, bath, and journal, prior to the Page time, was one that satisfied the condition of strong subadditivity. This means that the entropies of these three subsystems behaved in a way that is known to happen only for states called 'Markov chain states' [74]. Furthermore, at later times, the state of this combined system continues to meet all unitarity constraints, and, it was noted that this state also saturates the strong subadditivity condition. It is not clear what exactly is the cause of this saturation again, and further investigation may be fruitful. It may also be interesting to change the probability distribution of the random variables, or the conformal matter to more general cases.

Further, as highlighted in the work of [9], broadening this analysis to include scenarios involving an evaporating black hole could yield deeper insights. Such an exploration may help us understand how the reconstruction of the black hole's interior from the emitted Hawking radiation is influenced by the theoretical framework we use. Therefore, future research could focus on investigating these aspects in greater detail, and extended strong subadditivity could possibly play an important role for simplifications in this otherwise, relatively complicated problem involving backreaction.

The second part of this thesis focused on the Zassenhaus decomposition of exponential operators in the context of 1+1 dimensional conformal field theories, with a particular focus on half-sided translations. I examined the concept that $\mathcal{G}$ can be represented as the sum of $G + G'$, where these components are derived from entanglement Hamiltonians. However, it was shown that G and G' do not qualify as operators with well-defined properties. I then established that they can be replaced with operators $\hat{G}$ and $\hat{G}'$, which have clearly defined nested commutators. Through a central Zassenhaus expansion, I successfully derived an expression for $e^{is\mathcal{G}}$ in terms of $\hat{G}$, $\hat{G}'$, and their associated commutators, demonstrating that there exists an extensive class of operators that allows for this type of decomposition (5.3.1).

One possible observation that might be pertinent here concerns the non-self-adjoint nature of $\hat{G}$ and $\hat{G}'$. In section 5.2.3, we examined how conjugating a primary operator ϕ with these entities does not maintain the operator norm. This resembles scenarios in quantum mechanics where a particle is confined to half of the space, specifically in the interval $(0,\infty)$. The operator $-i\frac{d}{dx}$ fails to be self-adjoint for the space $\mathcal{L}^2(0,\infty)$, which consequently leads to the translation operator $e^{\frac{d}{dx}}$ being non-unitary. This can be demonstrated by the fact that $e^{\frac{d}{dx}}$ transforms elements of $\mathcal{L}^2(0,\infty)$ into a subset of itself through a rightward shift, indicating that the mapping is not onto. While such maps can be isometries that preserve inner products, they do not necessarily maintain operator norms during conjugation. This scenario is quite analogous to our current situation, suggesting that operators like $e^{i\hat{G}s}$ and $e^{i\hat{G}'s}$ might behave as isometries.

For future studies stemming from the second work, it could be insightful to explore the possibility of broadening the category of operators that permit the decomposition in (5.3.1) to include light-ray operators as discussed in [67]. An especially compelling context for investigating this decomposition would involve the entanglement Hamiltonian associated with a causal diamond, which can be examined using the methodologies developed in this research. This situation is significant for its possible links to entanglement dynamics among subregions $(-b_L, a_R)$ and $(-a_L, b_R)$, particularly as the intersecting region $(-a_L, a_R)$ decreases in size [refer to equation (5.3.2)]. Furthermore, it might be beneficial to assess if these results can be extended to conformal field theories within the AdS_2 framework, particularly in relation to applications to two-sided black holes in JT gravity.

It is also interesting to check whether the constructions here have connections tied to the split property [75]. The split property is related to the property of isotony in algebraic QFT, which means that if $R_1 \subset R_2$ as open regions of spacetime, the corresponding von Neumann algebras obey $\mathcal{A}(R_1) \subset \mathcal{A}(R_2)$. Both of these algebras are of type III_1 nature, as they correspond to regions in spacetime. The split property makes a statement on top of isotony, that if R_1 has a closure contained in R_2, then there exists a type I algebra, $\mathcal{A}_I$, satisfying

$$\mathcal{A}(R_1) \subset \mathcal{A}_I \subset \mathcal{A}(R_2). \tag{6.0.1}$$

This requirement of R_1's closure being contained in R_2 is like saying that there is a "gap" region between boundaries of R_1 and R_2.

In the study of spatial decomposition of operators, as we have seen, the decomposition, or the stitching was enabled through an overlap region between two causal diamonds, which is akin to the gap of split property. One could define R_2 as the usual right Rindler wedge, and R_1 as an algebra for a shifted wedge, with origin $(2a, 0)$. The fact that split property guarantees existence of a type I algebra $\mathcal{A}_I$ containing $\mathcal{A}(R_1)$ means that meaningful density matrices can be defined on $\mathcal{A}_I$ without regularization, for arbitrarily small but positive a. One of the things to study therefore would be how these well defined density matrices are different from the ill-defined ones on R_2.

Appendix

Difference of boost operators to obtain $\mathcal{G}$ for Rindler wedge

Let us use the Poincaré algebra generators to obtain difference between boost operators corresponding to different stationary points. The algebra is given by

$$[K, P] = iH, \quad [K, H] = iP, \quad [H, P] = 0. \tag{6.0.2}$$

The boost operator K preserves the origin $(t, x) = (0, 0)$, thus the boost transformation that preserves $(t, x) = (v_t, v_x)$ is given by

$$e^{itK(v_t,v_x)} = e^{i(v_t H - v_x P)} e^{iKt} e^{-i(v_t H - v_x P)}. \tag{6.0.3}$$

One can expand e^{iKt} in a Taylor series, and use Hadamard lemma and the Poincaré algebra repeatedly to get the answer. The first commutator in the series is

$$[v_t H - v_x P, K] = -i(v_t P - v_x H). \tag{6.0.4}$$

Using commutation relations,

$$[v_t H - v_x P, [v_t H - v_x P, P.K]] = -i[v_t H - v_x P, v_t P - v_x P.H] = 0. \tag{6.0.5}$$

The nested commutators vanish, and we get

$$K(v_t, v_x) = K + (v_t P - v_x H). \tag{6.0.6}$$

Because of the linearity of vector v in this expression, we see for any (x_0, t_0),

$$\mathcal{G} = 2aP^{+} \equiv a(H + P). \tag{6.0.7}$$

We can use another method of direct subtraction and using properties of stress tensor:

$$\mathcal{G} = \frac{1}{2\pi}(K_{\mathcal{M}} - K_{\mathcal{N}}) = \int (x - x_0)T_{00}(t_0, x)dx - \int (x - x_0 - a)T_{00}(t_0 - a, x)dx \tag{6.0.8}$$

We will restrict to conformal field theories, for which we can write $T_{00}(t, x) = T(x - t) + \overline{T}(x + t)$, that yields

$$\mathcal{G}^{-} = \int (x - x_0)[T(x - t_0) - T(x - t_0 + 2a)]dx = 2a \int T(x - t_0)dx \tag{6.0.9}$$

where we changed the variable $x \to x - 2a$ in the last step for the second integral. Using the fact that our conventions lead to $T(x) = T_{--}(x)$, this becomes

$$= 2a \int_{-\infty}^{\infty} T_{--}(x) dx \equiv 2aP^{+}. \tag{6.0.10}$$

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