SOME PROBLEMS IN HOMOGENIZATION

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Preface

We consider some problems in the homogenization of partial differential equations. The subject of homogenization deals with the process of obtaining the macroscopic or effective properties of materials having heterogeneities on a scale much smaller compared to the material dimensions. The following discussion, from the introduction of Bensoussan, Lions and Papanicolaou [6], is illustrative of the theory of homogenization. Consider a generic, well-posed boundary value problem, having coefficients which are rapidly oscillating (periodically), depending on a small parameter ε,

$$\mathcal{A}_\varepsilon u_\varepsilon = f \text{ in } \Omega \tag{0.0.1}$$

subject to appropriate boundary conditions. The presence of high frequency oscillations is troublesome. For example, in the numerical solution of this problem a very fine mesh has to be used, leading to costly numerical computations. Thus, the need arises for an asymptotic expansion of the solution. One such expansion is the two-scale asymptotic expansion,

$$u_\varepsilon = u^0 + \varepsilon u^1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u^2(x, \frac{x}{\varepsilon}) + \ldots \tag{0.0.2}$$

which is modelled on two separate scales, the macroscopic scale x, and the microscopic scale x/ε capturing the high frequency periodic oscillations. One, now, hopes that the u^i's in the asymptotic expansion can be obtained by solving some numerically friendly equations where these high frequency oscillations are absent. Usually, it is seen that u^0 satisfies a *homogenized equation*

$$\mathcal{A}u^0 = f \text{ in } \Omega \tag{0.0.3}$$

with appropriate boundary conditions. The most important aspect of the passage from (0.0.1) to (0.0.3) is the explicit analytical construction of $\mathcal{A}$ and not merely the assertion that it exists [1]. The construction requires, typically, the solution

[1] However, in problems which lack a periodic structure, it is not always possible to construct $\mathcal{A}$ explicitly. In this case, one is satisfied with showing the existence of a homogenized operator and with obtaining bounds for coefficients of this operator.

of a boundary value problem within a single period cell, usually called the *cell problem*. The problem of computing $\mathcal{A}$ from the cell problem and also the problem of calculating u^0 from (0.0.3) are, usually, numerically stable and cheap. The solution of the cell problem is also used to obtain the second term in the asymptotic expansion of u_ε. Thus, for example, if it is now shown that $u_\varepsilon(x)-u^0(x)$ or $u_\varepsilon(x)-u^0(x)-\varepsilon u^1(x,\frac{x}{\varepsilon})$ converges to zero in an appropriate sense, we will have obtained an approximation of u_ε by the terms in its asymptotic expansion. Also, this approximation is numerically cheap to compute.

To sum up, problems in homogenization are of the nature of obtaining the global behaviour of solutions of problems in partial differential equations having rapidly oscillating coefficients. The aim is always to identify a suitable homogenized problem whose solution approximates the solution of the original problem, for small oscillations.

The theory of homogenization has developed over the last three decades and is used systematically in solving many problems coming from Mechanics of Solids and Fluids, Geology, Engineering, and many other branches of Physics and Chemistry. The books by Bensoussan, Lions and Papanicolaou [6], Sanchez-Palencia [35], Bakhvalov and Panasenko [4] are classical treatises and treat a broad range of problems having a periodic structure, while that of Jikov, Kozlov and Oleinik [22] is a recent, comprehensive monograph on problems and methods in homogenization. Dal Maso [16] gives a detailed introduction to Γ- convergence. The appendix of his book is a comprehensive guide to the literature on homogenization. Oleinik, Shamaev and Yosifian [32] treat homogenization problems in elasticity theory, Hornung(ed.) [21] treats problems on flow and transport through porous media, and Conca, Planchard and Vanninathan [15] treat spectral problems in the asymptotic analysis of fluid-solid structures and also give an extensive bibliography on homogenization.

This thesis consists of two parts: the first, concerns the homogenization of a class of optimal control problems; in the second, we justify the second term in the asymptotic expansion for a flow in a partially fissured medium.

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This

thesis is dedicated to

my late father

and to

my personal Gods.

Contents

Part I

Homogenization of some optimal control problems

Chapter 1

Introduction

1.1 Introduction

We study the homogenization of a class of optimal control problems, under several situations, in this part of the thesis. To begin with, we introduce the class of optimal control problems and briefly review the existing literature on the homogenization of such problems. Following this, we list the various contexts in which we study these problems in the thesis and give an overview of the results obtained.

Let Ω be a bounded open set in $\mathbb{R}^n$. Let $0 < a < b$, $0 < c < d$, $N > 0$ be given constants. We denote by $M(a, b, \Omega)$ the set of all $n \times n$ matrices, $A = (a_{ij})$, whose entries are functions on Ω such that,

$$a\,|\xi|^2 \leq A(x)\xi.\xi \leq b\,|\xi|^2 \text{ a.e. } x.$$

for all $\xi = (\xi_i) \in \mathbb{R}^n$. Let $A \in M(a, b, \Omega)$ and $B \in M(c, d, \Omega)$ with B symmetric. Let U_{ad} be a closed convex subset of $L^2(\Omega)$ and let $f \in L^2(\Omega)$ be a given function. The basic optimal control problem is the following: Find $\theta^* \in U_{ad}$ such that,

$$\text{(P)} \qquad J(\theta^*) = \min_{\theta \in U_{ad}} J(\theta), \tag{1.1.1}$$

where the *cost functional*, $J(\theta)$, is defined by

$$J(\theta) = \frac{1}{2}\int_\Omega B\,\nabla u.\nabla u dx + \frac{N}{2}\int_\Omega \theta^2\, dx\,, \tag{1.1.2}$$

and the *state* $u = u(\theta)$ is the weak solution in $H_0^1(\Omega)$ of the boundary value problem,

$$\left.\begin{array}{rcll} -\mathrm{div}(A\nabla u) & = & f+\theta & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array}\right\} \qquad (1.1.3)$$

It can be shown (cf. Lions [28]) by direct methods in the calculus of variations that there is a unique optimal control, $\theta^* \in U_{ad}$, minimizing J over U_{ad}.

We will consider situations where the coefficients of this problem or the domain begin to vary rapidly with a parameter, $\varepsilon > 0$, which tends to zero. For example, let $A_\varepsilon \in M(a,b,\Omega)$ and $B_\varepsilon \in M(c,d,\Omega)$ be two sequences of matrices, where the B_ε's are assumed to be symmetric. For each ε, the optimal control problem whose coefficients are A_ε and B_ε has a unique optimal control θ_ε^*. From the assumptions on A_ε and B_ε, it can be shown that θ_ε^* is a bounded sequence in $L^2(\Omega)$ and so, for a subsequence, $\theta_\varepsilon^* \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$ for some $\theta^* \in U_{ad}$. The question of interest is, can θ^* be shown to be the optimal control of a *homogenized problem*, i.e. an optimal control problem of the same type, say, with coefficients A^* and B^* ? If the answer is yes, then identify the homogenized problem by an appropriate limiting procedure.

Kesavan and Vanninathan [27] consider the periodic case where the coefficients A_ε and B_ε oscillate periodically and obtain explicit expressions for the coefficients A^* and B^* of the homogenized problem. For the general case, i.e. when $A_\varepsilon \in M(a,b,\Omega)$ and $B_\varepsilon \in M(c,d,\Omega)$, with B_ε symmetric, are arbitrary sequences, Kesavan and Saint Jean Paulin [24] obtain the homogenized problem in the framework of H-convergence. The extension of this problem, in the case of perforated domains and where the states satisfy a Neumann condition on the boundary of holes, was solved by Kesavan and Saint Jean Paulin [25] in the framework of H_0-convergence. Here, one also needs to identify the correct space of controls U_{ad}^* for the homogenized problem, as for each ε the space of admissible controls U_{ad}^ε is different, being dependent on the domain Ω_ε.

1.2 Thesis Summary

Chapter 2 and Chapter 3 are devoted to the study of the problems considered in [24] and [25], and [27] respectively, but from new points of view which allow us to obtain some generalizations of the existing results. Chapters 4 and 5 are devoted to the study of the homogenization of the class of optimal control problems in two new situations, viz. those governed by elliptic systems and Dirichlet boundary value problems in perforated domains, respectively.

In Chapter 2, we first try to get to the essence of the two seemingly different homogenization procedures adopted in the papers [24] and [25] for a domain without holes and with holes, respectively. Essentially, we have a sequence of functionals J_ε each having a minimizer x_ε^* in a set $K_\varepsilon \subset L^2(\Omega)$ and $x_\varepsilon^* \rightharpoonup x^*$ weakly in $L^2(\Omega)$. Is x^* the minimizer of a functional J over a set $K \subset L^2(\Omega)$, where J and K can be chosen in a natural way? The question, in this generality, forms the subject of study of the theory of Γ- convergence. However, for the problems in question, the special nature of the J_ε's and K_ε's allows us to formulate and prove a lemma, which is in the spirit of Γ-convergence, giving an answer to this question. Then, the problems considered in [24] and [25] can be homogenized, again, in the framework of this lemma. Subsequently, the same lemma will be used as the framework in which to homogenize the optimal control problems considered in Chapters 3, 4 and 5.

A crucial step in the verification of the hypotheses of the lemma is the characterization of the limit of some energies associated with the state equations. This question of characterization was taken up by Kesavan and Saint Jean Paulin in [24] and [25], as a matter of independent interest. This is solved by first introducing an adjoint state equation, which is coupled to the state equation, and then homogenizing the resulting system. The characterization of the limit of energies is obtainable from a knowledge of the homogenized state-adjoint state system of equations. We ask the question, "can we, from a characterization of the limits of such energies, say what the homogenized state-adjoint system of equations is going to be ?" We show that this is possible and interestingly, the uniqueness of the coefficient B^* in the homogenized cost follows from this, with the knowledge that it is symmetric. Next, we give another expression for the matrix B^* obtained by Kesavan and Saint Jean Paulin [25]. In fact, we show that B^* is the distribution limit of $M_\varepsilon^t B_\varepsilon M_\varepsilon$, where the matrices M_ε are the *corrector matrices* of Murat [29] corresponding to the matrices A_ε. This has the advantage that it gives an upper bound for the matrix B^*, a question left open in [24]. Also, the symmetry of B^* is got for free, while, previously, it needed a careful proof. This new description is also simple as it does not involve too many test functions.

In Chapter 3, we recover the results of Kesavan and Vanninathan [27] in the periodic case, *directly*, using *two-scale convergence*. We show that the adjoint formulation can be bypassed by using a corrector result, which allows us to use the first two terms in the asymptotic expansion of the states instead of the states themselves in the computation of the limits of energies. This reduces the computation to one of taking limits of integrals of the form $\int_\Omega g(x, \frac{x}{\varepsilon})\, dx$ for functions, g, periodic in the second variable. This is easily done and an explicit formula for B^* is obtained, which agrees with the formula of Kesavan and Vanninathan [27] in the non-perforated case. Going further, we consider the case of a domain which is periodically perforated on several microscopic scales and having coefficients which have periodic oscillations on all these scales. The homogenized problem is identified, and explicit formulas for B^* and A^* are found using the method of multi-scale convergence introduced by Allaire and Briane [2].

In Chapter 4, we study the homogenization of optimal control problems governed by *elliptic systems* in perforated domains. The *principal difficulty* is to pass to the limits in the state equation which is now an elliptic system. H_0-convergence cannot handle this. To the best of our knowledge, the little literature on this problem that is available deals with the periodic case and that too, for non perforated domains (cf. [6]). The problem is resolved by developing the theory of H_b-convergence for *block matrices*, whose order exceeds the space dimension, in analogy with the usual H-convergence and H_0-convergence. Then, using the framework of H_b-convergence, the homogenization of the optimal control problem can be completed, by following the procedure outlined by Kesavan and Saint Jean Paulin [25] in the scalar case.

In Chapter 5, we obtain some results for the homogenization of the optimal control problem governed by Dirichlet boundary value problems in perforated domains. The nature of this problem is different from the one considered in [25], since we require the states to satisfy the homogeneous Dirichlet condition on the boundary of the holes. As a result, while dealing with the states during the process of homogenization it is enough to extend them by zero in the holes. However, the requirement that the states vanish on the boundary of the holes contributes to a lower order term, with a measure μ as coefficient, in the operator corresponding to the homogenized state equation, when the holes have a critical size and distribution, as was discovered by Cioranescu and Murat [12]. Under the same assumptions on the domain and with the assumption that the coefficients appearing in the state equation and cost functional are independent of ε, we show that the cost functional of the homogenized optimal control problem also picks up a lower order term which corresponds to a different measure. This measure is identified.

Chapter 2

General Results

2.1 Introduction

In this chapter, we first discuss the results of Kesavan and Saint Jean Paulin concerning the homogenization of some optimal control problems considered in [24], [25]. An analysis of the steps involved in homogenizing the problems provides us the inspiration for Lemma 2.1.1 which deals with the limits of minimizers of a sequence of functions. The role of the lemma is to identify crucial steps in the homogenization procedure. By using the framework of this lemma and by verifying the hypotheses involved, the homogenized problem can be found. This applies to the other cases/problems considered in the subsequent chapters as well. Next, we consider a question concerning the limit of energies which is of independent interest. Solving this question also leads to a proof of the uniqueness of the coefficients appearing in the cost functional of the limit problem. Finally, some properties of the coefficients appearing in the cost functional of the homogenized problem are discussed.

We now discuss the results of Kesavan and Saint Jean Paulin [24] concerning the homogenization of the optimal control problem over non perforated domains. Let Ω be a bounded domain in $\mathbb{R}^n$. Let ε be a parameter which tends to zero. $A_\varepsilon \in M(a, b, \Omega)$, $B_\varepsilon \in M(c, d, \Omega)$ are two sequences of matrices and it is assumed that B_ε's are symmetric. U_{ad}, the space of admissible controls, is a given closed convex subset of $L^2(\Omega)$. Let $f \in L^2(\Omega)$ be a given function and let N be a positive constant. For $\varepsilon > 0$ fixed, let θ_ε^* be the optimal control minimizing the functional,

$$(\mathrm{P}_\varepsilon) \qquad J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon dx + \frac{N}{2}\int_\Omega \theta^2 \, dx$$

over U_{ad}, where $u_\varepsilon = u_\varepsilon(\theta)$ is the solution of the state equation,

$$\left.\begin{aligned} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) &= f + \theta \quad \text{in } \Omega, \\ u_\varepsilon &= 0 \qquad\quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.1.1}$$

It can be shown that (up to a subsequence) $\theta_\varepsilon^* \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$. Now, is θ^* the optimal control of a *homogenized* optimal control problem? It was shown that θ^* minimizes the functional,

$$(\mathrm{P}^*) \qquad J(\theta) = \frac{1}{2}\int_\Omega B^* \nabla u . \nabla u dx + \frac{N}{2}\int_\Omega \theta^2 \, dx$$

over $\theta \in U_{ad}$, where $u = u(\theta)$ is the solution of,

$$\left.\begin{aligned} -\mathrm{div}(A^* \nabla u) &= f + \theta \quad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned}\right\}$$

where A^* is the H-limit of the sequence A_ε and B^* is given by (2.1.7) below. The H-convergence method introduced by Murat [30] and Tartar [36] is discussed briefly, and following it, a full description of the A^* and B^* are given.

Definition 2.1.1 *A sequence of matrices, $A_\varepsilon \in M(a, b, \Omega)$, is said to H-converge to a matrix, $A \in M(a', b', \Omega)$, if for every $g \in H^{-1}(\Omega)$, the solution v_ε of*

$$\left.\begin{aligned} -\mathrm{div}(A_\varepsilon \nabla v_\varepsilon) &= g \quad \text{in } \Omega \\ v_\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.1.2}$$

satisfies the weak convergences,

$$\left.\begin{aligned} v_\varepsilon &\rightharpoonup v \qquad \text{weakly in } H_0^1(\Omega), \\ A_\varepsilon \nabla v_\varepsilon &\rightharpoonup A\nabla v \quad \text{weakly in } L^2(\Omega)^n, \end{aligned}\right\} \tag{2.1.3}$$

where v is the solution of,

$$\left.\begin{aligned} -\mathrm{div}(A \nabla v) &= g \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.1.4}$$

We write $A_\varepsilon \xrightarrow{H} A$. ■

This definition along with the following compactness theorem is tailor-made for passing to the limit in the equation (2.1.1).

Theorem 2.1.1 *Every sequence, $A_\varepsilon \in M(a,b,\Omega)$, has a H-convergent subsequence. The H-limit for the subsequence belongs to $M(a,\frac{b^2}{a},\Omega)$.* ■

The following are some known facts about H-convergence: a sequence of matrices which H-converges, has a unique H-limit; H-convergence is local, i.e. if $F_\varepsilon \xrightarrow{H} F$ and $G_\varepsilon \xrightarrow{H} G$ and $F_\varepsilon = G_\varepsilon$ on some $\omega \subset\subset \Omega$ for all $\varepsilon > 0$, then $F = G$ on ω; lastly, let $A_\varepsilon \xrightarrow{H} A$, let $g_\varepsilon \longrightarrow g$ strongly in $H^{-1}(\Omega)$ and let v_ε solve (2.1.2) with right hand side g_ε - in this case also, the convergences (2.1.3) hold and v solves (2.1.4).

By Theorem 2.1.1, we may assume that the sequence (if necessary, by restricting to a subsequence) of matrices A_ε has a H-limit. Then, this limit, by the discussion above, is the desired A^*.

The description of B^* in the article [24] involves a few test functions. The first, are the sequences X_ε^k, $k = 1,2,...,n$, with the following properties (cf. Murat [29]),

$$\left.\begin{array}{rcll} X_\varepsilon^k & \rightharpoonup & x_k & \text{weakly in } H^1(\Omega), \\ A_\varepsilon \nabla X_\varepsilon^k & \rightharpoonup & A^* e_k & \text{weakly in } L^2(\Omega)^n, \\ \mathrm{div}(A_\varepsilon \nabla X_\varepsilon^k) & \subset\subset & H^{-1}(\Omega) & \end{array}\right\} \tag{2.1.5}$$

The symbol $\subset\subset$ denotes precompactness; here, the precompactness of the sequence $\mathrm{div}(A_\varepsilon \nabla X_\varepsilon^k)$ in $H^{-1}(\Omega)$. The existence of such X_ε^k can be deduced from the H-convergence of A_ε to the limit A^*. One also defines the sequences, $\psi_\varepsilon^k \in H_0^1(\Omega)$, $k = 1,2,...,n$, which solve

$$\left.\begin{array}{rcll} -\mathrm{div}(A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k) & = & 0 & \text{in } \Omega, \\ \psi_\varepsilon^k & = & 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{2.1.6}$$

For each k, the sequence ψ_ε^k is bounded in $H_0^1(\Omega)$ and it is assumed, without loss of generality, that ψ_ε^k converges weakly to some ψ^k in $H_0^1(\Omega)$ and that $A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k$ converges weakly in $L^2(\Omega)^n$. B^* is, then, defined (through its transpose [1]) as follows,

$$(B^*)^t e_k = \lim_{\varepsilon \to 0} (A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k) - (A^*)^t \nabla \psi^k. \tag{2.1.7}$$

[1] B_ε need not be symmetric for this approach.

The results in the perforated case obtained by Kesavan and Saint Jean Paulin in [25] are now discussed. For each $\varepsilon > 0$, the perforated domain Ω_ε is given to be $\Omega \setminus S_\varepsilon$, having characteristic function χ_ε; S_ε is assumed to be a closed subset of Ω with smooth boundary. The space of admissible controls also depends on ε and for each $\varepsilon > 0$, U_{ad}^ε is a closed convex subset of $L^2(\Omega_\varepsilon)$. For given $\varepsilon > 0$, the optimal control problem consists of minimizing the functional,

$$(\mathbf{P}_\varepsilon) \qquad J_\varepsilon(\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon dx + \frac{N}{2}\int_{\Omega_\varepsilon} \theta^2 \, dx$$

over U_{ad}^ε, where $u_\varepsilon = u_\varepsilon(\theta)$, is the solution of the state equation,

$$\left.\begin{array}{rcll} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) & = & f + \theta & \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla u_\varepsilon . n_\varepsilon & = & 0 & \text{on } \partial S_\varepsilon \\ u_\varepsilon & = & 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{2.1.8}$$

This problem has a unique minimizer θ_ε^* in U_{ad}^ε. It can be seen that if these minimizers are extended by zero in S_ε, then the extensions $\widetilde{\theta_\varepsilon^*}$ form a bounded sequence in $L^2(\Omega)$. It may be assumed (for a subsequence) that $\widetilde{\theta_\varepsilon^*} \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$. In this case, it was shown that the θ^* minimizes the functional

$$(\mathbf{P}^*) \qquad J(\theta) = \frac{1}{2}\int_{\Omega} B^* \nabla u . \nabla u dx + \frac{N}{2}\int_{\Omega} \frac{\theta^2}{\chi} \, dx$$

over a suitable U_{ad} (see the discussion below), where $u = u(\theta)$ is the solution of,

$$\left.\begin{array}{rcll} -\operatorname{div}(A^* \nabla u) & = & \chi f + \theta & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array}\right\}$$

A^* is the "H_0-limit" of the sequences A_ε and, B^* is given by (2.1.18) below; χ is the weak* limit of χ_ε in $L^\infty(\Omega)$. Notice the second term in the cost functional of the homogenized problem is different in the two cases, [24] and [25], considered.

Since, the space of admissible controls varies with ε, it is also required to prescribe the limiting space of admissible controls. One guesses that U_{ad} is the, so called, Kuratowski limit of the sequences U_{ad}^ε in the weak topology of $L^2(\Omega)$. This guess seems to be verified in the examples of U_{ad}^ε considered in [25]. When U_{ad}^ε is one of the following,

$$
\left.\begin{array}{rcl}
U^{\varepsilon}_{ad} & = & L^2(\Omega_\varepsilon), \\
U^{\varepsilon}_{ad} & = & \{\theta \in L^2(\Omega_\varepsilon | \theta \geq \psi \text{ in } \Omega_\varepsilon\}, \\
U^{\varepsilon}_{ad} & = & \{\theta \in L^2(\Omega_\varepsilon | \psi_1 \leq \theta \leq \psi_2 \text{ in } \Omega_\varepsilon\}, \\
U^{\varepsilon}_{ad} & = & \{\theta \in L^2(\Omega_\varepsilon | \int_{\Omega_\varepsilon} \theta^2 \, dx \leq 1\}
\end{array}\right\} \tag{2.1.9}
$$

the corresponding U_{ad} was shown to be the U_{ad}'s given below, in the same order,

$$
\left.\begin{array}{rcl}
U_{ad} & = & L^2(\Omega), \\
U_{ad} & = & \{\theta \in L^2(\Omega) | \theta \geq \chi\psi \text{ in } \Omega\}, \\
U_{ad} & = & \{\theta \in L^2(\Omega) | \chi\psi_1 \leq \theta \leq \chi\psi_2 \text{ in } \Omega\}, \\
U_{ad} & = & \{\theta \in L^2(\Omega) | \int_{\Omega} \frac{\theta^2}{\chi} \, dx \leq 1\}.
\end{array}\right\} \tag{2.1.10}
$$

It is assumed that the characteristic functions χ_ε satisfy

$$
\left.\begin{array}{r}
\chi_\varepsilon \overset{*}{\rightharpoonup} \chi \;\; \text{weak * in } L^\infty(\Omega), \\
\chi^{-1} \in L^\infty(\Omega).
\end{array}\right\} \tag{2.1.11}
$$

We now give a full description of the A^* and B^* after introducing the notion of H_0-convergence proposed by Briane, Damlamian and Donato [7].

The framework of H_0-convergence imposes some restriction on the geometry of the perforated domain Ω_ε by presupposing the existence of suitable extension operators. Let $V_\varepsilon \equiv \{u \in H^1(\Omega_\varepsilon) | u = 0 \text{ on } \partial\Omega\}$. It is assumed that there exist extension operators, $P_\varepsilon : V_\varepsilon \to H^1_0(\Omega)$ which are bounded uniformly with respect to ε, i.e.

$$
\left.\begin{array}{rcl}
R_\varepsilon(P_\varepsilon u) & = & u, \\
|\nabla P_\varepsilon u|_{0,\Omega} & \leq & C_0 |\nabla u|_{0,\Omega_\varepsilon}
\end{array}\right\} \text{ for all } u \in V_\varepsilon, \tag{2.1.12}
$$

where the symbol R_ε denotes the operator which restricts a function given on Ω to Ω_ε and, C_0 is a constant independent of ε. A sequence of holes S_ε for which the characteristic functions of Ω_ε satisfy (2.1.11) and there exist extension operators satisfying (2.1.12), is said to be an admissible sequence of holes for Ω.

Definition 2.1.2 *Let $A_\varepsilon \in M(a,b,\Omega)$ and S_ε be admissible in Ω. The pair $(A_\varepsilon, S_\varepsilon)$ is said to H_0-converge to a matrix $A \in M(a',b',\Omega)$ if for every $g \in H^{-1}(\Omega)$ the solution v_ε in V_ε of,*

$$\left.\begin{aligned} -\text{div}(A_\varepsilon \nabla v_\varepsilon) &= P_\varepsilon^* g \quad \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla v_\varepsilon . n_\varepsilon &= 0 \quad \text{on } \partial S_\varepsilon \\ v_\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.1.13}$$

satisfies the weak convergences,

$$\left.\begin{aligned} P_\varepsilon v_\varepsilon &\rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \\ Q_\varepsilon(A_\varepsilon \nabla v_\varepsilon) &\rightharpoonup A\nabla v \quad \text{weakly in } L^2(\Omega)^n, \end{aligned}\right\} \tag{2.1.14}$$

where v is the solution of

$$\left.\begin{aligned} -\text{div}(A\nabla v) &= g \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \blacksquare \end{aligned}\right\} \tag{2.1.15}$$

where Q_ε denotes the operator yielding the extension by zero over the holes. This definition and the following compactness theorem are exactly what are required to pass to the limit in the equations (2.1.8).

Theorem 2.1.2 *For every sequence, $A_\varepsilon \in M(a,b,\Omega)$, the sequence $(A_\varepsilon, S_\varepsilon)$ has a H_0-convergent subsequence. The H_0-limit for the subsequence belongs to $M(aC_0^{-2}, \frac{b^2}{a}, \Omega)$.* $\blacksquare$

H_0-convergence has all the properties of H-convergence, in a suitable form. One also has the independence of the H_0-limit on the actual choice of extension operators satisfying (2.1.12). Further, suppose that $(A_\varepsilon, S_\varepsilon) \xrightarrow{H_0} A$ and let $\chi_\varepsilon g_\varepsilon \longrightarrow g$ weakly in $L^2(\Omega)$, and let v_ε solve (2.1.13) with right hand side g_ε. The convergences (2.1.14) still hold and v solves (2.1.15). These properties of H_0-convergence are worth remembering. H-convergence is, really, a special case of H_0-convergence, but the assumptions involved are few.

Now, if A_ε is the sequence appearing in the problem ($\mathbf{P}_\varepsilon$), the conclusion is that $(A_\varepsilon, S_\varepsilon)$ can be assumed to have a H_0-limit, under the assumptions made on S_ε concerning its admissibility. This limit is the desired A^*.

The description of the matrix B^* involves a few test functions. The analogue of X_ε^k that was seen in the non perforated case is defined through the following means.

Let Ω' be a bounded open subset of $\mathbb{R}^n$ such that $\Omega \subset\subset \Omega'$. Extension operators for Ω'_ε can be obtained by first extending by P_ε to Ω and then by zero in $\Omega' \setminus \Omega$. With these extension operators, S_ε are admissible for Ω' also. As a function, the matrix A_ε is extended to Ω' by defining it to be $a\,I$ in $\Omega' \setminus \Omega$. The extension is also denoted by A_ε and they clearly belong to $M(a, b, \Omega')$. It may be assumed that $(A_\varepsilon, S_\varepsilon)$ has a H_0-limit A' in Ω'. By the local nature of H_0-convergence, A' restricted to Ω has to be the A^* above. Let $\phi \in D(\Omega')$ with $\phi \equiv 1$ in Ω. Then, the test functions X_ε^k, $k = 1, 2, ..., n$, are defined to be the the solutions of

$$\left.\begin{aligned} -\mathrm{div}(A_\varepsilon \nabla X_\varepsilon^k) &= -P_\varepsilon^* \mathrm{div}(A^* \nabla(\phi x_k)) && \text{in } \Omega'_\varepsilon, \\ A_\varepsilon \nabla X_\varepsilon^k . n_\varepsilon &= 0 && \text{on } \partial S_\varepsilon, \\ X_\varepsilon^k &= 0 && \text{on } \partial\Omega', \end{aligned}\right\} \tag{2.1.16}$$

By H_0-convergence, $P_\varepsilon X_\varepsilon^k$ converges weakly in $H_0^1(\Omega')$ to ϕx_k and hence to x_k when restricted to Ω. The test functions ψ_ε^k, $k = 1, 2, ..., n$ are, by definition, the solutions of

$$\left.\begin{aligned} -\mathrm{div}(A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k) &= 0 && \text{in } \Omega_\varepsilon, \\ (A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k).n_\varepsilon &= 0 && \text{on } \partial S_\varepsilon, \\ \psi_\varepsilon^k &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.1.17}$$

For each k, it can be seen that $P_\varepsilon \psi_\varepsilon^k$ is a bounded sequence in $H_0^1(\Omega)$ and therefore, it may be assumed that $P_\varepsilon \psi_\varepsilon^k$ converges weakly to some ψ^k in $H_0^1(\Omega)$ and that $Q_\varepsilon(A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k)$ converges weakly in $L^2(\Omega)^n$. Then B^* is given by

$$(B^*)^t e_k = \lim_{\varepsilon \to 0} Q_\varepsilon(A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k) - (A^*)^t \nabla \psi^k. \tag{2.1.18}$$

The above problems on homogenization boil down to essentially the following question: let F_ε be functionals defined on sets $K_\varepsilon \subset L^2(\Omega)$, having a minimizer $x_\varepsilon^* \in K_\varepsilon$ and, suppose that $x_\varepsilon^* \rightharpoonup x^*$ weakly in $L^2(\Omega)$. Is x^* the minimizer of a functional F defined over a set $K \subset L^2(\Omega)$, where F and K can be chosen in a natural way? A question of this generality is the subject of study of the theory of

Γ- convergence. But, for the problems in question, because of some properties that J_ε's and U^ε_{ad}'s have, the answer is provided by the lemma below which is in the spirit of Γ-convergence. We need to place the following assumptions on F_ε and K_ε: there exists a $K \subset L^2(\Omega)$ such that,

(P1) $x_\varepsilon \in K_\varepsilon$, $x_\varepsilon \rightharpoonup x$ weakly in $L^2(\Omega)$ implies $x \in K$.

(P2) For every $x \in K$, there exists a sequence, $x_\varepsilon \in K_\varepsilon$, such that $x_\varepsilon \rightharpoonup x$ weakly in $L^2(\Omega)$.

F_ε admits the decomposition, $F_\varepsilon \doteq F^1_\varepsilon + F^2_\varepsilon$ and there exist functionals F^1, F^2 on K such that:

(P3) For any $x_\varepsilon \in K_\varepsilon$, $x_\varepsilon \rightharpoonup x$ weakly in $L^2(\Omega)$ implies

$$\lim_{\varepsilon\to 0} F^1_\varepsilon(x_\varepsilon) = F^1(x) \text{ and,} \tag{2.1.19}$$

$$\underline{\lim}_{\varepsilon\to 0} F^2_\varepsilon(x_\varepsilon) \geq F^2(x). \tag{2.1.20}$$

(P4) For any $x \in K$, there exist a sequence $x_\varepsilon \in K_\varepsilon$ such that $x_\varepsilon \rightharpoonup x$ and $\lim_{\varepsilon\to 0} F^2_\varepsilon(x_\varepsilon) = F^2(x)$.

Lemma 2.1.1 *Under the assumptions made above, x^* is a minimizer in K for the function, $F \doteq F^1 + F^2$.*

Proof: For any $x \in K$, choose a sequence, $x_\varepsilon \in K_\varepsilon$ so that (P4) holds. Then, by (2.1.19) and (P4),

$$\begin{aligned} \lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) &= \lim_{\varepsilon\to 0} F^1_\varepsilon(x_\varepsilon) + \lim_{\varepsilon\to 0} F^2_\varepsilon(x_\varepsilon) \\ &= F^1(x) + F^2(x). \end{aligned} \tag{2.1.21}$$

On the other hand,

$$\begin{aligned} \underline{\lim}_{\varepsilon\to 0} F_\varepsilon(x^*_\varepsilon) &= \lim_{\varepsilon\to 0} F^1_\varepsilon(x^*_\varepsilon) + \underline{\lim}_{\varepsilon\to 0} F^2_\varepsilon(x^*_\varepsilon) \\ &\geq F^1(x^*) + F^2(x^*). \end{aligned} \tag{2.1.22}$$

Since we have, $F_\varepsilon(x^*_\varepsilon) \leq F_\varepsilon(x_\varepsilon)$, taking $\underline{\lim}_{\varepsilon\to 0}$ and using (2.1.21), (2.1.22), we get,

$$F(x^*) \doteq F^1(x^*) + F^2(x^*) \leq F^1(x) + F^2(x) \doteq F(x).$$

This is true for any $x \in K$. Thus x^* minimizes F over K. ■

Though for a general sequence x_ε in K_ε converging weakly to x, only (2.1.20) holds, we show that we have equality in (2.1.20) for the sequence of minimizers.

Proposition 2.1.1 *Let K_ε, K, F^1_ε, F^1, F^2_ε, F^2 be as above, satisfying the conditions (P1)-(P4). Let x^*_ε be the minimizer of F_ε in K_ε and let x^*_ε converge weakly to x^* in $L^2(\Omega)$. Then,*

$$\lim_{\varepsilon\to 0} F^2_\varepsilon(x^*_\varepsilon) = F^2(x^*).$$

Proof: Choose a sequence, $x_\varepsilon \in K_\varepsilon$ such that $x_\varepsilon \rightharpoonup x^*$ weakly in $L^2(\Omega)$ and $\lim_{\varepsilon\to 0} F^2_\varepsilon(x_\varepsilon) = F^2(x^*)$. Taking lim sup on either side of the inequality,

$$F^1_\varepsilon(x^*_\varepsilon) + F^2_\varepsilon(x^*_\varepsilon) \leq F^1_\varepsilon(x_\varepsilon) + F^2_\varepsilon(x_\varepsilon).$$

It follows from (2.1.19) and the choice of x^ε that,

$$F^1(x^*) + \overline{\lim}_{\varepsilon\to 0} F^2_\varepsilon(x^*_\varepsilon) \leq F^1(x^*) + F^2(x^*)$$

i.e. $\overline{\lim}_{\varepsilon\to 0} F^2_\varepsilon(x^*_\varepsilon) \leq F^2(x^*)$. On the other hand, by (2.1.20), $\underline{\lim}_{\varepsilon\to 0} F^2_\varepsilon(x^*_\varepsilon) \geq F^2(x^*)$. Therefore, $\lim_{\varepsilon\to 0} F^2_\varepsilon(x^*_\varepsilon) = F^2(x^*)$. ■

We now show an application of Lemma 2.1.1 by using it to the homogenization problem considered in [25], again, in the framework of this lemma, with the help of some facts from [25]. K_ε is chosen to be one of the U^ε_{ad} given in (2.1.9). For this, we note that any of these U^ε_{ad} may be thought of as a closed convex subset of $L^2(\Omega)$ by imbedding $L^2(\Omega_\varepsilon)$ in $L^2(\Omega)$, which is done by extending a function given on Ω_ε by zero in the holes. K is taken as the corresponding U_{ad} (cf. (2.1.10)). Also, take F_ε to be the cost functional J_ε and note that it is the sum of the functionals $F^1_\varepsilon(\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx$ and $F^2_\varepsilon(\theta) = \frac{N}{2}\int_{\Omega_\varepsilon} \theta^2 \, dx$, where $u_\varepsilon = u_\varepsilon(\theta)$ is the solution of,

$$\left.\begin{aligned} -\text{div}(A_\varepsilon \nabla u_\varepsilon) &= f + \theta && \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla u_\varepsilon . n_\varepsilon &= 0 && \text{on } \partial S_\varepsilon, \\ u_\varepsilon &= 0 && \text{on } \partial\Omega. \end{aligned}\right\}$$

Set,

$$F^1(\theta) = \frac{1}{2}\int_\Omega B^*\nabla u.\nabla u\, dx,$$

where $u = u(\theta)$ solves,

$$\left.\begin{array}{rcll} -\text{div}(A^*\nabla u) &=& \chi f + \theta & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right\}$$

where A^* is the H_0-limit of the sequence $(A_\varepsilon, S_\varepsilon)$ and B^* is given by (2.1.18). And set,

$$F^2(\theta) = \frac{1}{2}\int_\Omega \frac{\theta^2}{\chi}\, dx.$$

It is now verified that for these choices of K, F^1 and F^2 the hypotheses (P1)-(P4) are satisfied. (P1) is trivial except in the last example for K_ε in (2.1.9) and for this example (P1) is a consequence of the following lemma (cf. Proposition 2.2 [25]).

Lemma 2.1.2 *If $\theta_\varepsilon \in L^2(\Omega_\varepsilon)$ and $\widetilde{\theta_\varepsilon} \rightharpoonup \theta$ weakly in $L^2(\Omega)$, then*

$$\underline{\lim}_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \theta_\varepsilon^2\, dx \geq \int_\Omega \frac{\theta^2}{\chi}\, dx\,. \blacksquare$$

To verify (P2), given any $\theta \in K$, the choice $\theta_\varepsilon \doteq (\chi_\varepsilon/\chi)\theta$ belongs to K_ε and $\theta_\varepsilon \rightharpoonup \theta$ weakly in $L^2(\Omega)$. Thus, (P2) is also verified. Again from Lemma 2.1.2 it follows that F^2 defined above verifies (2.1.20) of (P3). To verify (P4), for any $\theta \in K$ we take θ_ε as above and by the idempotence of χ_ε, we get,

$$\begin{array}{rcl} \displaystyle\int_\Omega \chi_\varepsilon \theta_\varepsilon^2\, dx &=& \displaystyle\int_\Omega \chi_\varepsilon \frac{\theta^2}{\chi^2}\, dx \\ &\longrightarrow& \displaystyle\int_\Omega \frac{\theta^2}{\chi}\, dx\,. \end{array}$$

This shows that F^2 satisfies (P4). It remains to show that F^1 satisfies (2.1.19) of (P3), but, this is a consequence of the following fact about B^*, proved in [25].

Let $g_\varepsilon \in L^2(\Omega)$ be any sequence such that $\chi_\varepsilon g_\varepsilon \rightharpoonup g$ weakly in $L^2(\Omega)$ and let v_ε be the solution of,

$$\left.\begin{array}{rcll} -\text{div}(A_\varepsilon\nabla v_\varepsilon) &=& g_\varepsilon & \text{in } \Omega_\varepsilon, \\ A_\varepsilon\nabla v_\varepsilon.n_\varepsilon &=& 0 & \text{on } \partial S_\varepsilon \\ v_\varepsilon &=& 0 & \text{on } \partial\Omega. \end{array}\right\}$$

Then we have the following convergence of energies,

$$\int_{\Omega_\varepsilon} B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx \overset{\varepsilon \to 0}{\longrightarrow} \int_\Omega B^* \nabla v . \nabla v \, dx$$

where v is the solution of

$$\left.\begin{array}{rcll} -\text{div}(A^* \nabla v) & = & g & \text{in } \Omega, \\ v & = & 0 & \text{on } \partial\Omega. \end{array}\right\}$$

Note that, to verify (2.1.19) using the above it is enough to take $g_\varepsilon = f + \theta^\varepsilon$ for all sequences θ_ε in K_ε such that $\theta_\varepsilon \rightharpoonup \theta$ weakly in $L^2(\Omega)$. Now, Lemma 2.1.1 shows that θ^*, which is the limit of the optimal controls θ^*_ε, is the optimal control of the functional $F^1 + F^2$, where F^1 and F^2 are defined above.

The results in the non perforated case are a special case of the above; however, we are allowed to choose any closed convex set K we like, provided, we take all the K_ε 's equal to K.

Remark 2.1.1 *In the subsequent chapters, the choice of K_ε will remain the same, viz. one of the spaces listed in (2.1.9). Also, the form of F^2_ε will be similar to the one we have just considered. For these reasons, we will only need to identify a suitable F^1 and verify (2.1.19) for that function as the other hypotheses have already been verified for these choices of K_ε and F^2_ε.* ■

2.2 Convergence of Energies

We found, in the last part of the previous section, that the convergence of the energies associated with the state equations played a crucial role in the homogenization process, chiefly in identifying the function F^1 of Lemma 2.1.1. We shall address this question in some more detail now and we restrict ourselves to the non perforated case to convey our ideas better. The results, in the perforated case, are only stated as the proofs go through, *mutatis mutandis*.

As before, A_ε is a sequence of matrices in $M(a, b, \Omega)$ and B_ε is a sequence of symmetric matrices in $M(c, d, \Omega)$. It is assumed that the sequence A_ε has as its

H-limit the matrix A^*. We wish to compute the limit of

$$\int_\Omega B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx$$

where v_ε solves,

$$\left.\begin{array}{rcll} -\mathrm{div}(A\nabla v_\varepsilon) &=& g_\varepsilon & \text{in } \Omega, \\ v_\varepsilon &=& 0 & \text{on } \partial\Omega. \end{array}\right\}$$

and g_ε is a sequence in $L^2(\Omega)$ such that $g_\varepsilon \rightharpoonup g$ weakly in $L^2(\Omega)$. We know that $v_\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)$ and v solves,

$$\left.\begin{array}{rcll} -\mathrm{div}(A^*\nabla v) &=& g & \text{in } \Omega, \\ v &=& 0 & \text{on } \partial\Omega. \end{array}\right\}$$

It is desirable to express this limit in the form $\int_\Omega B\nabla v.\nabla v \, dx$ for a suitable symmetric matrix B and this should be independent of the sequences g_ε one may consider. To calculate these limits is not so easy as the integrand is a product of weakly convergent sequences. In the case when $B_\varepsilon = A_\varepsilon$ for all ε, we can compute the limit using the equations that v_ε and v solve. Indeed,

$$\begin{array}{rcl} \int_\Omega A_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx &=& \int_\Omega g_\varepsilon \, v_\varepsilon \, dx \\ &\longrightarrow& \int_\Omega g \, v \, dx \\ &=& \int_\Omega A^* \nabla v . \nabla v \, dx. \end{array}$$

We have only done some integration by parts and used the strong convergence of v_ε to v in $L^2(\Omega)$, which follows from the compact inclusion of $H^1(\Omega)$ in $L^2(\Omega)$(Rellich's compactness theorem).

When $B_\varepsilon \neq A_\varepsilon$, there is, usually, no direct way of obtaining the limit $\lim_{\varepsilon\to 0} \int_\Omega B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx$. In this case, introducing the *adjoint problem* can be of help and this is as follows. Let $p_\varepsilon \in H_0^1(\Omega)$ solve

$$\left.\begin{array}{rcll} -\mathrm{div}(A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) &=& 0 & \text{in } \Omega, \\ p_\varepsilon &=& 0 & \text{on } \partial\Omega. \end{array}\right\}$$

Then, p_ε is known as the adjoint state corresponding to the state v_ε. It can be seen that p_ε is a bounded in sequence in $H_0^1(\Omega)$. One can suppose that,

$$\begin{aligned} p_\varepsilon &\rightharpoonup p \text{ weakly in } H_0^1(\Omega), \\ z_\varepsilon \doteq A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon &\rightharpoonup z \text{ weakly in } L^2(\Omega)^n. \end{aligned}$$

Kesavan and Saint Jean Paulin [25] have shown that $z = (A^*)^t \nabla p - B^* \nabla v$ for the B^* they have defined through (2.1.7) (cf. Theorem 2.2.3). Using this, they show that (cf. Remark 3.3 [25] or Theorem 2.2.1 below)

$$\lim_{\varepsilon \to 0} \int_\Omega B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx = \int_\Omega B^* \nabla v . \nabla v \, dx.$$

The upshot is, if you know how to homogenize the state-adjoint state system of equations then it is possible to identify the limit of energies. There are still a few interesting questions which need to be answered.

Question 1: Is the matrix B^* given by (2.1.7) the unique matrix which appears in the limit of energies?

Question 2: Is there a converse to the statement, "if you know how to homogenize the state-adjoint state system of equations then you can identify the limit of energies" ?

The answer to the second question is provided by the following theorem. First, we write two statements concerning a matrix, B *not necessarily symmetric*, and the theorem will be about the equivalence of these two statements.

Statement-1 Let g_ε be any sequence in $H^{-1}(\Omega)$ such that $g_\varepsilon \longrightarrow g$ strongly in $H^{-1}(\Omega)$. Let v_ε be the solution of,

$$\left.\begin{aligned} -\text{div}(A_\varepsilon \nabla v_\varepsilon) &= g_\varepsilon \text{ in } \Omega, \\ v_\varepsilon &= 0 \text{ on } \partial\Omega. \end{aligned}\right\}$$

Let $v_\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)$. Then

$$\lim_{\varepsilon \to 0} \int_\Omega B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx = \int_\Omega B \nabla v . \nabla v \, dx \tag{2.2.1}$$

$$B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \rightharpoonup B \nabla v . \nabla v \text{ in } D'(\Omega). \tag{2.2.2}$$

Statement-2 Let g_ε be any sequence in $H^{-1}(\Omega)$ such that $g_\varepsilon \longrightarrow g$ strongly in $H^{-1}(\Omega)$. Let $(v_\varepsilon, p_\varepsilon) \in H_0^1(\Omega)^2$ solve the state-adjoint state equations,

$$\left.\begin{array}{rcll} -\mathrm{div}(A_\varepsilon \nabla v_\varepsilon) & = & g_\varepsilon & \text{in } \Omega, \\ -\mathrm{div}(A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) & = & 0 & \text{in } \Omega, \\ v_\varepsilon & = 0 = & p_\varepsilon & \text{on } \partial\Omega. \end{array}\right\}$$

Since $(v_\varepsilon, p_\varepsilon)$ is bounded in $H_0^1(\Omega)^2$, let

$$\left.\begin{array}{rcll} v_\varepsilon & \rightharpoonup & v & \text{weakly in } H_0^1(\Omega), \\ p_\varepsilon & \rightharpoonup & p & \text{weakly in } H_0^1(\Omega), \\ z_\varepsilon \doteq A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon & \rightharpoonup & z & \text{weakly in } L^2(\Omega)^n. \end{array}\right\}$$

Then

$$z = (A^*)^t \nabla p - B \nabla v. \blacksquare$$

Remark 2.2.1 *If Statement-2 is true for some B then it is to be noted that (v,p) solve the system,*

$$\left.\begin{array}{rcll} -\mathrm{div}(A^* \nabla v) & = & g & \text{in } \Omega, \\ -\mathrm{div}((A^*)^t \nabla p - B \nabla v) & = & 0 & \text{in } \Omega, \\ p & = 0 = & v & \text{on } \partial\Omega. \blacksquare \end{array}\right\} \tag{2.2.3}$$

The div-curl lemma (cf. Murat [30]) will be used to prove many of our results including Theorem 2.2.1 and hence, it is stated here.

Lemma 2.2.1 (The div-curl lemma) Let $\xi_\varepsilon, \eta_\varepsilon \in L^2(\Omega)^n$ be such that $\xi_\varepsilon \rightharpoonup \xi$ and $\eta_\varepsilon \rightharpoonup \eta$ weakly in $L^2(\Omega)^n$. Further, assume that the sequences $\mathrm{div}\xi_\varepsilon$ and $\mathrm{curl}\,\eta_\varepsilon$ are precompact in $H^{-1}(\Omega)$. Then,

$$\xi_\varepsilon . \eta_\varepsilon \rightharpoonup \xi . \eta \text{ in } D'(\Omega). \blacksquare \tag{2.2.4}$$

Theorem 2.2.1 *If B is a matrix for which Statement-2 is true then Statement-1 is true for B. The* converse *is true if B is also given to be symmetric.*

Proof: Suppose that B is a matrix for which Statement-2 is true. Let the hypotheses in Statement-1 hold. Let $p_\varepsilon \in H_0^1(\Omega)$ solve the adjoint-state equation,

$$\left.\begin{aligned} -\mathrm{div}(A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) &= 0 \quad \text{in } \Omega, \\ p_\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.2.5}$$

The uniform coercivity of A_ε implies that p_ε is bounded in $H_0^1(\Omega)$. For any subsequence, ε' of ε there is a further subsequence ε'' such that,

$$\begin{aligned} p_{\varepsilon''} &\rightharpoonup p'' \text{ weakly in } H_0^1(\Omega), \\ z_{\varepsilon''} \doteq A_{\varepsilon''}^t \nabla p_{\varepsilon''} - B_{\varepsilon''} \nabla v_{\varepsilon''} &\rightharpoonup z'' \text{ weakly in } L^2(\Omega)^n. \end{aligned}$$

By Statement-2, it follows that $z'' = A^{*t} \nabla p'' - B \nabla v$. By Remark 2.2.1 and uniqueness of solution to (2.2.3), it follows that p'' is independent of the subsequence and we denote it by p, the solution of (2.2.3). Now, by an integration by parts and using equation (2.2.5),

$$\begin{aligned} \int_\Omega B_{\varepsilon''} \nabla v_{\varepsilon''} . \nabla v_{\varepsilon''}\, dx &= \int_\Omega A_{\varepsilon''}^t \nabla p_{\varepsilon''} . \nabla v_{\varepsilon''}\, dx \\ &= \int_\Omega A_{\varepsilon''} \nabla v_{\varepsilon''} . \nabla p_{\varepsilon''}\, dx \\ &= < g_{\varepsilon''}, p_{\varepsilon''} >_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\longrightarrow < g, p >_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &= \int_\Omega A^* \nabla v . \nabla p\, dx \\ &= \int_\Omega B \nabla v . \nabla v\, dx \end{aligned}$$

where the last equality follows from (2.2.3) after an integration by parts. As every subsequence has a further subsequence converging to the same limit, $\int_\Omega B \nabla v . \nabla v\, dx$, this proves (2.2.1).

Rewriting

$$B_{\varepsilon''} \nabla v_{\varepsilon''} . \nabla v_{\varepsilon''} = -(A_{\varepsilon''}^t \nabla p_{\varepsilon''} - B_{\varepsilon''} \nabla v_{\varepsilon''}) . \nabla v_{\varepsilon''} + A_{\varepsilon''} \nabla v_{\varepsilon''} . \nabla p_{\varepsilon''}. \tag{2.2.6}$$

Note that,

$$\begin{aligned} \mathrm{div}(A_{\varepsilon''}^t \nabla p_{\varepsilon''} - B_{\varepsilon''} \nabla v_{\varepsilon''}) &= 0, \\ \mathrm{div}(A_{\varepsilon''} \nabla v_{\varepsilon''}) &= g_{\varepsilon''} \end{aligned}$$

and hence are precompact sequences in $H^{-1}(\Omega)$. So, we may apply the div-curl lemma in (2.2.6) to conclude,

$$\begin{aligned} B_{\varepsilon''}\nabla v_{\varepsilon''}.\nabla v_{\varepsilon''} &\rightharpoonup -(A^{*t}\nabla p - B\nabla v).\nabla v + A^{*}\nabla v.\nabla p \text{ in } D'(\Omega) \\ &= B\nabla v.\nabla v \end{aligned}$$

As the limit is independent of the subsequence, we conclude that,

$$B_\varepsilon \nabla v_\varepsilon.\nabla v_\varepsilon \rightharpoonup B\nabla v.\nabla v \text{ in } D'(\Omega).$$

Thus, we have shown that if B satisfies Statement-2, then it satisfies Statement-1 along with (2.2.2).

Conversely, suppose that B is symmetric and satisfies Statement-1. Now, let the hypotheses in Statement-2 hold. Let $\omega \subset\subset \Omega$ and let $\eta \in D(\Omega)$ be a cut-off function such that $\eta \equiv 1$ on ω. We define test functions $\eta_\varepsilon^k \in H_0^1(\Omega)$, $k = 1, 2, ..., n$, as solving

$$-\text{div}(A_\varepsilon \nabla \eta_\varepsilon^k) = -\text{div}(A^{*}\nabla(\eta x_k)) \text{ in } \Omega.$$

By the H-convergence of the matrices A_ε, we obtain,

$$\begin{aligned} \eta_\varepsilon^k &\rightharpoonup \eta x_k \text{ weakly in } H_0^1(\Omega), \\ A_\varepsilon \nabla \eta_\varepsilon^k &\rightharpoonup A^{*}\nabla(\eta x_k) \text{ weakly in } L^2(\Omega)^n. \end{aligned}$$

By superposition of the equations for v_ε and η_ε^k,

$$-\text{div}(A_\varepsilon \nabla(v_\varepsilon \pm \eta_\varepsilon^k)) = g_\varepsilon \pm (-\text{div}(A^{*}\nabla \eta x_k)) \text{ in } \Omega.$$

Therefore, by Statement-1,

$$B_\varepsilon(\nabla v_\varepsilon \pm \nabla \eta_\varepsilon^k).(\nabla v_\varepsilon \pm \nabla \eta_\varepsilon^k) \rightharpoonup B(\nabla v \pm \nabla(\eta x_k)).(\nabla v \pm \nabla(\eta x_k)) \text{ in } D'(\Omega).$$

Hence, using the polarization identity and the symmetry of B,

$$B_\varepsilon \nabla v_\varepsilon.\nabla \eta_\varepsilon^k \rightharpoonup B\nabla v.\nabla(\eta x_k) \text{ in } D'(\Omega). \quad (2.2.7)$$

Now, we obtain the distribution limit of $z_\varepsilon \doteq A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon$ in two ways. By the div-curl lemma applied to $z_\varepsilon.\nabla\eta_\varepsilon^k$, we get,

$$z_\varepsilon.\nabla\eta_\varepsilon^k \rightharpoonup z.\nabla(\eta x_k) \text{ in } D'(\Omega) \tag{2.2.8}$$

On the other hand,

$$\left.\begin{aligned} z_\varepsilon.\nabla\eta_\varepsilon^k &= A_\varepsilon\nabla\eta_\varepsilon^k.\nabla p_\varepsilon - B_\varepsilon\nabla v_\varepsilon.\nabla\eta_\varepsilon^k \\ &\rightharpoonup A^*\nabla(\eta x_k).\nabla p - B\nabla v.\nabla(\eta x_k) \text{ in } D'(\Omega), \end{aligned}\right\} \tag{2.2.9}$$

using (2.2.7) and applying the div-curl lemma to pass to the limit in the first term of the sum. So, from (2.2.8) and (2.2.9), we get,

$$z.e_k = A^{*t}\nabla p.e_k - B\nabla v.e_k \text{ in } D'(\omega)$$

Since this is true for all $\omega \subset\subset \Omega$, we have the desired conclusion. Thus B has been shown to satisfy Statement-2. ■

The answer to our first question is given by the following theorem.

Theorem 2.2.2 *Any symmetric B satisfying Statement-1 or, equivalently, Statement-2 is unique.*

Proof: Suppose B, B' are two symmetric matrices for which Statement-1 with (2.2.2) is true. For η_ε^k defined in the proof of previous theorem, we have for any $j,k \in \{1,2,...,n\}$,

$$\begin{aligned} B_\varepsilon\nabla\eta_\varepsilon^j.\nabla\eta_\varepsilon^k &\rightharpoonup B\nabla(\eta x_j).\nabla(\eta x_k) \text{ in } D'(\Omega) \text{ and,} \\ B_\varepsilon\nabla\eta_\varepsilon^j.\nabla\eta_\varepsilon^k &\rightharpoonup B'\nabla(\eta x_j).\nabla(\eta x_k) \text{ in } D'(\Omega). \end{aligned}$$

Therefore, $B \equiv B'$ for all $\omega \subset\subset \Omega$, and this proves the result.

Another proof would be to use (2.2.1) and the desired result follows directly from Lemma 22.5 of Dal Maso [16]. ■

Now, is there is a matrix B satisfying Statement-2 ? which of course implies B satisfies Statement-1. It was shown by Kesavan and Saint Jean Paulin in [24] that B^* defined through (2.1.7) satisfies Statement-2. We prove this, now, for the sake of completeness; also, our proof turns out to be much shorter than the original proof in [24].

Theorem 2.2.3 *B^* defined by (2.1.7) satisfies Statement-2.*

Proof: Let g_ε be any sequence in $H^{-1}(\Omega)$ such that $g_\varepsilon \longrightarrow g$ strongly in $H^{-1}(\Omega)$. Let $(v_\varepsilon, p_\varepsilon) \in H_0^1(\Omega)^2$ solve the state-adjoint state system of equations,

$$\left.\begin{aligned} \operatorname{div}(A_\varepsilon \nabla v_\varepsilon) &= g_\varepsilon \quad \text{in } \Omega, \\ \operatorname{div}(A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) &= 0 \quad \text{in } \Omega, \\ v_\varepsilon = 0 &= p_\varepsilon \quad \text{on } \partial\Omega. \end{aligned}\right\}$$

Since $(v_\varepsilon, p_\varepsilon)$ is bounded in $H_0^1(\Omega)^2$, we may suppose that,

$$\left.\begin{aligned} v_\varepsilon &\rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \\ p_\varepsilon &\rightharpoonup p \quad \text{weakly in } H_0^1(\Omega), \\ z_\varepsilon \doteq A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon &\rightharpoonup z \quad \text{weakly in } L^2(\Omega)^n. \end{aligned}\right\}$$

Let B^* be given by (2.1.7). We need to show that $z = (A^*)^t \nabla p - B^* \nabla v$. Let X_ε^k be the test sequences having the properties (2.1.5). Then,

$$\begin{aligned} z_\varepsilon . \nabla X_\varepsilon^k &= \nabla p_\varepsilon . A_\varepsilon \nabla X_\varepsilon^k - B_\varepsilon \nabla X_\varepsilon^k . \nabla v_\varepsilon \\ &= \nabla p_\varepsilon . A_\varepsilon \nabla X_\varepsilon^k - \{(B_\varepsilon \nabla X_\varepsilon^k + A_\varepsilon^t \nabla \psi_\varepsilon^k). \nabla v_\varepsilon - A_\varepsilon \nabla v_\varepsilon . \nabla \psi_\varepsilon^k\}. \end{aligned}$$

We may apply the div-curl lemma, use the convergence properties of X_ε^k, ψ_ε^k, v_ε, p_ε etc. and (2.1.7) to conclude that,

$$z_\varepsilon . \nabla X_\varepsilon^k \rightharpoonup A^* e_k . \nabla p - (B^*)^t e_k . \nabla v \text{ in } D'(\Omega).$$

On the other hand, directly from the div-curl lemma,

$$z_\varepsilon . \nabla X_\varepsilon^k \rightharpoonup z . e_k \text{ in } D'(\Omega).$$

So, we conclude that $z = (A^*)^t \nabla p - B^* \nabla v$ in $D'(\Omega)$. However, they are both $L^2(\Omega)^n$ functions and the sequence z_ε is bounded in $L^2(\Omega)^n$. Therefore, they are equal as $L^2(\Omega)^n$ functions as well. ■

Remark 2.2.2 *The symmetry of B^* follows from Theorem 2.3.1 in the next section. So, by the theorems we have just proved, the matrix appearing in the cost functional of the homogenized problem can only be the B^* which is defined by (2.1.7).* ■

Remark 2.2.3 *Though, in [25] it is observed that B^* satisfies Statement-2, they do not observe that B^* has the convergence property (2.2.2) while still observing (2.2.1). We have seen now that these two convergences characterize the homogenized state-adjoint system of equations.* ■

The versions of Theorem 2.2.1, Theorem 2.2.2, and Theorem 2.2.3 for the *perforated case* are now stated. Let $(A_\varepsilon, S_\varepsilon)$ H_0 converge to A^*. In the perforated case, Statement-1 and Statement-2 concerning a matrix B are to be replaced by,

Statement-3: Let g_ε be any sequence in $H^{-1}(\Omega)$ such that $g_\varepsilon \longrightarrow g$ strongly in $H^{-1}(\Omega)$. Let v_ε be the solution of,

$$\left.\begin{array}{rcll} -\text{div}(A_\varepsilon \nabla v_\varepsilon) &=& P_\varepsilon^* g_\varepsilon & \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla v_\varepsilon . n_\varepsilon &=& 0 & \text{on } \partial S_\varepsilon, \\ v_\varepsilon &=& 0 & \text{on } \partial\Omega. \end{array}\right\}$$

Let $P_\varepsilon v_\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)$. Then

$$\begin{array}{rcl} \displaystyle\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx &=& \displaystyle\int_\Omega B\nabla v . \nabla v \, dx \\ \chi_\varepsilon B_\varepsilon \nabla P_\varepsilon v_\varepsilon . \nabla P_\varepsilon v_\varepsilon &\rightharpoonup& B\nabla v.\nabla v \text{ in } D'(\Omega). \end{array}$$

Statement-4: Let g_ε be any sequence in $H^{-1}(\Omega)$ such that $g_\varepsilon \longrightarrow g$ strongly in $H^{-1}(\Omega)$. Let $(v_\varepsilon, p_\varepsilon) \in H_0^1(\Omega)^2$ solve the state-adjoint state equations,

$$\left.\begin{array}{rcll} -\text{div}(A_\varepsilon \nabla v_\varepsilon) &=& P_\varepsilon g_\varepsilon & \text{in } \Omega, \\ -A_\varepsilon \nabla v_\varepsilon . n_\varepsilon &=& 0 & \text{on } \partial S_\varepsilon, \\ \text{div}(A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) &=& 0 & \text{in } \Omega, \\ (A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon).n_\varepsilon &=& 0 & \text{on } \partial S_\varepsilon, \\ v_\varepsilon &=0=& p_\varepsilon & \text{on } \partial\Omega. \end{array}\right\}$$

Since $(P_\varepsilon v_\varepsilon, P_\varepsilon p_\varepsilon)$ is bounded in $H_0^1(\Omega)^2$, let

$$\left.\begin{array}{rcll} P_\varepsilon v_\varepsilon &\rightharpoonup& v & \text{weakly in } H_0^1(\Omega), \\ P_\varepsilon p_\varepsilon &\rightharpoonup& p & \text{weakly in } H_0^1(\Omega), \\ z_\varepsilon \doteq Q_\varepsilon(A_\varepsilon^t \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) &\rightharpoonup& z & \text{weakly in } L^2(\Omega)^n. \end{array}\right\}$$

Then

$$z = (A^*)^t \nabla p - B \nabla v. \blacksquare$$

The following theorems are true.

Theorem 2.2.4 *If B is a matrix for which Statement-4 is true then Statement-3 is true for B. The* converse *is true if B is also given to be* symmetric. ■

Theorem 2.2.5 *Any symmetric B satisfying Statement-3 or, equivalently, Statement-4 is unique.* ■

Theorem 2.2.6 *B^* defined by (2.1.18) satisfies Statement-4.* ■

A similar remark (cf. Remark 2.2.2) can be made about the uniqueness of B^* after showing that it is symmetric. The symmetry and some other properties of B^* are shown in [25] but with much difficulty as the expression (2.1.18) is not convenient. The next section deals with these problems.

2.3 Properties of B^*

The properties of B^* that are of interest to us are its symmetry, ellipticity and upper bound. We now reformulate B^*, given by (2.1.18), in a natural way as to give us some properties of B^* like symmetry and upper bound, easily; the question on the upper bound was left open in [24], [25]. The ellipticity of B^* was shown in [25] in a certain sense, but not of the matrix itself. So, we will also prove the ellipticity of B^*.

We recall the test functions, X_ε^k, which were defined through (2.1.16). Define, the corrector matrices, M_ε by,

$$M_\varepsilon e_k = \nabla P_\varepsilon X_\varepsilon^k, \text{ for } k = 1, 2, ..., n. \tag{2.3.1}$$

It is known (cf. Proposition 1.14 [7]) that they have the following properties,

$$\begin{aligned} M_\varepsilon &\rightharpoonup I \text{ weakly in } L^2(\Omega)^{n^2}, \\ \chi_\varepsilon A_\varepsilon M_\varepsilon &\rightharpoonup A^* \text{ weakly in } L^2(\Omega)^{n^2}, \\ \operatorname{div}(\chi_\varepsilon A_\varepsilon M_\varepsilon) &\subset\subset H^{-1}(\Omega). \end{aligned}$$

The proof of Theorem 2.3.1, below, will use the following lemma.

Lemma 2.3.1 *(cf. Proposition 1.13 [7]) Let $\xi_\varepsilon \in L^2(\Omega)^n$ be a sequence of vector fields such that the sequence, $Q_\varepsilon(\xi_\varepsilon)$, is bounded in $L^2(\Omega)^n$ and satisfies*

$$\left.\begin{array}{lcl} -\mathrm{div}(\xi_\varepsilon) & = & P_\varepsilon^* f_\varepsilon \ \textit{in}\ \Omega_\varepsilon, \\ \xi_\varepsilon.n_\varepsilon & = & 0 \ \textit{on}\ \partial S_\varepsilon \end{array}\right\} \tag{2.3.2}$$

and the sequence, f_ε, is in a compact subset of $H^{-1}(\Omega)$. Then, the sequence $\mathrm{div}(Q_\varepsilon\xi_\varepsilon)$ is in a compact subset of $H^{-1}(\Omega)$. ■

Theorem 2.3.1 *Let B^* be defined through (2.1.18). Then, B^* is the limit, in the distribution sense, of the sequence of matrices $\chi_\varepsilon M_\varepsilon^t B_\varepsilon M_\varepsilon$.*

Proof: It is enough to show, for any $j, k \in \{1, 2, ..., n\}$, that,

$$\chi_\varepsilon M_\varepsilon^t B_\varepsilon M_\varepsilon e_k.e_j \rightharpoonup B^* e_k.e_j \text{ in } D'(\Omega).$$

We first rewrite the left hand side as follows

$$\begin{aligned} \chi_\varepsilon M_\varepsilon^t B_\varepsilon M_\varepsilon e_k.e_j &= \chi_\varepsilon B_\varepsilon \nabla P_\varepsilon X_\varepsilon^k.\nabla P_\varepsilon X_\varepsilon^j \\ &= \chi_\varepsilon (A_\varepsilon^t \nabla P_\varepsilon \psi_\varepsilon^k + B_\varepsilon \nabla P_\varepsilon X_\varepsilon^k).\nabla P_\varepsilon X_\varepsilon^j - \chi_\varepsilon A_\varepsilon \nabla P_\varepsilon X_\varepsilon^j.\nabla P_\varepsilon \psi_\varepsilon^k \\ &= Q_\varepsilon (A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k).\nabla P_\varepsilon X_\varepsilon^j - \chi_\varepsilon A_\varepsilon M_\varepsilon e_j.\nabla P_\varepsilon \psi_\varepsilon^k. \end{aligned}$$

We are in a position to apply div-curl lemma provided we show that $\mathrm{div} Q_\varepsilon(A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k)$ is precompact in $H^{-1}(\Omega)$. But this follows by taking $\xi_\varepsilon = (A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k)$ and $f_\varepsilon = 0$ in the previous lemma. Therefore, we get

$$\begin{aligned} \chi_\varepsilon M_\varepsilon^t B_\varepsilon M_\varepsilon e_k.e_j &\rightharpoonup \lim_{\varepsilon\to 0} Q_\varepsilon (A_\varepsilon^t \nabla \psi_\varepsilon^k + B_\varepsilon \nabla X_\varepsilon^k).e_j - A^* e_j.\nabla \psi^k \text{ in } D'(\Omega) \\ &= B^* e_k.e_j \end{aligned}$$

from the definition of B^*. ■

As an immediate consequence we have,

Corollary 2.3.1 *B^* is symmetric.* ■

We now obtain bounds for the matrix B^*.

Corollary 2.3.2 $B^* \in M(c\,C_0^{-2}, d\,b^2/a^2, \Omega)$.

Proof: First we prove the upper bound. Let $\phi \in D(\Omega)$, $\phi \geq 0$ and let $\xi \in \mathbb{R}^n$. We have, using the bounds on A_ε and B_ε,

$$\int_\Omega \chi_\varepsilon B_\varepsilon M_\varepsilon \xi . M_\varepsilon \xi \phi \, dx \leq d/a \int_\Omega \chi_\varepsilon A_\varepsilon M_\varepsilon \xi . M_\varepsilon \xi \phi \, dx. \tag{2.3.3}$$

It can be shown, using the div-curl lemma and the properties of M_ε, that

$$\chi_\varepsilon M_\varepsilon^t A_\varepsilon M_\varepsilon \rightharpoonup A^* \text{ in } D'(\Omega).$$

We know from Theorem 2.3.1 that $\chi_\varepsilon M_\varepsilon^t B_\varepsilon M_\varepsilon$ converges in $D'(\Omega)$ to B^*. Therefore, we may pass to the limit as $\varepsilon \to 0$ in (2.3.3) and obtain

$$\int_\Omega B^* \xi . \xi \phi \, dx \leq d/a \int_\Omega A^* \xi . \xi \phi \, dx.$$

As this holds for any $\phi \geq 0$ in $D(\Omega)$, we conclude that $B^*(x)\xi.\xi \leq (d/a)\, A^*(x)\xi.\xi$ for almost every x in Ω. Since it is known that $A^*(x)\xi.\xi \leq b^2/a\,|\xi|^2$ a.e. $x \in \Omega$, we conclude that

$$B^*(x)\xi.\xi \leq d\,b^2/a^2\,|\xi|^2 \text{ a.e. } x \in \Omega \tag{2.3.4}$$

and for all $\xi \in \mathbb{R}^n$. This proves the upper bound.

A lower bound already exists (cf. Theorem 3.3 [25]) for the quadratic functional defined by B^* viz. ,

$$c\,C_0^{-2} \int_\Omega |\nabla v|^2 \, dx \leq \int_\Omega B^* \nabla v . \nabla v \, dx \text{ for all } v \in H_0^1(\Omega). \tag{2.3.5}$$

Then, it follows from Proposition 2.3.1 below, proved by Juan Casado Díaz [9], that the matrix B^* is itself elliptic and indeed

$$c\,C_0^{-2}|\xi|^2 \leq B^*(x)\xi.\xi \text{ a.e. } x \in \Omega.$$

This gives the lower bound. ■

Proposition 2.3.1 *Let $A = (a_{ij})$ be a symmetric matrix whose entries belong to $L^\infty(\Omega)$. Then,*

$$\int_\Omega A\nabla v.\nabla v\, dx \geq 0 \text{ for all } v \in H_0^1(\Omega) \tag{2.3.6}$$

implies that $A(x)\xi.\xi \geq 0$ a.e. in Ω and for all $\xi \in \mathbb{R}^n$.

Proof: Let $\xi \in \mathbb{R}^n$ and let $\phi \geq 0$ be any function in $C_0^1(\Omega)$. Consider the sequence v_ε, given by $v_\varepsilon = \varepsilon \cos(\varepsilon^{-1}\xi.x)\phi$. It is clear that for each $\varepsilon > 0$, the function v_ε belongs to $H_0^1(\Omega)$. Rewriting (2.3.6) with v_ε we get,

$$\begin{aligned} 0 \;&\leq\; \varepsilon^2 \int_\Omega A\nabla\phi.\nabla\phi \cos^2(\xi.x/\varepsilon)\, dx - 2\varepsilon \int_\Omega A\nabla\phi.\xi \cos(\xi.x/\varepsilon)\sin(\xi.x/\varepsilon)\phi\, dx \\ &\quad + \int_\Omega A\xi.\xi \sin^2(\xi.x/\varepsilon)\phi^2\, dx. \end{aligned}$$

Similarly, starting with $\varepsilon \sin(\varepsilon^{-1}\xi.x)\phi$ one gets

$$\begin{aligned} 0 \;&\leq\; \varepsilon^2 \int_\Omega A\nabla\phi.\nabla\phi \sin^2(\xi.x/\varepsilon)\, dx + 2\varepsilon \int_\Omega A\nabla\phi.\xi \cos(\xi.x/\varepsilon)\sin(\xi.x/\varepsilon)\phi\, dx \\ &\quad + \int_\Omega A\xi.\xi \cos^2(\xi.x/\varepsilon)\phi^2\, dx. \end{aligned}$$

Adding these two inequalities gives

$$0 \leq \varepsilon^2 \int_\Omega A\nabla\phi.\nabla\phi\, dx + \int_\Omega A\xi.\xi\phi^2\, dx.$$

We may let $\varepsilon \to 0$ to obtain

$$0 \leq \int_\Omega A\xi.\xi\phi^2\, dx$$

for all $\phi \in C_0^1(\Omega)$. From this it follows, by standard arguments from measure theory, that $A(x)\xi.\xi \geq 0$ a.e. $x \in \Omega$. ■

Remark 2.3.1 *The original proof of Juan Casado Díaz uses the test sequence v_ε defined by, $v_\varepsilon = \varepsilon\psi(\varepsilon^{-1}\xi.x)\phi$ where ψ is the roof function,*

$$\psi(t) = \begin{cases} t & \text{if} \quad t \in [0, 1/2], \\ 1 - t & \text{if} \quad t \in [1/2, 1]. \end{cases}$$

We have modified it to resemble the proof of a result due to Dal Maso (cf. Lemma 22.5 [16]) where it is essentially shown that if (2.3.6) changed to an equality then A is the zero matrix in the almost everywhere sense. ■

Chapter 3

Periodic Case

3.1 Introduction

In this chapter, we study the homogenization of the optimal control problem with periodically oscillating coefficients and posed over periodically perforated domains. We obtain formulae for the homogenized coefficients, *directly*, employing a corrector result, using the method of two-scale convergence. This recovers the formulae of Kesavan and Vanninathan [27] in the non perforated case. In Section 3.4, generalizations of these formulae are obtained by considering the situation where there are several(well separated) scales, using the method of multi-scale convergence proposed by Allaire and Briane [2]. The results of this chapter appeared in Kesavan and Rajesh [23].

First, some notations and definitions.

• Periodic function spaces on $\mathbb{R}^n$ with the unit cell, Y, as period will be denoted by the subscript $\#$. For e.g., $C(\overline{\Omega}, C_\#(Y))$ will denote the space of continuous functions on $\overline{\Omega} \times \mathbb{R}^n$ which are Y-periodic in the second variable.

• The optimal control problem will have as coefficients, $A_\varepsilon(x) = A(x, \frac{x}{\varepsilon})$, $B_\varepsilon(x) = B(x, \frac{x}{\varepsilon})$, where $A(x, y) \in M(a, b, \Omega \times \mathbb{R}^n)$, $B(x, y) \in M(c, d, \Omega \times \mathbb{R}^n)$, with $A, B \in C(\overline{\Omega}, L^\infty_\#(Y))$.

• A periodically perforated domain, Ω_ε, is obtained from Ω by removing a set T_ε, consisting of holes, where,

$$T_\varepsilon = \bigcup_{\mathbf{k}\in\mathbb{Z}^n} \varepsilon(\mathbf{k}+T)$$

for some T, closed, $\subset Y$ with Lipschitz boundary. It is assumed that Ω_ε is connected. The boundary of Ω_ε has two parts -the *interior boundary* given by,

$$\partial_{\text{int}}\Omega_\varepsilon = \bigcup_{\mathbf{k}\in\mathbb{Z}^n} \{\partial\varepsilon(\mathbf{k}+T) | \varepsilon(\mathbf{k}+T) \subset \Omega\}.$$

and the *exterior* boundary, $\partial_{\text{ext}}\Omega_\varepsilon = \partial\Omega_\varepsilon \setminus \partial_{\text{int}}\Omega_\varepsilon$. The *material part* in the unit cell is $Y^* = Y \setminus T$ and is assumed to have non-zero Lebesgue measure, m^*. Note that m^* is also the L^∞ weak* limit of the sequence χ_ε.

• Q_ε or $\tilde{}$ will be used to denote the operator which extends a function given on Ω_ε by zero in the holes.

For $\varepsilon > 0$ fixed, the optimal control problem consists of minimizing the cost functional

$$(P_\varepsilon) \qquad J_\varepsilon(\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon dx + \frac{N}{2}\int_{\Omega_\varepsilon} \theta^2\, dx$$

over $\theta \in U^\varepsilon_{ad}$ where $u_\varepsilon = u_\varepsilon(\theta)$ is the solution of the state equation,

$$\left.\begin{aligned} -\text{div}(A_\varepsilon \nabla u_\varepsilon) &= f + \theta \quad \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla u_\varepsilon . n_\varepsilon &= 0 \qquad \text{on } \partial_{int}\Omega_\varepsilon, \\ u_\varepsilon &= 0 \qquad \text{on } \partial_{ext}\Omega_\varepsilon. \end{aligned}\right\}$$

U^ε_{ad} is taken to be one of (2.1.9). Set,

$$\begin{aligned} F^1_\varepsilon(\theta) &= \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon\, dx \text{ and,} \\ F^2_\varepsilon(\theta) &= \frac{N}{2}\int_{\Omega_\varepsilon} \theta^2\, dx \end{aligned}$$

The homogenization of $(\mathbf{P}_\varepsilon)$ is performed in the framework of Lemma 2.1.1. As already remarked (cf. Remark 2.1.1), the limiting space of controls is the corresponding U_{ad} given by (2.1.10) and F^2 is the function defined in Section 2.1. We need to identify F^1 which will satisfy (2.1.19). This involves two things-first, to pass to the limit in the following equations,

$$\left.\begin{aligned} -\text{div}(A_\varepsilon \nabla v_\varepsilon) &= f_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla v_\varepsilon . n_\varepsilon &= 0 \quad \text{on } \partial_{int}\Omega_\varepsilon, \\ v_\varepsilon &= 0 \quad \text{on } \partial_{ext}\Omega_\varepsilon. \end{aligned}\right\} \tag{3.1.1}$$

for any sequence $f_\varepsilon \in L^2(\Omega)$, with $\chi_\varepsilon f_\varepsilon \rightharpoonup m^* f$ weakly in $L^2(\Omega)$; and second, to obtain the limit of the energies,

$$\int_{\Omega_\varepsilon} B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx.$$

This will be done using the method of two-scale convergence which is described in the next section.

3.2 The Two-Scale Method

In this section, we shall discuss the formal two-scale method and its counterpart, the two-scale convergence method. We shall, mostly, recall various results found in the literature or small modifications of these, without proof. For the proofs, Allaire [1] or Conca, Planchard and Vanninathan [15] is a suitable reference.

In the homogenization of problems with a periodic micro structure, the solution is, usually, assumed to have a two-scale asymptotic expansion,

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + ..., \tag{3.2.1}$$

where each $u_i(x, y)$ is assumed $Y-$*periodic*. By formal expansion of the differential equation and by equating the coefficients of various powers of ε, it is possible to obtain the homogenized equation that u_0 solves. The coefficients of this equation and the next term u_1 in the asymptotic expansion are obtained by solving some *cell problems*. The formal calculations are made rigorous by proving the convergence of u_ε to u_0 in a suitable topology (usually *weak*). Previously, the convergence results were proved by the energy method which consists of some clever manipulations of the equations and carefully chosen test functions. A recent method, which is very suitable to handle convergences in the homogenization of periodic micro-structures, is the method of *two-scale convergence*, proposed by Nguetseng [31] and refined by Allaire [1]. At the basis of this method is the following *averaging principle* (cf. Lemma 5.2 [1] or Lemma 5.3, Ch. III [15])

Lemma 3.2.1 *If $\phi(x,y) \in C(\overline{\Omega}, C_\#(Y))$, then*

$$\int_\Omega \phi(x, \frac{x}{\varepsilon})\, dx \overset{\varepsilon \to 0}{\longrightarrow} \int_\Omega \int_Y \phi(x,y)\, dy\, dx. \blacksquare \qquad (3.2.2)$$

Remark 3.2.1 *It is easy to see that if u_ε has the asymptotic expansion (3.2.1) where the u_i' s are smooth, then*

$$\int_\Omega u_\varepsilon \phi(x, \frac{x}{\varepsilon})\, dx \overset{\varepsilon \to 0}{\longrightarrow} \int_\Omega \int_Y u_0(x,y)\phi(x,y)\, dy\, dx \qquad (3.2.3)$$

for all $\phi \in C(\overline{\Omega}, C_\#(Y))$. Thus we obtain the first term in the asymptotic expansion of u_ε. $\blacksquare$

This leads to the following definition,

Definition 3.2.1 *A sequence u_ε of functions which satisfies (3.2.3) is said to two-scale converge to u_0 and we write $u_\varepsilon \overset{2-s}{\longrightarrow} u_0(x,y)$.* $\blacksquare$

The following compactness result helps us to obtain the first term in the asymptotic expansion of u_ε whenever the sequence, u_ε, is bounded in $L^2(\Omega)$.

Theorem 3.2.1 *For each bounded sequence u_ε in $L^2(\Omega)$ one can extract a subsequence and, there exists a function $u_0(x,y) \in L^2(\Omega \times Y)$ such that this subsequence two-scale converges to u_0.*

A few properties of two-scale convergence are listed below. Let u_ε be a sequence in $L^2(\Omega)$.

1. For any two-scale convergent sequence its two-scale limit is unique.
2. If $u_\varepsilon \longrightarrow u$ strongly in $L^2(\Omega)$, then $u_\varepsilon \overset{2-s}{\longrightarrow} u(x)$.
3. If $u_\varepsilon \overset{2-s}{\longrightarrow} u_0(x,y)$, then $u_\varepsilon \rightharpoonup \int_Y u_0(x,y)\, dy$ weakly in $L^2(\Omega)$. Then it follows

from the uniform boundedness principle that any two-scale convergent sequence, u_ε is bounded in $L^2(\Omega)$.

Property 3 above shows that two-scale convergence yields something weak and is not quite enough to pass to the limit in integrals involving the product of two weakly convergent sequences in $L^2(\Omega)$. To handle this situation we need to have a stronger two-scale convergence. This leads to the following definition.

Definition 3.2.2 *A measurable function $\psi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ which is Y-periodic in the variable y is said to be admissible if*

$$\int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right)^2 dx \longrightarrow \int_\Omega \int_Y \psi(x,y)^2 \, dy \, dx. \tag{3.2.4}$$

More generally, let u_ε be a sequence in $L^2(\Omega)$ which two-scale converges to $u_0(x,y)$. It is said to be admissible if

$$\int_\Omega u_\varepsilon^2 \, dx \longrightarrow \int_\Omega \int_Y u_0(x,y)^2 \, dy \, dx. \blacksquare \tag{3.2.5}$$

Remark 3.2.2 *Though the most general condition under which $\psi(x,y)$ is admissible is not known, it is known that if ψ belongs to one of the spaces $L^2(\Omega, C_\#(Y))$, $C_c(\Omega, L^\infty_\#(Y))$ or $C(\overline{\Omega}, L^\infty_\#(Y))$ then it is admissible (cf. Allaire [1]). Moreover, $\psi(x, \frac{x}{\varepsilon}) \xrightarrow{2-s} \psi(x,y)$.* $\blacksquare$

We, then, have the following strong convergences,

Theorem 3.2.2 *(Allaire [1]) Let $u_\varepsilon \xrightarrow{2-s} u_0(x,y)$ and assume that u_ε is an admissible sequence. If v_ε is any sequence such that $v_\varepsilon \xrightarrow{2-s} v_0(x,y)$ then,*

$$u_\varepsilon v_\varepsilon \rightharpoonup \int_Y u_0(x,y) v_0(x,y) \, dy \text{ in } D'(\Omega) \text{ and,} \tag{3.2.6}$$

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon v_\varepsilon \, dx = \int_\Omega \int_Y u_0(x,y) v_0(x,y) \, dy \, dx. \tag{3.2.7}$$

Further, if $u_0(x, \frac{x}{\varepsilon}) \xrightarrow{2-s} u_0(x,y)$ and u_0 is an admissible function then,

$$\lim_{\varepsilon \to 0} \left\| u_\varepsilon - u_0(x, \frac{x}{\varepsilon}) \right\|_{2,\Omega} = 0 \,. \qquad \blacksquare \tag{3.2.8}$$

Remark 3.2.3 *We observe that, due to Remark 3.2.2 and Theorem 3.2.2, any* ϕ *from the spaces* $L^2(\Omega, C_\#(Y))$, $C_c(\Omega, L^\infty_\#(Y))$ *or* $C(\overline{\Omega}, L^\infty_\#(Y))$ *can be used in the definition of two-scale convergence. Besides, as these spaces are dense in* $L^2(\Omega \times Y)$ *the compactness result, Theorem 3.2.1, is also valid if in the definition of two-scale convergence we substitute one of the admissible spaces for* $C(\overline{\Omega}, C_\#(Y))$. ■

To obtain more terms in the asymptotic expansion of u_ε, we need higher regularity of u_ε than $L^2(\Omega)$. In fact,

Theorem 3.2.3 *Let* u_ε *be a bounded sequence in* $H^1(\Omega)$ *converging weakly to a function* $u \in H^1(\Omega)$. *Then, there exists* $u_1(x,y) \in L^2(\Omega, H^1_\#(Y)/\mathbb{R})$ *such that, up to a subsequence,* ∇u_ε *two-scale converges to* $\nabla_x u(x) + \nabla_y u_1(x,y)$. ■

We, thus, obtain the second term, u_1 in the asymptotic expansion of u_ε. Now, we prove a result which turns out to be quite useful in several proofs in the next section.

Theorem 3.2.4 *Let* u_ε *be a bounded sequence in* $L^2(\Omega)$ *such that* $u_\varepsilon \xrightarrow{2-s} u_0(x,y)$ *and let* $\phi \in C(\overline{\Omega}, L^\infty_\#(Y))$. *Then,*

$$u_\varepsilon \phi(x, \frac{x}{\varepsilon}) \xrightarrow{2-s} u_0(x,y)\phi(x,y). \tag{3.2.9}$$

Proof: Let $\psi \in C(\overline{\Omega}, C_\#(Y))$. We note that $\phi\psi \in C(\overline{\Omega}, L^\infty_\#(Y))$ and so, by Remark 3.2.2, $\phi(x, \frac{x}{\varepsilon})\psi(x, \frac{x}{\varepsilon}) \xrightarrow{2-s} \phi(x,y)\psi(x,y)$ and $\phi\psi$ is an admissible function. Therefore, by (3.2.7) of Theorem 3.2.2, we get,

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon \phi(x, \frac{x}{\varepsilon})\psi(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_Y u_0(x,y)\phi(x,y)\psi(x,y)\, dx\, dy\, . ■$$

Corollary 3.2.1 *Let* u_ε, u_0 *and* ϕ *be as in the previous theorem. Let* v_ε *be an admissible sequence which two-scale converges to* $v_0(x,y)$. *Then,*

$$\int_\Omega u_\varepsilon \phi(x, \frac{x}{\varepsilon}) v_\varepsilon\, dx \longrightarrow \int_\Omega \int_Y u_0 \phi v_0\, dx\, dy. \tag{3.2.10}$$

Proof: By the previous theorem, $u_\varepsilon \phi\left(x, \frac{x}{\varepsilon}\right) \xrightarrow{2-s} u_0(x,y)\phi(x,y)$. Also, we are given that $v_\varepsilon \xrightarrow{2-s} v_0(x,y)$ and that v_ε is an admissible sequence. Therefore, (3.2.10) follows from Theorem 3.2.2. ■

3.3 Homogenization-Two Scales

We now resume the discussion from where we left off in Section 3.1. The homogenization of the equation, (3.1.1), was done by Allaire [1], using two-scale convergence. This is recalled.

Let f_ε be a sequence in $L^2(\Omega)$ such that $\chi_\varepsilon f_\varepsilon \rightharpoonup m^* f$ weakly in $L^2(\Omega)$ and let v_ε be the solution of (3.1.1) in,

$$V_\varepsilon = \{u \in H^1(\Omega_\varepsilon) | u = 0 \text{ on } \partial_{ext}\Omega_\varepsilon\}$$

equipped with the inner product, $< u, u >_{V_\varepsilon} \doteq \int_{\Omega_\varepsilon} |\nabla u|^2\, dx$. It can be shown, using the ellipticity, that $\mathrm{Sup}_\varepsilon \|v_\varepsilon\|_{V_\varepsilon} < \infty$. Further, a uniform Poincaré inequality,

$$|v_\varepsilon|_{0,\Omega_\varepsilon} \leq C\, \|v_\varepsilon\|_{V_\varepsilon}$$

holds for some C independent of ε (cf. Lemma A.4, Allaire and Murat [3]). Thus, $\widetilde{v_\varepsilon}$ and $\widetilde{\nabla v_\varepsilon}$ are bounded in $L^2(\Omega)$ and leads to the following theorem which is similar to Theorem 3.2.3.

Theorem 3.3.1 *[1] For some $v \in H_0^1(\Omega)$ and $v_1 \in L^2(\Omega, H^1_\#(Y)/\mathbb{R})$,*

$$\begin{aligned} \widetilde{v_\varepsilon} &\xrightarrow{2-s} \chi(y)v(x), \\ \widetilde{\nabla v_\varepsilon} &\xrightarrow{2-s} \chi(y)(\nabla_x v + \nabla_y v_1(x,y)) \end{aligned}$$

up to a subsequence, where $\chi(y)$ is the characteristic function of Y^*. ■

Remark 3.3.1 *Note that $\widetilde{v_\varepsilon} \rightharpoonup m^* v$ weakly in $L^2(\Omega)$.* ■

Further,

Theorem 3.3.2 *[1] The v, v_1 solve the two-scale homogenized problem,*

$$\left.\begin{aligned} -\mathrm{div}_y(A(x,y)(\nabla_x v + \nabla_y v_1(x,y))) &= 0 \text{ in } \Omega \times Y^*, \\ A(x,y)(\nabla_x v + \nabla_y v_1(x,y)).n_y &= 0 \text{ on } \Omega \times \partial Y^* \setminus \partial Y, \\ -\mathrm{div}_x \left(\int_Y \chi(y) A(x,y)(\nabla_x v + \nabla_y v_1(x,y))\, dy \right) &= m^* f \text{ in } \Omega. \blacksquare \end{aligned}\right\} \tag{3.3.1}$$

Remark 3.3.2 *In fact,* $v_1 \in L^2(\Omega, C^1_\#(Y)/\mathbb{R})$*, the extra regularity coming from the smoothness of the coefficients* $a_{ij}(x,y)$. ■

It is possible to decouple the equations (3.3.1) for v and v_1 by setting

$$v_1(x,y) = \frac{\partial v}{\partial x_i} X^i(x,y), \tag{3.3.2}$$

where, $X^i(x,y)$ solve the following periodic boundary value problem in Y^* :

$$\left.\begin{array}{rcll} -\mathrm{div}_y(A(x,y)(e_i + \nabla_y X^i(x,y))) &=& 0 & \text{in } Y^* \text{ a.e } x, \\ A(x,y)(e_i + \nabla_y X^i(x,y)).n_y &=& 0 & \text{on } \Omega \times \partial Y^* \setminus \partial Y, \\ \int_{Y^*} X^i(x,y)dy &=& 0 & \text{a.e. } x, \\ y &\mapsto& X^i(x,y) & \text{is Y-periodic.} \end{array}\right\} \tag{3.3.3}$$

Defining the matrix A^* by,

$$A^*_{ij}(x) = \int_Y \chi(y)(a_{ij}(x,y) + a_{ik}(x,y)\frac{\partial X^j}{\partial y_k}(x,y))\,dy\,, \tag{3.3.4}$$

it can be verified that v solves the homogenized problem,

$$\left.\begin{array}{rcll} -\mathrm{div}_x(A^*(x)\nabla v) &=& m^* f & \text{in } \Omega, \\ v &=& 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{3.3.5}$$

It is now to be shown that, there exists a B^*, independent of the sequence f_ε, such that the limit, $\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon\, dx$, can be written as $\int_\Omega B^* \nabla v . \nabla v\, dx$. For this, we use a corrector result which will be proved immediately after the following lemma.

Lemma 3.3.1 *Let* v_ε *be the solution of (3.1.1) and* (v, v_1) *be as in Theorems 3.3.1 and 3.3.2. Then* $\chi_\varepsilon\left(\nabla_x v(x) + \nabla_y v_1(x, \frac{x}{\varepsilon})\right)$ *is an admissible sequence.*

Proof: Note that by Remark 3.3.2, $\nabla_x v(x) + \nabla_y v_1(x,y) \in L^2(\Omega, C_\#(Y))$. Hence, by Remark 3.2.2, it is admissible and further,

$$\xi(x, \frac{x}{\varepsilon}) \doteq \nabla_x v(x) + \nabla_y v_1(x, \frac{x}{\varepsilon}) \xrightarrow{2-s} \nabla_x v(x) + \nabla_y v_1(x,y) \doteq \xi(x,y)\,. \tag{3.3.6}$$

Now, since $\chi \in L^\infty_\#(Y)$, by applying Theorem 3.2.4, we get,

$$\chi_\varepsilon \left(\nabla_x v(x) + \nabla_y v_1(x, \frac{x}{\varepsilon})\right) \xrightarrow{2-s} \chi(y) \left(\nabla_x v(x) + \nabla_y v_1(x, y)\right). \tag{3.3.7}$$

Since χ_ε is idempotent, to prove the theorem it is enough to show that

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \chi_\varepsilon |\nabla_x v(x) + \nabla_y v_1(x, \frac{x}{\varepsilon})|^2 \, dx = \int_\Omega \int_Y \chi(y) |\nabla_x v(x) + \nabla_y v_1(x, y)|^2 \, dy \, dx.$$

which readily follows from Theorem 3.2.2 using (3.3.7)and the admissibility of $\nabla_x v(x) + \nabla_y v_1(x, y)$. ■

The corrector result is as follows,

Theorem 3.3.3 *Let v_ε, v, v_1 and ξ be as in the previous lemma. Then,*

$$\lim_{\varepsilon \to 0} |\widetilde{\nabla v_\varepsilon} - \chi_\varepsilon \left(\nabla_x v(x) + \nabla_y v_1(x, \frac{x}{\varepsilon})\right) |_{0,\Omega} = 0. \tag{3.3.8}$$

Proof: Set $r_\varepsilon = \widetilde{\nabla v_\varepsilon} - \chi_\varepsilon \xi\left(x, \frac{x}{\varepsilon}\right)$, where ξ is as in (3.3.6). Then,

$$\begin{aligned}
&\alpha |r_\varepsilon|^2_{0,\Omega} \\
&\leq \int_{\Omega_\varepsilon} A_\varepsilon \left(\nabla v_\varepsilon - \xi\left(x, \frac{x}{\varepsilon}\right)\right) . \left(\nabla v_\varepsilon - \xi\left(x, \frac{x}{\varepsilon}\right)\right) \, dx \\
&\leq \int_{\Omega_\varepsilon} f_\varepsilon v_\varepsilon dx - \int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \widetilde{\nabla v_\varepsilon} . \xi\left(x, \frac{x}{\varepsilon}\right) \, dx \\
&\quad - \int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \xi\left(x, \frac{x}{\varepsilon}\right) . \widetilde{\nabla v_\varepsilon} \, dx + \int_\Omega \chi_\varepsilon A\left(x, \frac{x}{\varepsilon}\right) \xi\left(x, \frac{x}{\varepsilon}\right) . \xi\left(x, \frac{x}{\varepsilon}\right) \, dx. \quad (3.3.9)
\end{aligned}$$

where we have used the idempotency of $\chi_\varepsilon = \chi(\frac{x}{\varepsilon})$ and (3.1.1). Note that from Theorem 3.3.1, $\widetilde{\nabla v_\varepsilon} \xrightarrow{2-s} \chi(y)\xi(x, y)$; $A \in C(\overline{\Omega}, L^\infty_\#(Y))$ and; as already observed, $\xi\left(x, \frac{x}{\varepsilon}\right)$ is an admissible sequence. Using these as inputs in Corollary 3.2.1 we get the following convergences as $\varepsilon \to 0$,

$$\left.\begin{aligned}
&\int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \widetilde{\nabla v_\varepsilon} . \xi\left(x, \frac{x}{\varepsilon}\right) \, dx \\
&\int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \xi\left(x, \frac{x}{\varepsilon}\right) . \widetilde{\nabla v_\varepsilon} \, dx \\
&\int_\Omega \chi(\frac{x}{\varepsilon}) A\left(x, \frac{x}{\varepsilon}\right) \xi\left(x, \frac{x}{\varepsilon}\right) . \xi\left(x, \frac{x}{\varepsilon}\right) \, dx
\end{aligned}\right\} \longrightarrow \int_\Omega \int_Y \chi(y) A(x, y) \xi(x, y) . \xi(x, y) \, dy \, dx$$

Taking $\overline{\lim}_{\varepsilon\to 0}$ in (3.3.9) we get,

$$\overline{\lim}_{\varepsilon\to 0}\alpha|r_\varepsilon|^2_{0,\Omega} \leq \overline{\lim}_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f_\varepsilon v_\varepsilon\, dx - \int_\Omega\int_Y \chi(y)A(x,y)\xi(x,y).\xi(x,y)\, dy\, dx. \tag{3.3.10}$$

We note, using (3.3.1)-(3.3.5), that

$$\int_\Omega\int_Y \chi(y)A(x,y)\xi(x,y).\xi(x,y)\, dy\, dx = \int_\Omega m^* f v\, dx. \tag{3.3.11}$$

We still need to compute $\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f_\varepsilon v_\varepsilon$, but it was shown by Allaire, Murat and Nandakumar (cf. Lemma A.3 [3]) that, if

$$\begin{aligned} \text{Sup}_\varepsilon \|u_\varepsilon\|_{V_\varepsilon} &< \infty \\ \widetilde{v_\varepsilon} &\rightharpoonup m^* v \text{ weakly in } L^2(\Omega) \\ \widetilde{f_\varepsilon} &\rightharpoonup m^* f \text{ weakly in } L^2(\Omega), \end{aligned}$$

then,

$$\int_{\Omega_\varepsilon} f_\varepsilon v_\varepsilon dx \longrightarrow \int_\Omega m^* f v dx. \tag{3.3.12}$$

Therefore, it follows from (3.3.10), (3.3.11) and (3.3.12) that $|r_\varepsilon|_{0,\Omega} \longrightarrow 0$ as $\varepsilon \to 0$ completing the proof. ■

Remark 3.3.3 *It is easy to see that the sum of an admissible sequence with something which converges strongly to zero in $L^2(\Omega)$ is admissible. Therefore, it follows from Lemma 3.3.1 and Theorem 3.3.3 that $\widetilde{\nabla v_\varepsilon}$ is also an admissible sequence.* ■

We, finally, prove the convergence of energies to a suitable energy.

Theorem 3.3.4 *Let v_ε, v be as before solving (3.1.1) and (3.3.5) respectively. Then*

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B_\varepsilon \nabla v_\varepsilon.\nabla v_\varepsilon\, dx = \int_\Omega B^*\nabla v(x).\nabla v(x)\, dx \tag{3.3.13}$$

where,

$$B^*_{ij}(x) = \int_Y \chi(y)B(x,y)\nabla_y(y_i + X^i(x,y)).\nabla_y(y_j + X^j(x,y))\, dy. \tag{3.3.14}$$

Proof: As a consequence of Theorem 3.3.3 we may write

$$\int_{\Omega_\varepsilon} B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx = \int_\Omega B_\varepsilon \widetilde{\nabla} v_\varepsilon . \widetilde{\nabla} v_\varepsilon dx$$

$$= \int_\Omega \chi_\varepsilon B_\varepsilon (\nabla_x v(x) + \nabla_y v_1(x, \frac{x}{\varepsilon})).(\nabla_x v(x) + \nabla_y v_1(x, \frac{x}{\varepsilon}))\, dx + o(1).$$

Then arguing as in Theorem 3.3.3 it can be shown that,

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} B_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx$$

$$= \int_\Omega \int_{Y^*} B(x,y)(\nabla_x v(x) + \nabla_y v_1(x,y)).(\nabla_x v(x) + \nabla_y v_1(x,y))\, dx\, dy. \quad (3.3.15)$$

Then, (3.3.13) follows from (3.3.15) and (3.3.14) due to the relation $v_1(x,y) = \frac{\partial v}{\partial x_i} X^i(x,y)$. ■

Remark 3.3.4 *In the case where there are no perforations, by taking $\chi(y) \equiv 1$ in (3.3.14) we recover the formula of Kesavan and Vanninathan [27].* ■

From Theorem 3.3.4, it follows by taking

$$F^1(\theta) = \frac{1}{2} \int_\Omega B^* \nabla v . \nabla v \, dx$$

where $v = v(\theta)$ solves

$$\left.\begin{aligned} -\text{div}(A^* \nabla v) &= m^* f + \theta \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\}$$

that F^1 verifies (2.1.19) of Lemma 2.1.1. For this, it suffices to take f_ε to be equal to $f + (\chi_\varepsilon / m^*)\theta$, for some $\theta \in U_{ad}$ in Theorem 3.3.4. Thus, by Lemma 2.1.1, the homogenized problem is the following; minimize the cost functional,

$$(\text{P}^*) \qquad J(\theta) = \frac{1}{2} \int_\Omega B^* \nabla u . \nabla u dx + \frac{N}{2\, m^*} \int_\Omega \theta^2 \, dx\,,$$

over $\theta \in U_{ad}$ and where $u = u(\theta)$ solves,

$$\left.\begin{aligned} -\text{div}(A^* \nabla u) &= m^* f + \theta \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\}$$

where A^* and B^* are given by (3.3.4) and (3.3.14) respectively.

3.4 The Case of Multiple Scales

We now consider optimal control problems on multi-scale periodically perforated domains and whose coefficients have oscillations on several microscopic scales.

Let $a_1(\varepsilon), a_2(\varepsilon), ..., a_m(\varepsilon)$ be m microscopic scales and assumed to be well-separated i.e. there exists $r > 0$ such that for $i \in 1, 2, ..., m$,

$$\lim_{\varepsilon \to 0} \frac{1}{a_{i-1}(\varepsilon)} \left[\frac{a_i(\varepsilon)}{a_{i-1}(\varepsilon)}\right]^r = 0. \tag{3.4.1}$$

and the macroscopic scale $a_0(\varepsilon) \equiv 1$.

Example: $a_i(\varepsilon) = \varepsilon^{k_i}$, where $0 < k_1 < k_2 < ... < k_m$. ■

For any function $\phi(x, y_1, .., y_m)$, which is Y-periodic in y_i for all i, the scaled function $\phi\left(x, \frac{x}{a_1(\varepsilon)}, ... \frac{x}{a_m(\varepsilon)}\right)$ is denoted by $[\phi]_\varepsilon$.

Let $A \in M(a, b, \Omega \times Y^m)$ and $B \in M(c, d, \Omega \times Y^m)$ with B symmetric. Assume, further, that $A, B \in L^\infty(\Omega, C_\#(Y^m))$. We consider the optimal control problem having coefficients $[A]_\varepsilon, [B]_\varepsilon$ on a multi-scale periodically perforated domain, Ω_ε, which is defined as follows.

Let T_i, $i = 1, 2, ...m$, be closed subsets of the unit cell, Y and having smooth boundary. Set,

$$T^\varepsilon = \bigcup_{\mathbf{k} \in \mathbb{Z}^n} \bigcup_{i=1}^{m} a_i(\varepsilon)(\mathbf{k} + T_i)$$

which is the region occupied by the holes. Then, $\Omega_\varepsilon \doteq \Omega \setminus T^\varepsilon$ and is assumed to be connected. The *interior boundary* of Ω_ε comprises of the boundary of holes strictly contained in Ω and is denoted by $\partial_{int}\Omega_\varepsilon$. The *exterior boundary* is the set, $\partial_{ext}\Omega_\varepsilon = \partial\Omega_\varepsilon \setminus \partial_{int}\Omega_\varepsilon$.

Following Lemma 2.1.1, it is enough to homogenize the state equation and to find the limit of associated energies. We accomplish this using the multi-scale convergence method introduced by Allaire and Briane [2]. To begin with we recall the notion of multi-scale convergence and a few results from their paper.

Definition 3.4.1 *A sequence $u_\varepsilon \in L^2(\Omega)$ is said to $(m+1)$ -scale converge to a function $u \in L^2(\Omega \times Y^m)$ if*

$$\int_\Omega u_\varepsilon [\phi]_\varepsilon \, dx \longrightarrow \int_\Omega \int_Y \cdots \int_Y (u\phi)(x, y_1, .., y_m) dy_m..dy_1 dx \tag{3.4.2}$$

for all $\phi \in L^2(\Omega, C_\#(Y^m))$. We write $u_\varepsilon \overset{(m+1)-s}{\longrightarrow} u(x, y_1, .., y_m)$. ∎

The definition makes sense because of the following compactness theorem.

Theorem 3.4.1 *(cf. Theorem 1.4 [2]) From each bounded sequence in $L^2(\Omega)$ one can extract a subsequence which $(m+1)-$scale converges to a limit in $L^2(\Omega \times Y^m)$.* ∎

The proof of the theorem uses the fact(cf. Donato [19]) that if ϕ is any function in $L^2(\Omega, C_\#(Y^m))$ then

$$\lim_{\varepsilon \to 0} \int_\Omega [\phi]_\varepsilon^2 \, dx = \int_\Omega \int_Y \cdots \int_Y \phi^2(x, y_1, .., y_m) \, dy_m..dy_1 dx. \tag{3.4.3}$$

Also one knows(cf. Allaire and Briane [2]) that $[\phi]_\varepsilon$ (m+1)-scale converges to $\phi(x, y_1, .., y_m)$. Such functions are said to be *admissible functions*. One can introduce, like in the two-scale case, the notion of an admissible sequence and obtain the following version of Theorem 3.2.2 for the product of two (m+1)-scale convergent sequences at least one of which is an admissible sequence (cf. Theorem 1.5 [2]).

Theorem 3.4.2 *Let u_ε be a sequence of functions in $L^2(\Omega)$ which (m+1)-scale converges to $u(x, y_1, .., y_m)$ and satisfies*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon^2 \, dx = \int_\Omega \int_Y \cdots \int_Y u^2(x, y_1, .., y_m) \, dy_m..dy_1 dx.$$

Then for any sequence v_ε which (m+1)-scale converges to $v(x, y_1, .., y_m)$, one has

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon v_\varepsilon \, dx = \int_\Omega \int_Y \cdots \int_Y u(x, y_1, .., y_m) v(x, y_1, .., y_m) \, dy_m..dy_1 dx.$$ ∎

Remark 3.4.1 *A sequence like u_ε in the above theorem is said to be an admissible sequence. Clearly a sequence of functions which converges strongly in L^2 is an admissible sequence. Also by the above theorem, the sum of two (m+1)-scale convergent sequences which are admissible is also an admissible sequence.* ∎

One proves a version of Theorem 3.2.4 as follows.

Theorem 3.4.3 *Let u_ε be a bounded sequence in $L^2(\Omega)$ such that $u_\varepsilon \overset{(m+1)-s}{\longrightarrow} u(x, y_1, .., y_m)$. Let $\phi \in L^\infty(\Omega, C_\#(Y^m))$. Then $u_\varepsilon[\phi]_\varepsilon \overset{(m+1)-s}{\longrightarrow} (u\phi)(x, y_1, .., y_m)$.*

Proof: Let ψ be any function in $L^2(\Omega, C_\#(Y^m))$. Then $\phi\psi$ also belongs to this space. Hence, by the definition of (m+1)-scale convergence,

$$\begin{aligned}
\int_\Omega u_\varepsilon[\phi]_\varepsilon[\psi]_\varepsilon\, dx &= \int_\Omega u_\varepsilon[\phi\psi]_\varepsilon\, dx \\
&\longrightarrow \int_\Omega\int_Y ... \int_Y u(x, y_1, .., y_m)\ (\phi\psi)(x, y_1, .., y_m)\, dy_m..dy_1 dx \\
&= \int_\Omega\int_Y ... \int_Y (u\phi)(x, y_1, .., y_m)\ \psi(x, y_1, .., y_m)\, dy_m..dy_1 dx.
\end{aligned}$$

This completes the proof. ■

3.5 Homogenization-Multiple Scales

To obtain the homogenized coefficients, A^* and B^*, in the limiting optimal control problem we follow the same steps as in the two-scale case. Let f_ε be a sequence in $L^2(\Omega)$ such that $\chi_\varepsilon f_\varepsilon \rightharpoonup m^* f$, where m^* is the material part of Y. Let u_ε be the weak solution of the following elliptic boundary value problem,

$$\begin{aligned}
-\mathrm{div}([A]_\varepsilon \nabla u_\varepsilon) &= f_\varepsilon \text{ in } \Omega_\varepsilon, \\
[A]_\varepsilon \nabla u_\varepsilon . n_\varepsilon &= 0 \text{ on } \partial_{int}\Omega_\varepsilon, \\
u &= 0 \text{ on } \partial_{ext}\Omega_\varepsilon.
\end{aligned}$$

It is required to homogenize these equations and to find the limit of the sequence $\int_{\Omega_\varepsilon}[B]_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon\, dx$. The homogenization of the equations (3.5.1) was done by Allaire and Briane [2] and it is summarized in the following theorem (cf. Theorem 3.4 [2]).

Theorem 3.5.1 *Denote by $\tilde{}$ the extension by zero in the holes $\Omega \setminus \Omega_\varepsilon$. Then*

$$\begin{aligned}
\widetilde{u_\varepsilon} &\overset{(m+1)-s}{\longrightarrow} u(x)\chi(y_1, ..., y_m), \\
\widetilde{\nabla u_\varepsilon} &\overset{(m+1)-s}{\longrightarrow} \left(\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k\right) \chi(y_1, ..., y_m)
\end{aligned} \tag{3.5.1}$$

where $\chi(y_1, ..., y_m) \doteq \prod_{i=1}^m \chi_i(y_i)$ *and* χ_i *is the characteristic function of the set* Y_i^*. *Also,* $(u, u_1, ..., u_m)$ *is the unique solution in*

$$V = H_o^1(\Omega) \times \prod_{i=1}^{m} (L^2(\Omega \times Y^{i-1}, H^1_\#(Y^*))$$

of the (m+1)-scale homogenized problem:

$$\left.\begin{array}{rcl}
-\mathrm{div}_{y_m}(A(\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k)) & = & 0 \ \mathit{in}\ Y_m^*, \\
A(\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k).n & = & 0 \ \mathit{on}\ \partial T_n, \\
\int_Y \chi_m(y_m) u_m \, dy_m & = & 0, \\
-\mathrm{div}_{y_j} \int_Y .. \int_Y \prod_{j+1}^m \chi_k(y_k) A(\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k) dy_m .. dy_{j+1} & = & 0 \ \mathit{in}\ Y_j^*, \\
\int_Y .. \int_Y \prod_{j+1}^m \chi_k(y_k) A(\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k) dy_m .. dy_{j+1}).n & = & 0 \ \mathit{on}\ \partial T_j, \\
\int_{Y_j} \chi_j(y_j) u_j dy_j & = & 0 \\
\mathit{for}\ j=1,\ 2,\ ...,\ m\text{-}1\ \mathit{and\ finally,} & & \\
-\mathrm{div}_x \int_Y .. \int_Y \prod_1^m \chi_k(y_k) A(\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k) dy_m .. dy_1 & = & m^* f \ \mathit{in}\ \Omega, \\
u & = & 0 \ \mathit{on}\ \partial\Omega.
\end{array}\right\} \tag{3.5.2}$$

m^*, *here, is* $\int_Y .. \int_Y \prod_1^m \chi_k(y_k) dy_m .. dy_1$. ■

Remark 3.5.1 *It is possible to decouple the (m+1)-scale homogenized problem by setting*

$$u_j(x, y_1, ..., y_m) = \frac{\partial u}{\partial x_l} + \sum_{k=1}^{j-1} \frac{\partial u_k}{\partial y_{k_l}} w^{j,l}(x, y_1, ..., y_m) \tag{3.5.3}$$

for $j = m, m-1, ..., 1$ *successively. For all* $l \in \{1, 2, ..., n\}$, *the* $w_{j,l}$ *are obtained successively for* $j = m, m-1, ..., 1$ *by solving the periodic boundary value problems in the cells with holes,* Y_j^*,

$$\left.\begin{array}{rcl}
-\mathrm{div}_{y_j}(A^j(e_l + \nabla w^{j,l}) & = & 0 \quad \mathit{in}\ Y_j^*, \\
A^j(e_l + \nabla w^{j,l}).n & = & 0 \quad \mathit{on}\ \partial T_j, \\
\int_Y \chi_j(y_j) w^{j,l}(x, y_1, ..., y_j) \, dy_j & = & 0.
\end{array}\right\} \tag{3.5.4}$$

for $w^{j,l} \in L^2(\Omega \times Y^{j-1}, H^1_\#(Y_j^*))$ *and the* A^j*'s are obtained successively in conjunction with (3.5.4) as follows,*

$$\begin{aligned} A^m &= A \\ A^{j-1}e_l &= \int_{Y_j} \chi_j(y_j) A^j (e_l + \nabla_{y_j} w^{j,l}) dy_j \text{ for all } l. \end{aligned} \quad (3.5.5)$$

The homogenized problem that u *solves is,*

$$\begin{aligned} -\text{div}(A^* \nabla u) &= m^* f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

where $A^* \doteq A^0$ *given by (3.5.5) for* $j = 1$. ■

We now wish to compute the limit of the energies $\int_{\Omega_\varepsilon} [B]_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx$. Note that this can be written as $\int_\Omega [B]_\varepsilon \widetilde{\nabla u_\varepsilon} . \widetilde{\nabla u_\varepsilon} \, dx$. From Theorem 3.4.3 and Theorem 3.4.2 it follows that

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_\Omega [B]_\varepsilon \widetilde{\nabla u_\varepsilon} . \widetilde{\nabla u_\varepsilon} \, dx \\ &= \int_\Omega \int_Y \cdots \int_Y B \, (\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k) . (\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k) dy_m .. dy_1 dx \quad (3.5.6) \end{aligned}$$

provided we can show that $\widetilde{\nabla u_\varepsilon}$ is an admissible sequence. To prove this we require the solutions u_k of the cell equations to be more regular, i.e. they belong to $L^2(\Omega, C^1_\#(Y^k))$ and we can assume this provided the coefficient matrix A is sufficiently smooth (cf. Allaire and Briane [2]). Under these assumptions we show that $\widetilde{\nabla u_\varepsilon}$ is an admissible sequence.

Lemma 3.5.1 $\chi_\varepsilon(\nabla_x u(x) + \sum_{k=1}^m [\nabla_{y_k} u_k]_\varepsilon)$ *is an admissible sequence.*

Proof: The regularity assumptions on u_k's imply that $\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k$ is an admissible function. We also note that

$$\chi_\varepsilon(\nabla_x u + \sum_{k=1}^m [\nabla_{y_k} u_k]_\varepsilon) \overset{(m+1)-s}{\longrightarrow} \chi(x, y_1, ..., y_m)(\nabla_x u + \sum_{k=1}^m \nabla_{y_k} u_k).$$

Therefore, by Theorem 3.4.2

$$\int_\Omega (\chi_\varepsilon(\nabla_x u(x) + \sum_{k=1}^{m}[\nabla_{y_k} u_k]_\varepsilon))^2\,dx$$
$$= \int_\Omega \chi_\varepsilon(\nabla_x u(x) + \sum_{k=1}^{m}[\nabla_{y_k} u_k]_\varepsilon).(\nabla_x u(x) + \sum_{k=1}^{m}[\nabla_{y_k} u_k]_\varepsilon)\,dx$$
$$\longrightarrow \int_\Omega\int_Y \cdots \int_Y \chi(x, y_1, .., y_m)(\nabla_x u + \sum_{k=1}^{m}\nabla_{y_k} u_k).(\nabla_x u + \sum_{k=1}^{m}\nabla_{y_k} u_k)\,dy_m..dy_1 dx.$$

This completes the proof. ■

Under the same regularity assumptions one can prove, as in the case of two-scales (cf. Theorem 3.3.3), the following corrector result;

$$\widetilde{\nabla_x u_\varepsilon} - \chi_\varepsilon(\nabla_x u(x) + \sum_{k=1}^{m}[\nabla_{y_k} u_k]_\varepsilon) \text{ converges strongly in } L^2(\Omega) \text{ to } 0.$$

From Lemma 3.5.1, the corrector result and Remark 3.4.1 it follows that $\widetilde{\nabla u_\varepsilon}$ is an admissible sequence. This justifies (3.5.6). The left hand side of (3.5.6) may be now be written as $\int_\Omega B^* \nabla u.\nabla u\,dx$ using (3.5.3) and the following iterative formulae,

$$B^m = B,$$
$$B^{k-1}e_i.e_j = \int_Y \chi_k(y_k)B^k(e_i + \nabla_{y_k} w^{k,i}).(e_j + \nabla_{y_k} w^{k,j})dy_k.$$

for $k = m, m-1, ..., 1$ and we set $B^* \doteq B^0$ obtained by this process.

Thus, the homogenized optimal control problem, $(\mathbf{P}^*)$, is as given at the end of the Section 3.3 with m^*, A^* and B^* obtained in this section.

Chapter 4

Elliptic Systems

4.1 Introduction

In this chapter, we study the homogenization of an optimal control problem governed by elliptic systems on perforated domains. We need some notations in order to state the problem and we do this first by recording all the notations that will be used in this chapter.

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $0 < a < b$ be constants. We define $M_n^m(a, b, \Omega)$ to be the class of $nm \times nm$ block matrices, $\boldsymbol{C} = ((C_{ij}))\, i, j = 1, 2, ..., m$, where each block $C_{ij} \in L^\infty(\Omega)^{n\times n}$ and for a.e. $x \in \Omega$ we have,

$$a|\boldsymbol{\xi}|^2 \leq \boldsymbol{C}(x)\boldsymbol{\xi}.\boldsymbol{\xi} \text{ and } |\boldsymbol{C}(x)\boldsymbol{\xi}| \leq b|\boldsymbol{\xi}| \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^{nm}. \tag{4.1.1}$$

In the sequel, the Greek indices, α, β will take values in $\{1, 2, ..., n\}$ and the Latin indices, i, j will take values in $\{1, 2, ..., m\}$. Thus the $(\alpha, \beta)^{th}$ element of $(i, j)^{th}$ block in $\boldsymbol{C}$ will be denoted by $C_{ij}^{\alpha\beta}$. Let $\underline{u} = (u_1, u_2, ..., u_m)$, $\underline{v} = (v_1, v_2, ..., v_m)$ be $\mathbb{R}^m$ valued vector fields on Ω. Then, we set,

$$\begin{aligned} \boldsymbol{D}\underline{u} &\equiv (\nabla u_1, \nabla u_2, ..., \nabla u_m) \text{ and,} \\ \boldsymbol{D}\underline{u}.\boldsymbol{D}\underline{v} &\equiv \sum_{i=1}^{m} \nabla u_i.\nabla v_i. \end{aligned}$$

For an $\mathbb{R}^n$ valued vector field on Ω, $\boldsymbol{\eta} = (\eta^1, \eta^2, ..., \eta^n)$, we recall that,

$$\begin{aligned} \operatorname{div}\boldsymbol{\eta} &= \sum_{\alpha=1}^{n} \frac{\partial \eta^\alpha}{\partial x^\alpha}, \\ \operatorname{curl}\boldsymbol{\eta} &= \left(\left(\frac{\partial \eta^\alpha}{\partial x^\beta} - \frac{\partial \eta^\beta}{\partial x^\alpha} \right) \right). \end{aligned}$$

For an $(\mathbb{R}^n)^m$ valued vector field on Ω, $\boldsymbol{\zeta} = (\overline{\zeta}_1, \overline{\zeta}_2, ..., \overline{\zeta}_m)$, where each $\overline{\zeta}_i(x) \in R^n$, we define,

$$\begin{aligned} \operatorname{div}\boldsymbol{\zeta} &= (\operatorname{div}\overline{\zeta}_1, \operatorname{div}\overline{\zeta}_2, ..., \operatorname{div}\overline{\zeta}_m), \\ \operatorname{curl}\boldsymbol{\zeta} &= (\operatorname{curl}\overline{\zeta}_1, \operatorname{curl}\overline{\zeta}_2, ..., \operatorname{curl}\overline{\zeta}_m). \end{aligned}$$

The product $\boldsymbol{C\zeta}$ will be written in the block form $((\boldsymbol{C\zeta})_i)_{i=1}^m$ where $(\boldsymbol{C\zeta})_i = (\sum_{j=1}^m C_{ij}\overline{\zeta}_j)$, $i = 1, 2, ..., m$.

For $\varepsilon > 0$, the perforated domain Ω_ε is defined to be $\Omega \setminus S_\varepsilon$, where S_ε is a closed subset of Ω with smooth boundary. χ_ε is the characteristic function of Ω_ε. Let $\boldsymbol{A}_\varepsilon$ be a sequence in $M_n^m(a, b, \Omega)$ and let $\boldsymbol{B}_\varepsilon$ be a sequence of symmetric matrices in $M_n^m(c, d, \Omega)$ for some constants $0 < c < d$. Let $K \in M_1^m(a, b, \Omega)$ and let $N \in M_1^m(c, d, \Omega)$ be symmetric. The space of *admissible controls* is taken to be $U_{ad}^\varepsilon = L^2(\Omega_\varepsilon)^m$ or some analogue of those defined in (2.1.9). Let $\underline{f} \in L^2(\Omega)^m$ be a given function.

For each $\varepsilon > 0$, the optimal control problem consists of minimizing the cost functional

$$(\mathrm{P}_\varepsilon) \qquad J_\varepsilon(\underline{\theta}) = \frac{1}{2}\int_{\Omega_\varepsilon} \boldsymbol{B}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon . \boldsymbol{D}\underline{u}_\varepsilon \, dx + \frac{1}{2}\int_{\Omega_\varepsilon} N\underline{\theta}.\underline{\theta} \, dx \, ,$$

over $\underline{\theta} \in U_{ad}^\varepsilon$ and where the *state*, $\underline{u}_\varepsilon = \underline{u}_\varepsilon(\underline{\theta})$, is the solution of the *elliptic system*,

$$\left. \begin{aligned} -\operatorname{div}(\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon) + K\underline{u}_\varepsilon &= \underline{f} + \underline{\theta} \text{ in } \Omega_\varepsilon, \\ (\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon)_i . n_\varepsilon &= 0 \text{ on } \partial S_\varepsilon, \\ \underline{u}_\varepsilon &= 0 \text{ on } \partial\Omega. \end{aligned} \right\}$$

There exists a unique optimal control $\underline{\theta}_\varepsilon^*$ for this problem. It can be assumed that $\widetilde{\underline{\theta}_\varepsilon^*}$ converges weakly in $L^2(\Omega)^m$ to some $\underline{\theta}^*$ and it is required to identify the homogenized problem for which $\underline{\theta}^*$ is the minimizer. We once again work in the framework of Lemma 2.1.1 (or an appropriate modification) and set,

$$\begin{aligned} F_\varepsilon^1(\underline{\theta}) &= \frac{1}{2}\int_{\Omega_\varepsilon} \boldsymbol{B}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon . \boldsymbol{D}\underline{u}_\varepsilon \, dx \text{ and,} \\ F_\varepsilon^2(\underline{\theta}) &= \frac{1}{2}\int_{\Omega_\varepsilon} N\underline{\theta}.\underline{\theta}\, dx. \end{aligned}$$

As before, it is easy to identify U_{ad} as an analogue of those given in (2.1.10). The F^2 which verifies (2.1.20) and (P4) of the lemma is,

$$F^2(\underline{\theta}) = \frac{1}{2}\int_\Omega \frac{N\underline{\theta}.\underline{\theta}}{\chi}\, dx$$

where χ is the L^∞ weak* limit of the sequence χ_ε (χ^{-1} is assumed to be in $L^\infty(\Omega)$). This follows, as in the scalar case, from Lemma 4.1.1 below.

Lemma 4.1.1 *Let $\underline{\theta}^\varepsilon$ be a sequence in $L^2(\Omega_\varepsilon)m$ such that $\widetilde{\underline{\theta}^\varepsilon} \rightharpoonup \underline{\theta}$ weakly in $L^2(\Omega)^m$. Let N be the symmetric matrix mentioned above. Then*

$$\liminf_{\varepsilon\to 0} \int_\Omega N\widetilde{\underline{\theta}^\varepsilon}.\widetilde{\underline{\theta}^\varepsilon}\, dx \geq \int_\Omega \frac{N\underline{\theta}.\underline{\theta}}{\chi}\, dx. \tag{4.1.2}$$

Proof: By the hypotheses on χ, $(\chi_\varepsilon/\chi)\underline{\theta} \in L^2(\Omega)^m$. Since, N is a positive matrix the functional $\Phi(\underline{\phi}) = \int_\Omega N\underline{\phi}.\underline{\phi}\, dx$ is convex. Therefore,

$$\begin{aligned} \Phi(\widetilde{\underline{\theta}^\varepsilon}) - \Phi(\frac{\chi_\varepsilon\underline{\theta}}{\chi}) &\geq \int_\Omega N\frac{\chi_\varepsilon\underline{\theta}}{\chi}.(\widetilde{\underline{\theta}^\varepsilon} - \frac{\chi_\varepsilon\underline{\theta}}{\chi})\, dx \\ &= \int_\Omega N\frac{\underline{\theta}}{\chi}.(\widetilde{\underline{\theta}^\varepsilon} - \frac{\chi_\varepsilon\underline{\theta}}{\chi})\, dx. \end{aligned} \tag{4.1.3}$$

The right hand side in (4.1.3) tends to 0 as $\varepsilon \to 0$, since $\widetilde{\underline{\theta}^\varepsilon} \rightharpoonup \underline{\theta}$ weakly in $L^2(\Omega)^m$. Further,

$$\begin{aligned} \Phi(\frac{\chi_\varepsilon\underline{\theta}}{\chi}) &= \int_\Omega \chi_\varepsilon \frac{N\underline{\theta}.\underline{\theta}}{\chi^2}\, dx \\ &\longrightarrow \int_\Omega \frac{N\underline{\theta}.\underline{\theta}}{\chi}\, dx. \end{aligned}$$

From these one concludes (4.1.2). ■

It remains to identify a F^1 verifying (2.1.19). This shall be done in the next section by developing a notion of H-convergence for block matrices and then we can proceed along the lines of the scalar case to obtain the homogenized problem.

4.2 H_b-convergence

Continuing our discussion, we need to homogenize the equations,

$$\left.\begin{aligned} -\mathrm{div}(\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon) &= \underline{f}_\varepsilon \text{ in } \Omega_\varepsilon, \\ (\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon)_i.n_\varepsilon &= 0 \text{ on } \partial S_\varepsilon, \quad i = 1,2,...,m, \\ \underline{u}_\varepsilon &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{4.2.1}$$

for any sequence, $\underline{f}_\varepsilon \in L^2(\Omega)^m$, such that $\chi_\varepsilon \underline{f}_\varepsilon \rightharpoonup \underline{f}$ weakly in $L^2(\Omega)^m$. And, we need to obtain the limits of associated energies,

$$\int_{\Omega_\varepsilon} \boldsymbol{B}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon.\boldsymbol{D}\underline{u}_\varepsilon \, dx. \tag{4.2.2}$$

To the best of our knowledge, the homogenization of problem (4.2.1) has not been studied except in the periodic case and that too only in non perforated domains (cf. Besoussan, Lions and Papanicolaou [6]). In order to homogenize (4.2.1) we need to develop a notion of H_0-convergence for *block matrices* on perforated domains. This is done now and we shall call this H_b-convergence; the subscript, b stands for *block*. Certain additional hypotheses are required on the geometry of the domains, as in the case of H_0-convergence (cf. Briane, Damlamian and Donato [7]).

The holes S_ε are said to be *admissible* if,

(H.1) Whenever $\chi_\varepsilon \rightharpoonup \chi$ weakly* in $L^\infty(\Omega)$, we have $\chi > 0$ almost everywhere.

(H.2) Let $V_\varepsilon \equiv \{u \in H^1(\Omega_\varepsilon) | u = 0 \text{ on } \partial\Omega\}$ be equipped with the norm, $\|u\|_{V_\varepsilon} \equiv |\nabla u|_{0,\Omega_\varepsilon}$. There exists a sequence of extension operators $P_\varepsilon : V_\varepsilon \to H_0^1(\Omega)$ and a constant C_0 independent of ε such that,

$$|\nabla P_\varepsilon u|_{0,\Omega} \leq C_0 |\nabla u|_{0,\Omega_\varepsilon} \text{ for all } u \in V_\varepsilon.$$

Remark 4.2.1 *Examples of such holes are spherical holes of size ε periodically distributed in space with period 2ε (cf. [13]). Therefore, we have extension operators, $\wp_\varepsilon : V_\varepsilon^m \to H_0^1(\Omega)^m$ which are simply the extension operators P_ε applied to each component. Note that,*

$$|\boldsymbol{D}\wp_\varepsilon \underline{u}|_{0,\Omega} \leq C_0 |\boldsymbol{D}\underline{u}|_{0,\Omega_\varepsilon} \textit{ for all } \underline{u} \in (V_\varepsilon)^m. \blacksquare \tag{4.2.3}$$

Remark 4.2.2 *The operators $\wp_\varepsilon$ have the following interesting property which will be used in the next section. If $\underline{v} \in H_0^1(\Omega)^m$, then*

$$\wp_\varepsilon(\underline{v}_{|\Omega_\varepsilon}) \rightharpoonup \underline{v} \text{ weakly in } H_0^1(\Omega)^m \tag{4.2.4}$$

A proof of this can be found in the paper of Briane, Damlamian and Donato (cf. Lemma 2.1 [7]). ■

In what follows, $\wp_\varepsilon^* : H^{-1}(\Omega)^m \to (V_\varepsilon^*)^m$ will denote the adjoint operator of $\wp_\varepsilon$. For $\underline{f} \in H^{-1}(\Omega)^m$, let $\underline{u}_\varepsilon \in V_\varepsilon^m$ be the solution of the boundary value problem,

$$\left.\begin{aligned} -\mathrm{div}(\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon) &= \wp_\varepsilon^* \underline{f} \text{ in } \Omega_\varepsilon, \\ (\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon)_i.n_\varepsilon &= 0 \text{ on } \partial S_\varepsilon, \quad i = 1, 2, ..., m, \\ \underline{u}_\varepsilon &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{4.2.5}$$

where n_ε is the outward normal on ∂S_ε. We have the following definition:

Definition 4.2.1 *Let A_ε be a sequence in $M_n^m(a, b, \Omega)$ and $\{S_\varepsilon\}$ be an admissible family of holes. The pair $(\boldsymbol{A}_\varepsilon, S_\varepsilon)$ is said to H_b converge to a matrix $\boldsymbol{A} \in M_n^m(a', b', \Omega)$ for some constants $0 < a' < b'$ if the following holds:*
For any $\underline{f} \in H^{-1}(\Omega)^m$, the sequence of solutions $\underline{u}_\varepsilon$ of the boundary value problems (4.2.5), satisfies,

$$\left.\begin{aligned} \wp_\varepsilon \underline{u}_\varepsilon &\rightharpoonup \underline{u} \text{ weakly in } H_0^1(\Omega)^m, \\ Q_\varepsilon(\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon) &\rightharpoonup \boldsymbol{A}\boldsymbol{D}\underline{u} \text{ weakly in } L^2(\Omega)^{nm} \end{aligned}\right\} \tag{4.2.6}$$

as $\varepsilon \to 0$, where $\underline{u}$ is the solution in $H_0^1(\Omega)^m$ of

$$\left.\begin{aligned} -\mathrm{div}(\boldsymbol{A}\boldsymbol{D}\underline{u}) &= \underline{f} \text{ in } \Omega, \\ \underline{u} &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{4.2.7}$$

We write $(\boldsymbol{A}_\varepsilon, S_\varepsilon) \xrightarrow{H_b} \boldsymbol{A}$. ■

Remark 4.2.3 *For $m = 1$, this coincides with the H_0-convergence of Briane, Damlamian and Donato [7].* ■

Our main theorem is the following and this directly leads to the homogenization of the state equation.

Theorem 4.2.1 *(Compactness) Let A_ε be a sequence in $M_n^m(a,b,\Omega)$ and let S_ε be an admissible sequence of holes. Then there exists a subsequence indexed by ε' and $A \in M_n^m(a/C_0^2, b^2/a, \Omega)$ such that the pair $(A_{\varepsilon'}, S_{\varepsilon'}) \xrightarrow{H_b} A$.* ■

The next section is devoted to proving the theorem.

4.3 Proof of the Main Theorem

The following propositions are required in proving the Main Theorem.

Proposition 4.3.1 *Let $\underline{u}_\varepsilon$ be the solution of (4.2.5). Then the following estimates hold:*

$$\left.\begin{array}{rcl} \|\wp_\varepsilon \underline{u}_\varepsilon\|_{H_0^1(\Omega)^m} & \leq & c_1 \|\underline{f}\|_{H^{-1}(\Omega)^m}, \\ |Q_\varepsilon(A_\varepsilon D\underline{u}_\varepsilon)|_{0,\Omega} & \leq & c_2 \|\underline{f}\|_{H^{-1}(\Omega)^m} \end{array}\right\} \qquad (4.3.1)$$

where $c_1 = C_0^2/a$ and $c_2 = C_0 b/a$.)

Proof: Note that, by Poincaré's inequality, $|\nabla.|_{0,\Omega}$ is a norm equivalent to the original norm on $H_0^1(\Omega)$. So we assume that $H_0^1(\Omega)$ is equipped with this equivalent norm. Then,

$$\left.\begin{array}{rcl} \|\wp_\varepsilon \underline{u}_\varepsilon\|_{H_0^1(\Omega)^m} & \equiv & |D\wp_\varepsilon \underline{u}_\varepsilon|_{0,\Omega} \\ & \leq & C_0 |D\underline{u}_\varepsilon|_{0,\Omega_\varepsilon} \end{array}\right\} \qquad (4.3.2)$$

Now, using ellipticity and (4.3.2), we get,

$$\begin{array}{rcl} a|D\underline{u}_\varepsilon|_{0,\Omega_\varepsilon}^2 & \leq & \displaystyle\int_{\Omega_\varepsilon} A_\varepsilon D\underline{u}_\varepsilon . D\underline{u}_\varepsilon \, dx \\ & = & <\wp_\varepsilon^* \underline{f}, \underline{u}_\varepsilon> \\ & = & <\underline{f}, \wp_\varepsilon \underline{u}_\varepsilon> \\ & \leq & \|\underline{f}\|_{H^{-1}(\Omega)^m} \|\wp_\varepsilon \underline{u}_\varepsilon\|_{H_0^1(\Omega)^m} \\ & \leq & C_0 \|\underline{f}\|_{H^{-1}(\Omega)^m} |D\underline{u}_\varepsilon|_{0,\Omega_\varepsilon}. \end{array}$$

Therefore, we get,

$$|D\underline{u}_\varepsilon|_{0,\Omega_\varepsilon} \leq C_0 a^{-1} \|\underline{f}\|_{H^{-1}(\Omega)^m}.$$

From this and (4.3.2), the first estimate follows. By using the estimate for $|D\underline{u}_\varepsilon|_{0,\Omega_\varepsilon}$ we get,

$$\begin{aligned} |Q_\varepsilon(A_\varepsilon D\underline{u}_\varepsilon)|_{0,\Omega} &= |A_\varepsilon D\underline{u}_\varepsilon|_{0,\Omega_\varepsilon} \\ &\leq b|D\underline{u}_\varepsilon|_{0,\Omega_\varepsilon} \\ &\leq C_0 b a^{-1} \|\underline{f}\|_{H^{-1}(\Omega)^m}. \blacksquare \end{aligned}$$

The proofs of the following propositions are straightforward adaptations of those found in Briane, Damlamian and Donato [7] for the case $m = 1$.

Proposition 4.3.2 *Let $\xi_\varepsilon \in L^2(\Omega_\varepsilon)^{nm}$ be a sequence of vector fields such that the sequence, $Q_\varepsilon(\xi_\varepsilon)$, is bounded in $L^2(\Omega)^{nm}$ and satisfies*

$$\left.\begin{aligned} -\text{div}(\xi_\varepsilon) &= \wp_\varepsilon^* \underline{f}_\varepsilon \text{ in } \Omega_\varepsilon, \\ (\xi_\varepsilon)_i . n_\varepsilon &= 0 \text{ on } \partial S_\varepsilon \text{ for } i=1,2,\ldots,m, \end{aligned}\right\} \quad (4.3.3)$$

and the sequence, $\underline{f}_\varepsilon$, is in a compact subset of $H^{-1}(\Omega)^m$. Then, the sequence $\text{div}(Q_\varepsilon \xi_\varepsilon)$ *is in a compact subset of $H^{-1}(\Omega)^m$.* ■

Proposition 4.3.3 *(div-curl lemma) Let ζ_ε and ξ_ε be two sequences of vector fields in $L^2(\Omega)^{nm}$ such that* $\text{div}\zeta_\varepsilon$ *is in a compact set in $H^{-1}(\Omega)^m$ and* $\text{curl}\xi_\varepsilon$ *is in a compact set in $H^{-1}(\Omega)^{nm}$. Furthermore, suppose that $\zeta_\varepsilon \rightharpoonup \zeta$ and $\xi_\varepsilon \rightharpoonup \xi$ weakly in $L^2(\Omega)^{nm}$. Then,*

$$\zeta_\varepsilon . \xi_\varepsilon \longrightarrow \zeta . \xi \text{ in } D'(\Omega). \blacksquare$$

The following abstract theorem will be used in the proof of the main theorem (cf. Murat [29]).

Proposition 4.3.4 *Let V be a separable Banach space and let W be a reflexive Banach space. Let T_ε be a sequence of linear operators $T_\varepsilon : V \to W$ such that $|||T_\varepsilon|||_{L(V,W)} \leq d$ for a constant d. Then, there exists a subsequence indexed by ε' and a linear operator $T : V \to W$ such that $T_{\varepsilon'} v \rightharpoonup Tv$ weakly in W for all $v \in V$.* ■

We now prove our main theorem. Our arguments are along the lines of those of Murat([29], [30]) and Briane, Damlamian and Donato ([7]) for the cases of $H-$ and H_0- convergences respectively.

Proof of Theorem 1.1: It is done in several steps.

Step 1: Let $\Omega \subset\subset \Omega'$. Note that the sequence S_ε is admissible for the domain Ω' also. Just observe that the extension operators P_ε can be extended by zero on $\Omega' \setminus \Omega$ and (H.2) holds for the new extension with Ω replaced by Ω'. Extend $\boldsymbol{A}_\varepsilon$ by $a\boldsymbol{I}$ on $\Omega' \setminus \Omega$ and we denote the extension by $\boldsymbol{A}_\varepsilon$ again. Note that $\boldsymbol{A}_\varepsilon \in M_n^m(a, b, \Omega')$. Let $\Omega'_\varepsilon = \Omega' \setminus S_\varepsilon$ and,

$$Z_\varepsilon \equiv \{\underline{v} \in H^1(\Omega'_\varepsilon)^m : \underline{v} = 0 \text{ on } \partial\Omega'\}.$$

Define a sequence of operators $T_\varepsilon : H^{-1}(\Omega')^m \to H_0^1(\Omega')^m$ by $T_\varepsilon \underline{g} = \wp_\varepsilon \underline{v}_\varepsilon$ for $\underline{g} \in H^{-1}(\Omega')^m$, where $\underline{v}_\varepsilon$ is the solution in Z_ε of the equations

$$\left.\begin{array}{rcl} -\text{div}(\boldsymbol{A}_\varepsilon^t \boldsymbol{D}\underline{v}_\varepsilon) &=& \wp_\varepsilon^* \underline{g} \text{ in } \Omega'_\varepsilon, \\ (\boldsymbol{A}_\varepsilon^t \boldsymbol{D}\underline{v}_\varepsilon)_i . n_\varepsilon &=& 0 \text{ on } \partial S_\varepsilon, \quad i = 1, 2, ..., m, \\ \underline{v}_\varepsilon &=& 0 \text{ on } \partial\Omega'. \end{array}\right\} \tag{4.3.4}$$

By Proposition 4.3.1, we have the estimate,

$$||T_\varepsilon \underline{g}||_{H_0^1(\Omega')^m} \le a^{-1} C_0^2 \, ||\underline{g}||_{H^{-1}(\Omega')^m}.$$

Therefore, by Proposition 4.3.4, we can extract a subsequence indexed by ε' and find an operator $T : H^{-1}(\Omega')^m \to H_0^1(\Omega')^m$ such that

$$T_{\varepsilon'} \underline{g} \rightharpoonup T\underline{g} \text{ weakly in } H_0^1(\Omega')^m \tag{4.3.5}$$

for all $\underline{g} \in H^{-1}(\Omega')^m$.

Step 2: We show that T is coercive. Define subspaces W_ε of Z_ε by,

$$W_\varepsilon = \{\underline{v} \in Z_\varepsilon | (\boldsymbol{A}_\varepsilon^t \boldsymbol{D}\underline{v})_i . n_\varepsilon = 0 \text{ on } \partial S_\varepsilon, i = 1, 2, ..., m\}.$$

W_ε is a closed subspace of Z_ε. Define the operators $C_\varepsilon : W_\varepsilon \to Z_\varepsilon^*$ by,

$$C_\varepsilon \underline{v} \equiv -\text{div}(\boldsymbol{A}_\varepsilon^t \boldsymbol{D}\underline{v}) \text{ for } \underline{v} \in W_\varepsilon.$$

Then, for $\underline{u} \in W_\varepsilon, \underline{v} \in Z_\varepsilon$, we have,

$$\begin{aligned} | < C_\varepsilon \underline{u}, \underline{v} > | &= \int_{\Omega_\varepsilon} \boldsymbol{A}_\varepsilon^t \boldsymbol{D}\underline{u}.\boldsymbol{D}\underline{v}\, dx \\ &\leq |\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{v}|_{0,\Omega'_\varepsilon} |\boldsymbol{D}\underline{u}|_{0,\Omega'_\varepsilon} \\ &\leq b|\boldsymbol{D}\underline{u}|_{0,\Omega'_\varepsilon} |\boldsymbol{D}\underline{v}|_{0,\Omega'_\varepsilon} \\ &= b\,\|\underline{u}\|_{W_\varepsilon} \|\underline{v}\|_{Z_\varepsilon}. \end{aligned}$$

Therefore, we have the following bound for the operator norms of the sequence C_ε;

$$||C_\varepsilon||_{L(W_\varepsilon, Z_\varepsilon^*)} \leq b. \tag{4.3.6}$$

Let R_ε be the restriction operators, $R_\varepsilon \underline{u} \equiv \underline{u}_{|_{\Omega'_\varepsilon}}$ from $H_0^1(\Omega')^m$ to Z_ε. For any $\underline{u} \in H_0^1(\Omega')^m$,

$$\|R_\varepsilon \underline{u}\|_{Z_\varepsilon} = |\boldsymbol{D}\underline{u}|_{0,\Omega'_\varepsilon} \leq |\boldsymbol{D}\underline{u}|_{0,\Omega'}.$$

So,

$$||R_\varepsilon||_{L(H_0^1(\Omega')^m, Z_\varepsilon)} \leq 1. \tag{4.3.7}$$

Let $\underline{g} \in H^{-1}(\Omega')^m$ and let $\underline{v}_\varepsilon$ be the solution of (4.3.4). Then,

$$\begin{aligned} (C_\varepsilon \circ R_\varepsilon \circ T_\varepsilon)\underline{g} &= (C_\varepsilon \circ R_\varepsilon)(\wp_\varepsilon \underline{v}_\varepsilon) \\ &= C_\varepsilon(\underline{v}_\varepsilon) \\ &= \wp_\varepsilon^* \underline{g}. \end{aligned}$$

Therefore, using the estimates (4.3.6)-(4.3.7),

$$\left.\begin{aligned} \left\|\wp_\varepsilon^* \underline{g}\right\|_{Z_\varepsilon^*} &= \left\|(C_\varepsilon \circ R_\varepsilon \circ T_\varepsilon)\underline{g}\right\|_{Z_\varepsilon^*} \\ &\leq b\left\|T_\varepsilon \underline{g}\right\|_{H_0^1(\Omega')^m}. \end{aligned}\right\} \tag{4.3.8}$$

Let $\underline{w}$ be any function in $H_0^1(\Omega')^m$ be such that $\|\underline{w}\|_{H_0^1(\Omega')^m} \leq 1$. Then, by (4.3.7),

$$\left\|\underline{w}_{|_{\Omega'_\varepsilon}}\right\|_{Z_\varepsilon} \leq \|\underline{w}\|_{H_0^1(\Omega')^m} \leq 1.$$

Therefore,

$$\begin{aligned} \left\|\wp_\varepsilon^* \underline{g}\right\|_{Z_\varepsilon^*} &\geq | < \wp_\varepsilon^* \underline{g}, \underline{w}_{|_{\Omega'_\varepsilon}} > | \\ &= | < \underline{g}, \wp_\varepsilon(\underline{w}_{|_{\Omega'_\varepsilon}}) > |. \end{aligned}$$

Letting $\varepsilon \to 0$ and using (4.2.4), we get,

$$\underline{\lim}_{\varepsilon\to 0} \left\|\wp_\varepsilon^* \underline{g}\right\|_{Z_\varepsilon^*} \geq | < \underline{g}, \underline{w} > |.$$

As this holds for every $\underline{w} \in H_0^1(\Omega')^m$ whose norm is less than 1, we conclude that

$$\underline{\lim}_{\varepsilon\to 0} \left\|\wp_\varepsilon^* \underline{g}\right\|_{Z_\varepsilon^*} \geq \left\|\underline{g}\right\|_{H^{-1}(\Omega')^m}. \tag{4.3.9}$$

Now,

$$\begin{aligned}
< T_\varepsilon \underline{g}, \underline{g} >_{H_0^1(\Omega')^m, H^{-1}(\Omega')^m} &= < \wp_\varepsilon \underline{v}_\varepsilon, \underline{g} >_{H_0^1(\Omega')^m, H^{-1}(\Omega')^m} \\
&= < \underline{v}_\varepsilon, \wp_\varepsilon^* \underline{g} >_{Z_\varepsilon, Z_\varepsilon^*} \\
&= \int_{\Omega'_\varepsilon} \boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{v}_\varepsilon . \boldsymbol{D}\underline{v}_\varepsilon \, dx \\
&\geq a|\boldsymbol{D}\underline{v}_\varepsilon|^2_{0,\Omega'_\varepsilon} \\
&\geq aC_0^{-2}|\boldsymbol{D}\wp_\varepsilon \underline{v}_\varepsilon|^2_{0,\Omega'} \\
&= aC_0^{-2}\left\|T_\varepsilon \underline{g}\right\|^2_{H_0^1(\Omega')^m}.
\end{aligned}$$

Therefore, passing to the limit for the subsequence indexed by ε' and using (4.3.8)-(4.3.9), we get,

$$< T\underline{g}, \underline{g} > \geq aC_0^{-2}b^{-2}\left\|\underline{g}\right\|^2_{H^{-1}(\Omega')^m}. \tag{4.3.10}$$

Thus, the operator T has been shown to be coercive.

Step 3: <u>Construction of test functions and computation of H_b limit.</u>

Since T is coercive, by the Lax-Milgram theorem,

$$T^{-1} : H_0^1(\Omega')^m \to H^{-1}(\Omega')^m$$

exists. Define vector fields $\underline{P}_i^\alpha : \Omega' \to R^m$ for $i = 1, 2, ..., m$ and $\alpha = 1, 2, ..., n$ by, $\underline{P}_i^\alpha(x)$ to be the vector in R^m whose only non-vanishing component is the i th component and is x^α. Then note that, $\boldsymbol{e}_i^\alpha \doteq \boldsymbol{D}\underline{P}_i^\alpha = (\overline{0}, ..., e^\alpha, ..., \overline{0})$ where e_α is the αth standard basis vector in R^n.

Let $\phi \in D(\Omega')$ be such that $\phi \equiv 1$ on Ω. Define the sequence of test functions,

$$\underline{\zeta}_i^{\alpha,\varepsilon} \equiv T_\varepsilon \circ T^{-1}(\underline{P}_i^\alpha \phi). \tag{4.3.11}$$

Note that, by (4.3.5),

$$\underline{\zeta}_i^{\alpha,\varepsilon'} \rightharpoonup T \circ T^{-1}(\underline{P}_i^\alpha \phi) = \underline{P}_i^\alpha \phi \text{ weakly in } H_0^1(\Omega')^m.$$

Therefore, restricting to Ω,

$$\underline{\zeta}_i^{\alpha,\varepsilon'} \rightharpoonup \underline{P}_i^\alpha \text{ weakly in } H^1(\Omega)^m. \tag{4.3.12}$$

Since $Q_{\varepsilon'}(\boldsymbol{A}_{\varepsilon'}^t \boldsymbol{D}\underline{\zeta}_i^{\alpha,\varepsilon'})$ is bounded in $L^2(\Omega)^{nm}$, there exists functions $\boldsymbol{\eta}_i^\alpha$ and a further subsequence ε'' such that,

$$Q_{\varepsilon''}(\boldsymbol{A}_{\varepsilon''}^t \boldsymbol{D}\underline{\zeta}_i^{\alpha,\varepsilon''}) \rightharpoonup \boldsymbol{\eta}_i^\alpha \text{ weakly in } L^2(\Omega)^{nm} \tag{4.3.13}$$

for all i and α.

By definition of $\underline{\zeta}_i^{\alpha,\varepsilon}$ and from Proposition 4.3.2, we conclude that $\text{div}Q_\varepsilon(\boldsymbol{A}_\varepsilon^t \boldsymbol{D}\underline{\zeta}_i^{\alpha,\varepsilon})$ is in a compact set in $H^{-1}(\Omega')^m$. Consequently, when restricted to Ω,

$$-\text{div}(Q_\varepsilon \boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{\zeta}_i^{\alpha,\varepsilon}) \text{ is in a compact set in } H^{-1}(\Omega)^m. \tag{4.3.14}$$

Define $\boldsymbol{A}$ through the relation,

$$\boldsymbol{A}^t \boldsymbol{e}_i^\alpha \equiv \boldsymbol{\eta}_i^\alpha. \tag{4.3.15}$$

Step 4: We shall now show that $(\boldsymbol{A}_{\varepsilon''}, S_{\varepsilon''}) \xrightarrow{H_b} \boldsymbol{A}$. For convenience we denote ε'' by ε.

Define the operators, $G_\varepsilon : H^{-1}(\Omega)^m \to H_0^1(\Omega)^m$ and $H_\varepsilon : H^{-1}(\Omega)^m \to L^2(\Omega)^{nm}$ by, $G_\varepsilon \underline{f} = \wp_\varepsilon \underline{u}_\varepsilon$ and $H_\varepsilon = Q_\varepsilon(\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon)$ where $\underline{u}_\varepsilon$ is the solution of

$$\begin{aligned} -\text{div}(\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon) &= \wp_\varepsilon^* \underline{f} \text{ in } \Omega_\varepsilon, \\ (\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon)_i . n_\varepsilon &= 0 \text{ on } \partial S_\varepsilon \text{ for } i =, 1, 2, ..., m, \\ \underline{u}_\varepsilon &= 0 \text{ on } \partial\Omega. \end{aligned}$$

By Proposition 4.3.1, the operator sequences G_ε and H_ε are bounded and hence, by Proposition 4.3.4, there exists a subsequence ε' of ε and operators G and H such that,

$$\begin{aligned} G_{\varepsilon'}\underline{f} &\rightharpoonup G\underline{f} \text{ weakly in } H_0^1(\Omega)^m, \\ H_{\varepsilon'}\underline{f} &\rightharpoonup H\underline{f} \text{ weakly in } L^2(\Omega)^{nm}, \end{aligned}$$

for all $\underline{f} \in H^{-1}(\Omega)^m$. Set $\underline{u} = G\underline{f}$ and $\boldsymbol{\xi} = H\underline{f}$. Restating the above, we get,

$$\left.\begin{aligned} \wp_{\varepsilon'}\underline{u}_{\varepsilon'} &\rightharpoonup \underline{u} \text{ weakly in } H_0^1(\Omega)^m, \\ Q_{\varepsilon'}(\boldsymbol{A}_{\varepsilon'}\boldsymbol{D}\underline{u}_{\varepsilon'}) &\rightharpoonup \boldsymbol{\xi} \text{ weakly in } L^2(\Omega)^m. \end{aligned}\right\} \tag{4.3.16}$$

We now show that $\boldsymbol{\xi} = \boldsymbol{AD}\underline{u}$ where $\boldsymbol{A}$ was defined in Step 3. Let $\psi \in D(\Omega)$.

$$\begin{aligned} \int_\Omega Q_{\varepsilon'}(\boldsymbol{A}_{\varepsilon'}\boldsymbol{D}\underline{u}_{\varepsilon'}).\boldsymbol{D}\underline{\zeta}_i^{\alpha,\varepsilon'}\psi\,dx &= \int_{\Omega_\varepsilon} \boldsymbol{A}_{\varepsilon'}\boldsymbol{D}\underline{u}_{\varepsilon'}.\boldsymbol{D}\underline{\zeta}_i^{\alpha,\varepsilon'}\psi\,dx \\ &= \int_{\Omega_\varepsilon} \boldsymbol{D}\underline{u}_{\varepsilon'}.\boldsymbol{A}^t_{\varepsilon'}\boldsymbol{D}\underline{\zeta}_i^{\alpha,\varepsilon'}\psi\,dx \\ &= \int_\Omega \boldsymbol{D}\wp_{\varepsilon'}\underline{u}_{\varepsilon'}.Q_{\varepsilon'}(\boldsymbol{A}^t_{\varepsilon'}\boldsymbol{D}\underline{\zeta}_i^{\alpha,\varepsilon'})\psi\,dx \end{aligned}$$

By applying Proposition 4.3.2, we conclude that $-\text{div}(Q_\varepsilon(\boldsymbol{A}_\varepsilon\boldsymbol{D}\underline{u}_\varepsilon))$ is in a compact set in $H^{-1}(\Omega)^m$. Therefore, by this and (4.3.14) and applying div-curl lemma we can pass to the limit as $\varepsilon \to 0$ in the above, and we get,

$$\int_\Omega \boldsymbol{\xi}.\boldsymbol{e}_i^\alpha\psi\,dx = \int_\Omega \boldsymbol{D}\underline{u}.\boldsymbol{A}^t\boldsymbol{e}_i^\alpha\psi\,dx.$$

As this holds for all $\psi \in D(\Omega)$ we conclude that $\boldsymbol{\xi} = \boldsymbol{AD}\underline{u}$. Now, we show that $\underline{u}$ satisfies the equations

$$\left.\begin{aligned} -\text{div}(\boldsymbol{AD}\underline{u}) &= \underline{f} \text{ in } \Omega \\ \underline{u} &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{4.3.17}$$

Set $\boldsymbol{\xi}_\varepsilon \equiv (\boldsymbol{A}_\varepsilon\boldsymbol{D}\underline{u}_\varepsilon)$ and let $\Psi \in D(\Omega)^m$. We have,

$$\begin{aligned} < -\text{div}\boldsymbol{\xi}_\varepsilon, \Psi_{|\Omega_\varepsilon} > &= \int_{\Omega_\varepsilon} \boldsymbol{A}_\varepsilon\boldsymbol{D}\underline{u}_\varepsilon.\boldsymbol{D}\Psi\,dx \\ &= \int_\Omega Q_\varepsilon(\boldsymbol{\xi}_\varepsilon).\boldsymbol{D}\Psi\,dx. \end{aligned}$$

Passing to the limit for the subsequence ε' using div-curl lemma and (4.3.16), we get,

$$\lim_{\varepsilon' \to 0} < -\mathrm{div}\boldsymbol{\xi}_{\varepsilon'}, \Psi_{|\Omega_{\varepsilon'}} >= \int_\Omega \boldsymbol{\xi}.\boldsymbol{D}\Psi \, dx.$$

On the other hand

$$\begin{aligned} < -\mathrm{div}\boldsymbol{\xi}_\varepsilon, \Psi_{|\Omega_\varepsilon} > &= < \wp_\varepsilon^* \underline{f}, \Psi_{|\Omega_\varepsilon} > \\ &= < \underline{f}, \wp_\varepsilon(\Psi_{|\Omega_\varepsilon}) > \end{aligned}$$

Note however that $\wp_\varepsilon(\Psi_{|\Omega_\varepsilon}) \rightharpoonup \Psi$ weakly in $H_0^1(\Omega)^m$. Thus the limit of the above sequence is also equal to $< \underline{f}, \Psi >$.

Thus we get, $\int_\Omega \boldsymbol{\xi} \boldsymbol{D}\Psi \, dx =< \underline{f}, \Psi >$ for all $\Psi \in D(\Omega)^m$. From this we conclude that

$$-\mathrm{div}\boldsymbol{\xi} = \underline{f} \text{ i.e. } -\mathrm{div}(\boldsymbol{A}\boldsymbol{D}\underline{u}) = \underline{f}.$$

The argument of this step can be applied to any subsequence of ε and by uniqueness of the solution to (4.3.17) we conclude that (4.3.16) holds not only for the sequence ε' but also for the entire sequence ε.

Step 5: <u>Bounds for $\boldsymbol{A}$</u>.

We can show, as in Step 2, that the operator G is coercive and its inverse, $G^{-1} = -\mathrm{div}(\boldsymbol{A}\boldsymbol{D}(\,.\,))$. For any $\psi \in D(\Omega)$,

$$\begin{aligned} \int_{\Omega_\varepsilon} \boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon . \boldsymbol{D}\underline{u}_\varepsilon \psi^2 \, dx &\geq a \int_{\Omega_\varepsilon} |\boldsymbol{D}\underline{u}_\varepsilon|^2 \psi^2 \, dx \\ &\geq ab^{-2} \int_{\Omega_\varepsilon} |\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon|^2 \psi^2 \, dx. \end{aligned}$$

That is,

$$\int_\Omega Q_\varepsilon(\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon).\boldsymbol{D}\wp_\varepsilon(\underline{u}_\varepsilon)\psi^2 \, dx \;\geq\; ab^{-2} \int_\Omega |Q_\varepsilon(\boldsymbol{A}_\varepsilon \boldsymbol{D}\underline{u}_\varepsilon)|^2 \psi^2 \, dx.$$

Passing to the limit as $\varepsilon \to 0$ and observing that the left hand side has a limit while the right hand side can be handled using weak lower semi-continuity of norm, we obtain,

$$\int_\Omega \boldsymbol{A}\boldsymbol{D}\underline{u}.\boldsymbol{D}\underline{u}\psi^2 \, dx \geq ab^{-2} \int_\Omega |\boldsymbol{A}\boldsymbol{D}\underline{u}|^2 \psi^2 \, dx.$$

We can choose $\underline{f}$ and hence $\underline{u}$ such that $\underline{u} = \underline{P}_i^\alpha$ on the support of ψ. Then we conclude, using Cauchy-Schwarz inequality, that,

$$|\boldsymbol{A}\boldsymbol{e}_i^\alpha\psi|_{0,\Omega}^2 \leq b^2a^{-1}|\psi\boldsymbol{e}_i^\alpha|_{0,\Omega}|\boldsymbol{A}\boldsymbol{e}_i^\alpha\psi|_{0,\Omega}.$$

Therefore,

$$\begin{aligned} |\boldsymbol{A}\boldsymbol{e}_i^\alpha\psi|_{0,\Omega} &\leq b^2a^{-1}|\psi\boldsymbol{e}_i^\alpha|_{0,\Omega} \\ &= b^2a^{-1}|\psi|_{0,\Omega} \end{aligned}$$

for all $\psi \in D(\Omega)$ (and by density for all $\psi \in L^2(\Omega)$) and $i = 1, 2, ..., m$ and $\alpha = 1, 2, ..., n$. By duality, we conclude that $\boldsymbol{A} \in L^\infty(\Omega)^{nm}$ and $\|\boldsymbol{A}\|_\infty \leq b^2a^{-1}$.

Again starting from,

$$\begin{aligned} \int_{\Omega_\varepsilon} \boldsymbol{A}_\varepsilon\boldsymbol{D}\underline{u}_\varepsilon.\boldsymbol{D}\underline{u}_\varepsilon\,dx &\geq a\int_{\Omega_\varepsilon}|\boldsymbol{D}\underline{u}_\varepsilon|^2\,dx \\ &\geq aC_0^{-2}|\boldsymbol{D}\wp_\varepsilon\underline{u}_\varepsilon|_{0,\Omega}^2 \end{aligned}$$

and taking limits we find,

$$\int_\Omega \boldsymbol{A}\boldsymbol{D}\underline{u}.\boldsymbol{D}\underline{u}\,dx \geq aC_0^{-2}\int_\Omega|\boldsymbol{D}\underline{u}|^2\,dx$$

for all $\underline{u} \in H_0^1(\Omega)^m$. Therefore, $\boldsymbol{A} \geq aC_0^{-2}\boldsymbol{I}$ a.e. (for a proof cf. Lemma 22.5, Dal Maso [16]).

Thus, $\boldsymbol{A} \in M_n^m(aC_0^{-2}, b^2a^{-1}, \Omega)$. This concludes the proof of Theorem 4.2.1. ■

Remark 4.3.1 *Note that we have assumed $|\boldsymbol{A}_\varepsilon(x)\boldsymbol{\lambda}| \leq b|\boldsymbol{\lambda}|$ a.e. $x \in \Omega$, following [29] and not $\boldsymbol{A}_\varepsilon^{-1}(x) \geq b^{-1}\boldsymbol{I}$ a.e. $x \in \Omega$, which would have been the analogue of the corresponding condition in [7]. So, though we define the operators T_ε in a manner similar to Briane, Damlamian and Donato [7], the proof of the coercivity of T needs different arguments from those of [7], as can be seen in Step 2.* ■

Remark 4.3.2 *H_b- convergence cannot be obtained from H_0- convergence because of the presence of coupling in the highest order terms. Thus, we need to construct new test functions (cf. Step 3). Similar sequences have been constructed in the periodic case by Bensoussan, Lions and Papanicolaou [6].* ■

Remark 4.3.3 *It can be shown that H_b-convergence enjoys properties similar to H_0-convergence (cf. [7]). In particular, it is independent of the choice of extension operators, is local in nature, and the H_b limit of the transpose of a sequence of matrices is equal to the transpose of the H_b limit. Also, the existence of local correctors can be proved. As the statements and proofs of these results are only minor modifications of the corresponding ones for H_0- convergence (cf. [7]), we omit them.* ■

Remark 4.3.4 *Let $(A_\varepsilon, S_\varepsilon) \xrightarrow{H_b} A$. Let $\underline{f_\varepsilon} \in L^2(\Omega)^m$, such that $\chi_\varepsilon \underline{f_\varepsilon} \rightharpoonup \chi \underline{f}$ weakly in $L^2(\Omega)^m$. Then, the solutions $\underline{u}_\varepsilon$ of (4.2.1), corresponding to right hand side $\underline{f_\varepsilon}$, converge to the solution $\underline{u}$ of (4.2.7), corresponding to the right hand side $\chi \underline{f}$, and the convergences (4.2.6) hold. The proof is along the lines of Theorem 1.5 [7].* ■

4.4 The Homogenized Problem

In view of Theorem 4.2.1, it can be assumed that A_ε has a H_b limit. This is the desired A^*. The matrix B^* is obtained in terms of the test sequences $\underline{\zeta}_i^{\alpha,\varepsilon} \in H^1(\Omega)^m$ defined as follows.

Let Ω' be a bounded open set in R^n containing the closure of Ω. Extend A_ε by aI on $\Omega' \setminus \Omega$ and call it C_ε. Notice that S_ε form an admissible set of holes for Ω' also. Without loss of generality, let $(C_\varepsilon, S_\varepsilon) \xrightarrow{H_b} C$. By locality, C restricted to Ω is A^*. Let $\phi \in D(\Omega')$, with $\phi \equiv 1$ on Ω. Then, $\underline{\zeta}_i^{\alpha,\varepsilon}$ are taken to be the solutions in $H_0^1(\Omega')$ of

$$\left.\begin{aligned} -\mathrm{div}(C_\varepsilon D\underline{\zeta}_i^{\alpha,\varepsilon}) &= \wp_\varepsilon^*(-\mathrm{div}(CD(\phi \underline{P}_i^\alpha))) \text{ in } \Omega'_\varepsilon, \\ C_\varepsilon D\underline{\zeta}_i^{\alpha,\varepsilon}.n_\varepsilon &= 0 \text{ on } \partial S_\varepsilon,\ i = 1, 2, ..., m, \\ \underline{u}_\varepsilon &= 0 \text{ on } \partial\Omega'. \end{aligned}\right\} \tag{4.4.1}$$

Then, it follows from H_b convergence that,

$$\left.\begin{aligned} \wp_\varepsilon \underline{\zeta}_i^{\alpha,\varepsilon} &\rightharpoonup \underline{P}_i^\alpha \text{ weakly in } H^1(\Omega)^m \\ Q_\varepsilon(A_\varepsilon D\underline{\zeta}_i^{\alpha,\varepsilon}) &\rightharpoonup A^* e_i^\alpha \text{ weakly in } L^2(\Omega)^{nm} \\ \mathrm{div} Q_\varepsilon(A_\varepsilon^t \underline{\zeta}_i^{\alpha,\varepsilon}) &\subset\subset H^{-1}(\Omega)^m. \end{aligned}\right\} \tag{4.4.2}$$

for $i = 1, 2, ..., m$ and $\alpha = 1, 2, ..., n$.

The *corrector matrices*, $\boldsymbol{M}_\varepsilon$, are defined as follows,

$$\boldsymbol{M}_\varepsilon \boldsymbol{e}_i^\alpha = \boldsymbol{D}\varphi_\varepsilon(\underline{\zeta}_i^{\alpha,\varepsilon}). \tag{4.4.3}$$

Then, $\boldsymbol{B}^*$ is given by the formula

$$\boldsymbol{B}^* = \lim_{\varepsilon \to 0} \chi_\varepsilon \boldsymbol{M}_\varepsilon^t \boldsymbol{B}_\varepsilon \boldsymbol{M}_\varepsilon \text{ in } D'(\Omega). \tag{4.4.4}$$

Let $F^1(\underline{\theta})$ be defined as follows.

$$F^1(\theta) = \frac{1}{2}\int_\Omega \boldsymbol{B}^* \boldsymbol{D}\underline{u} \,.\, \boldsymbol{D}\underline{u}\, dx$$

where $\underline{u} = \underline{u}(\underline{\theta})$ is the solution of

$$\left.\begin{aligned} -\mathrm{div}(\boldsymbol{A}^* \boldsymbol{D}\underline{u}) + K\underline{u} &= \chi \underline{f} + \underline{\theta} \text{ in } \Omega, \\ \underline{u} &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{4.4.5}$$

It can be verified, as in the scalar case, by first introducing and homogenizing the state-adjoint state systems of equations that $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ defined above are the coefficients for the homogenized system. This implies, as in the scalar case, that the energies converge to an appropriate energy in which the matrices $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ appear naturally. All these show that F^1 defined above will satisfy (2.1.19) for a suitably modified Lemma 2.1.1. So, this and the discussion in Section 4.1, where F^2 and U_{ad} have been defined, show that $\underline{\theta}^*$ is the minimizer of the functional $F^1 + F^2$ over the domain U_{ad}.

In the *periodic case*, it is possible to give an explicit formula for $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$. Let S be a closed subset Y with Lipschitz boundary. We then define a periodically perforated domain as in Chapter 3 and we assume that Ω_ε is a connected set. Let $\boldsymbol{A} \in M_n^m(a, b, \mathbb{R}^n)$ and $\boldsymbol{B} \in M_n^m(c, d, \mathbb{R}^n)$ be Y-periodically defined block matrices and $\boldsymbol{B}$ is assumed to be symmetric. Define the sequences $\boldsymbol{A}_\varepsilon$ and $\boldsymbol{B}_\varepsilon$ as follows:

$$\boldsymbol{A}_\varepsilon(x) = \boldsymbol{A}(\frac{x}{\varepsilon}),\ \boldsymbol{B}_\varepsilon(x) = \boldsymbol{B}(\frac{x}{\varepsilon}).$$

We consider the homogenization of the problem (P_ε) defined with these coefficients on periodically perforated domains. To obtain the homogenized coefficients we need to define test functions, $\underline{\zeta}_i^\alpha$, which solve the following periodic boundary value problem in the basic perforated cell:

$$\left.\begin{aligned} -\mathrm{div}(\boldsymbol{A}^t\boldsymbol{D}(\underline{P}_i^\alpha + \underline{\zeta}_i^\alpha)) &= 0 \text{ in } Y \setminus S, \\ (\boldsymbol{A}^t\boldsymbol{D}(\underline{P}_i^\alpha + \underline{\zeta}_i^\alpha))_j.n &= 0 \text{ on } \partial S, j = 1,2,...,m, \\ y &\mapsto \underline{\zeta}_i^\alpha(y) \text{ Y-periodic} \end{aligned}\right\} \tag{4.4.6}$$

for $i = 1,2,...,m$ and $\alpha = 1,2,...,n$. It can be shown by calculations similar to those in Kesavan and Rajesh [23] or in Section 3.3 - 3.4 of the thesis, that the homogenized coefficients $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ are those given below

$$\begin{aligned} \boldsymbol{A}^* e_i^\alpha . e_j^\beta &= \int_{Y\setminus S} \boldsymbol{A}(y)\boldsymbol{D}(\underline{P}_i^\alpha(y) + \underline{\zeta}_i^\alpha(y)).\boldsymbol{D}(\underline{P}_j^\beta(y) + \underline{\zeta}_j^\beta(y))\, dy, \\ \boldsymbol{B}^* e_i^\alpha . e_j^\beta &= \int_{Y\setminus S} \boldsymbol{B}(y)\boldsymbol{D}(\underline{P}_i^\alpha(y) + \underline{\zeta}_i^\alpha(y)).\boldsymbol{D}(\underline{P}_j^\beta(y) + \underline{\zeta}_j^\beta(y))\, dy \end{aligned}$$

for $i, j = 1,2,...,m$ and $\alpha, \beta = 1,2,...,n$.

Chapter 5

The Dirichlet Problem

5.1 Introduction

In this chapter, we consider the homogenization of optimal control problems governed by Dirichlet boundary value problems on periodically perforated domains. The definition of the periodically perforated domain Ω_ε differs from the one in Chapter 3 in that the size of the holes bear a ratio, $a_\varepsilon : \varepsilon$, to the length of the scaled cell εY. We do not consider the situation where the coefficients appearing in the state equation and in the cost functional oscillate. This is a problem still open to investigation. The optimal control problem that we want to homogenize is the following: Let $B \in M(c, d, \Omega)$ be symmetric, N be a given positive constant and $g \in L^2(\Omega)$ be a given function. We take the space of *admissible controls*, U^ε_{ad}, to be one of (2.1.9). For $\varepsilon > 0$ fixed, we find θ^*_ε which minimizes the cost functional

$$(\mathrm{P}_\varepsilon) \qquad J_\varepsilon(\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} B\nabla u_\varepsilon.\nabla u_\varepsilon \, dx + \frac{N}{2}\int_{\Omega_\varepsilon} \theta^2 \, dx \tag{5.1.1}$$

over $\theta \in U^\varepsilon_{ad}$ and where the *state* $u_\varepsilon \equiv u_\varepsilon(\theta)$ is the solution of the Dirichlet problem,

$$\left.\begin{aligned} -\Delta u_\varepsilon &= g + \theta \text{ in } \Omega_\varepsilon, \\ u_\varepsilon &= 0 \text{ on } \Omega_\varepsilon. \end{aligned}\right\} \tag{5.1.2}$$

Once again, the homogenization is performed in the framework of Lemma 2.1.1 by setting,

$$\begin{aligned} F_\varepsilon^1(\theta) &= \frac{1}{2}\int_{\Omega_\varepsilon} B\nabla u_\varepsilon.\nabla u_\varepsilon \, dx \text{ and,} \\ F_\varepsilon^2(\theta) &= \frac{N}{2}\int_{\Omega_\varepsilon} \theta^2 \, dx \end{aligned}$$

The corresponding F^2 and U_{ad} are as before (cf. Section 2.1). It is required to identify the F^1 which satisfies the lemma. This can be split into two parts-one, to homogenize the following Dirichlet boundary value problems:

Let $f_\varepsilon \in L^2(\Omega)$ such that $\chi_\varepsilon f_\varepsilon \rightharpoonup f$ weakly in $L^2(\Omega)$ or $f_\varepsilon \in H^{-1}(\Omega)$ be such that $f_\varepsilon \longrightarrow f$ strongly in $H^{-1}(\Omega)$. Let $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$ solve,

$$\left.\begin{aligned} -\Delta u_\varepsilon &= f_\varepsilon \text{ in } \Omega_\varepsilon, \\ u_\varepsilon &= 0 \text{ on } \partial\Omega_\varepsilon. \end{aligned}\right\} \tag{5.1.3}$$

Two, to obtain the limit, $\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B\nabla u_\varepsilon.\nabla u_\varepsilon \, dx$ in a suitable form.

The homogenization of (5.1.3) has been studied by Cioranescu and Murat [11], [12]. It was shown that the asymptotic behaviour of the solutions of (5.1.3) depends on the size of the holes. A critical size c_ε is found so that:

a) if $a_\varepsilon >> c_\varepsilon$, i.e. the hole size tends to zero slower than c_ε, then $\widetilde{u_\varepsilon} \longrightarrow 0$ strongly in $H_0^1(\Omega)$.

b) if $a_\varepsilon = O(c_\varepsilon)$, then there exists a measure μ such that $\widetilde{u_\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and u solves the following Dirichlet boundary value problem having an *extra* lower order term,

$$\left.\begin{aligned} -\Delta u + \mu u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{5.1.4}$$

The lower order term is known in the literature as the 'strange term'.

c) if $a_\varepsilon << c_\varepsilon$, i.e. the hole size tends to zero faster than c_ε, then $\widetilde{u_\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and u solves the equation

$$\left.\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{5.1.5}$$

In the above, $\tilde{}$ denotes the extension by zero onto the holes. Since, the u_ε vanish on the boundary of Ω_ε the extension $\widetilde{u_\varepsilon}$ also belongs to $H_0^1(\Omega)$.

In cases b) and c) above, these results are proved in [12] by constructing a sequence of test functions, $w_\varepsilon \in H^1(\Omega)$, with the following properties:

CM1) $w_\varepsilon = 0$ in $\Omega \setminus \Omega_\varepsilon$,

CM2) $w_\varepsilon \rightharpoonup 1$ weakly in $H^1(\Omega)$ and,

CM3) For any sequence $v_\varepsilon \in H^1(\Omega)$ with $v_\varepsilon = 0$ in $\Omega \setminus \Omega_\varepsilon$ and such that $v_\varepsilon \rightharpoonup v$ weakly in $H^1(\Omega)$ and for any $\phi \in D(\Omega)$, we have,

$$\int_{\Omega_\varepsilon} \nabla w_\varepsilon . \nabla(\phi v_\varepsilon)\, dx \longrightarrow <\mu, \phi v> .$$

for some constant μ ($\mu = 0$ in the case c). We will, henceforth, refer to these conditions jointly as [CM] conditions.

In fact, the homogenized equations corresponding to (5.1.3) can be shown to be (5.1.4) with just the assumptions that a sequence $w_\varepsilon \in H^1(\Omega)$ and a distribution $\mu \in W^{-1,\infty}(\Omega)$ exist satisfying [CM], or just CM1 and CM2; as it has been shown by Juan Casado Díaz [9] that CM1 and CM2 together imply CM3. There are more general geometries than those considered by us in the thesis which satisfy [CM], as can be seen from numerous examples in [12].

We also have the fact that μ is a positive measure from,

$$<\mu, \phi> = \lim_{\varepsilon \to 0} \int_\Omega |\nabla w_\varepsilon|^2 \phi\, dx.$$

This gives the existence and uniqueness of solution to (5.1.4).

Now, we would like to characterize the limit of the energies $\int_{\Omega_\varepsilon} B\nabla u_\varepsilon . \nabla u_\varepsilon\, dx$ as $\varepsilon \to 0$. We observe that if $c_\varepsilon = o(a_\varepsilon)$, i.e $c_\varepsilon << a_\varepsilon$, then $\int_{\Omega_\varepsilon} B\nabla u_\varepsilon . \nabla u_\varepsilon\, dx \to 0$ which follows from the fact that $\widetilde{u_\varepsilon} \longrightarrow 0$ *strongly* in $H_0^1(\Omega)$. We, therefore, look at hole sizes which are much smaller or comparable to the critical size. Then, we can work under the general hypotheses of the existence of a sequence w_ε as above. Under these assumptions, we shall obtain a characterization of the limits of energies in Section 5.2 and derive conclusions for the optimal control problem in Section 5.3. We end this section with the following remark.

Remark 5.1.1 *If $B = I$, then the limit of the energies take the following form*

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \stackrel{\varepsilon\to 0}{\longrightarrow} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega u^2 \, d\mu.$$

This can be proved (cf. [30]) by a simple integration by parts and by using equations (5.1.3) and (5.1.4). ■

5.2 Strange Term for the Energy

Let Ω_ε be a general perforated domain. Assume that there exists a sequence, w_ε satisfying the [CM] conditions. We show that there exists a subsequence ε' of ε and a distribution μ_B such that, for u_ε as given by (5.1.3) we have the following convergence,

$$\int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'}.\nabla u_{\varepsilon'} \, dx \to \int_\Omega B\nabla u.\nabla u \, dx + < \mu_B, u^2 >$$

where u solves the Dirichlet problem (5.1.4). Further, we show that

$$B\nabla\widetilde{u_{\varepsilon'}}.\nabla\widetilde{u_{\varepsilon'}} \longrightarrow B\nabla u.\nabla u + u^2\mu_B \text{ in } D'(\Omega).$$

To define μ_B we need to define a sequence of test functions, which we do in the following lemma.

Lemma 5.2.1 *Let $\psi_\varepsilon \in H_0^1(\Omega_\varepsilon)$ be the solution of the boundary value problem,*

$$\left.\begin{aligned} -\Delta\psi_\varepsilon &= -\operatorname{div}(B\nabla w_\varepsilon) \text{ in } \Omega_\varepsilon \\ \psi_\varepsilon &= 0 \text{ on } \partial\Omega_\varepsilon. \end{aligned}\right\} \qquad (5.2.1)$$

Then, the sequence, $\widetilde{\psi_\varepsilon}$, is bounded in $H_0^1(\Omega)$.

Proof: Multiplying (5.2.1) by ψ_ε and integrating by parts we get,

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla\psi_\varepsilon|^2 \, dx &= \int_{\Omega_\varepsilon} B\nabla w_\varepsilon.\nabla\psi_\varepsilon \, dx \\ &\leq d\,|\nabla w_\varepsilon|_{0,\Omega_\varepsilon} |\nabla\psi_\varepsilon|_{0,\Omega_\varepsilon}. \end{aligned}$$

Therefore, since w_ε is a bounded sequence in $H^1(\Omega)$,

$$|\nabla\widetilde{\psi_\varepsilon}|_{0,\Omega} = |\nabla\psi_\varepsilon|_{0,\Omega_\varepsilon} \leq d\,|\nabla w_\varepsilon|_{0,\Omega} \leq C$$

where C is a generic constant. This completes the proof. ■

So, by the lemma, H^1 boundedness of w_ε and the bound for B in $L^\infty(\Omega)$, we also deduce that the sequence, $\nabla\widetilde{\psi_\varepsilon} - B\nabla w_\varepsilon$ is bounded in $L^2(\Omega)$. Hence, there exists a subsequence ε' of ε and a function $\psi \in H_0^1(\Omega)$ such that,

$$\left.\begin{array}{rcl} \widetilde{\psi_{\varepsilon'}} & \rightharpoonup & \psi \text{ weakly in } H_0^1(\Omega), \\ \nabla\widetilde{\psi_{\varepsilon'}} - B\nabla w_{\varepsilon'} & \rightharpoonup & \nabla\psi \text{ weakly in } L^2(\Omega)^n. \end{array}\right\} \quad (5.2.2)$$

Define, $\mu_B \in D'(\Omega)$ by,

$$\mu_B \equiv -\Delta\psi + \psi\mu. \quad (5.2.3)$$

Note that the definition of μ_B depends only on w_ε and B.

Proposition 5.2.1 *Let μ_B be given by (5.2.3). Let $u_{\varepsilon'}$ be the solution of the Dirichlet problem (5.1.3) in $\Omega_{\varepsilon'}$, such that $u_{\varepsilon'} \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Let $p_{\varepsilon'} \in H_0^1(\Omega_{\varepsilon'})$ be the solution of:*

$$\left.\begin{array}{rcl} -\Delta p_{\varepsilon'} & = & -\mathrm{div}(B\nabla u_{\varepsilon'}) \text{ in } \Omega_{\varepsilon'}, \\ p_{\varepsilon'} & = & 0 \text{ on } \partial\Omega_{\varepsilon'}. \end{array}\right\} \quad (5.2.4)$$

Then, $\widetilde{p_{\varepsilon'}} \rightharpoonup p$ weakly in $H_0^1(\Omega)$ and p is the solution of,

$$\left.\begin{array}{rcl} -\Delta p + p\mu & = & -\mathrm{div}(B\nabla u) + u\mu_B \text{ in } \Omega, \\ p & = & 0 \text{ on } \partial\Omega. \end{array}\right\} \quad (5.2.5)$$

Proof: It can be shown, as in Lemma 5.2.1, that $\widetilde{p_{\varepsilon'}}$ is bounded in $H_0^1(\Omega)$. So there is a subsequence ε'' of ε' and $p \in H_0^1(\Omega)$ such that,

$$\begin{array}{rcl} \widetilde{p_{\varepsilon''}} & \rightharpoonup & p \text{ weakly in } H_0^1(\Omega), \\ \eta_{\varepsilon''} \doteq \nabla\widetilde{p_{\varepsilon''}} - B\nabla\widetilde{u_{\varepsilon''}} & \rightharpoonup & \nabla p - B\nabla u \doteq \eta \text{ weakly in } L^2(\Omega)^n. \end{array}$$

We need to show that

$$-\mathrm{div}\eta + p\mu = u\mu_B.$$

Let $\phi \in D(\Omega)$. We note that $\phi w_{\varepsilon''}$ vanishes on $\partial\Omega_{\varepsilon''}$. So using this as a test function in the equation, $-\text{div}\eta_{\varepsilon''} = 0$, which holds in $\Omega_{\varepsilon''}$ we get

$$\int_{\Omega_{\varepsilon''}} \eta_{\varepsilon''}.\nabla w_{\varepsilon''}\ \phi\, dx \ = \ -\int_{\Omega_{\varepsilon''}} \eta_{\varepsilon''}.\nabla\phi\ \ w_{\varepsilon''}\, dx.$$

Therefore, since $w_{\varepsilon''} \to 1$ strongly in $L^2(\Omega)$ and $\eta_{\varepsilon''} \rightharpoonup \eta$ weakly in $L^2(\Omega)^n$,

$$\left.\begin{array}{rcl} \lim_{\varepsilon''\to 0} \int_{\Omega_{\varepsilon''}} \eta_{\varepsilon''}.\nabla w_{\varepsilon''}\ \phi\, dx & = & -\int_\Omega \eta.\nabla\phi\, dx \\ & = & <\text{div}\eta, \phi>. \end{array}\right\}. \tag{5.2.6}$$

Again,

$$\begin{aligned} \int_{\Omega_{\varepsilon''}} \eta_{\varepsilon''}.\nabla w_{\varepsilon''}\ \phi\, dx \ &= \ \int_{\Omega_{\varepsilon''}} \nabla p_{\varepsilon''}.\nabla w_{\varepsilon''}\ \phi\, dx - \int_{\Omega_{\varepsilon''}} B\nabla u_{\varepsilon''}.\nabla w_{\varepsilon''}\ \phi\, dx \\ &\equiv \ I_{\varepsilon''} + J_{\varepsilon''}. \end{aligned}$$

Now,

$$\begin{aligned} I_{\varepsilon''} \ &= \ \int_{\Omega_{\varepsilon''}} \nabla p_{\varepsilon''}.\nabla w_{\varepsilon''}\ \phi\, dx \\ &= \ \int_{\Omega_{\varepsilon''}} \nabla w_{\varepsilon''}.\nabla(p_{\varepsilon''}\phi)\, dx - \int_{\Omega_{\varepsilon''}} p_{\varepsilon''}\nabla w_{\varepsilon''}.\nabla\phi\, dx \end{aligned}$$

Therefore, using properties CM2 and CM3 of $w_{\varepsilon''}$, the weak convergence of $\widetilde{p_{\varepsilon''}}$ in $H_0^1(\Omega)$ and its strong convergence in $L^2(\Omega)$, we get,

$$\lim_{\varepsilon''\to 0} I_{\varepsilon''} = <\mu, \phi p>. \tag{5.2.7}$$

Now,

$$\begin{aligned} J_{\varepsilon''} \ &= \ -\int_{\Omega_{\varepsilon''}} B\nabla u_{\varepsilon''}.\nabla w_{\varepsilon''}\ \phi\, dx \\ &= \ -\int_{\Omega_{\varepsilon''}} B\nabla w_{\varepsilon''}.\nabla u_{\varepsilon''}\ \phi\, dx \\ &= \ \int_{\Omega_{\varepsilon''}} (\nabla\psi_{\varepsilon''} - B\nabla w_{\varepsilon''}).\nabla u_{\varepsilon''}\ \phi\, dx - \int_{\Omega_{\varepsilon''}} \nabla\psi_{\varepsilon''}.\nabla u_{\varepsilon''}\ \phi\, dx \\ &\equiv \ K_{\varepsilon''} + L_{\varepsilon''}. \end{aligned}$$

We have, by (5.2.1),

$$\begin{aligned} 0 &= \; < \mathrm{div}(\nabla\psi_{\varepsilon''} - B\nabla w_{\varepsilon''}), \phi u_{\varepsilon''} > \\ &= - \int_{\Omega_{\varepsilon''}} (\nabla\psi_{\varepsilon''} - B\nabla w_{\varepsilon''}).\nabla\phi \; u_{\varepsilon''}\, dx - \int_{\Omega_{\varepsilon''}} (\nabla\psi_{\varepsilon''} - B\nabla w_{\varepsilon''}).\nabla u_{\varepsilon''} \; \phi\, dx \\ &= -\int_{\Omega_{\varepsilon''}} (\nabla\psi_{\varepsilon''} - B\nabla w_{\varepsilon''}).\nabla\phi \; u_{\varepsilon''}\, dx - K_{\varepsilon''}. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{\varepsilon''\to 0} K_{\varepsilon''} &= -\lim_{\varepsilon''\to 0} \int_\Omega (\nabla\widetilde{\psi_{\varepsilon''}} - B\nabla w_{\varepsilon''}).\nabla\phi \; \widetilde{u_{\varepsilon''}}\, dx \\ &= -\int_\Omega \nabla\psi.\nabla\phi \; u\, dx. \end{aligned}$$

Therefore,

$$\lim_{\varepsilon''\to 0} K_{\varepsilon''} = \int_\Omega \nabla u.\nabla\psi \; \phi\, dx + < u\Delta\psi, \phi > . \tag{5.2.8}$$

$$\begin{aligned} L_{\varepsilon''} &= -\int_{\Omega_{\varepsilon''}} \nabla\psi_{\varepsilon''}.\nabla u_{\varepsilon''} \; \phi\, dx \\ &= -\int_{\Omega_{\varepsilon''}} \nabla u_{\varepsilon''}.\nabla(\psi_{\varepsilon''}\phi)\, dx + \int_{\Omega_{\varepsilon''}} \nabla u_{\varepsilon''}.\nabla\phi \; \psi_{\varepsilon''}\, dx \\ &= -\int_\Omega \chi_{\varepsilon''} f_{\varepsilon''} \widetilde{\psi_{\varepsilon''}} \, \phi\, dx + \int_\Omega \nabla\widetilde{u_{\varepsilon''}}.\nabla\phi \; \widetilde{\psi_{\varepsilon''}}\, dx. \end{aligned}$$

Therefore, using (5.2.2), we have,

$$\lim_{\varepsilon''\to 0} L_{\varepsilon''} = -\int_\Omega f\,\phi\,\psi\, dx + \int_\Omega \nabla u.\nabla\phi \; \psi\, dx,$$

which, using (5.1.4), gives,

$$\left.\begin{aligned} \lim_{\varepsilon''\to 0} L_{\varepsilon''} &= \; < \Delta u - u\mu, \psi\phi > + \int_\Omega \nabla u.\nabla\phi \; \psi\, dx \\ &= -\int_\Omega \nabla u.\nabla\psi \; \phi\, dx - < u\mu, \phi\,\psi > \end{aligned}\right\} \tag{5.2.9}$$

after an integration by parts. From (5.2.6)- (5.2.9), we get,

$$\begin{aligned} < \mathrm{div}\eta, \phi > &= \; < p\mu, \phi > + < u\Delta\psi, \phi > - < u\psi\mu, \phi > \\ &= \; < p\mu, \phi > - < u\mu_B, \phi > . \end{aligned}$$

Since the above holds for all $\phi \in D(\Omega)$, we have $-\Delta p + p\mu = -\text{div}(B\nabla u) + u\mu_B$, i.e. p satisfies (5.2.5). Since μ is a positive measure, the solution to (5.2.5) is unique, and therefore, it follows that the entire sequence $\widetilde{p_{\varepsilon'}} \rightharpoonup p$ weakly in $H_0^1(\Omega)$. This completes the proof of the proposition. ■

We now prove our main theorem.

Theorem 5.2.1 *Let ε' be the subsequence of ε chosen prior to Proposition 5.2.1. u_ε be the solution of the Dirichlet problem (5.1.3). Let μ_B be given by (5.2.3). Then,*

$$\int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'}.\nabla u_{\varepsilon'}\, dx \quad \to \quad \int_\Omega B\nabla u.\nabla u\, dx + < \mu_B, u^2 > \qquad (5.2.10)$$

and,

$$B\nabla u_{\varepsilon'}.\nabla u_{\varepsilon'} \quad \longrightarrow \quad B\nabla u.\nabla u + u^2\mu_B \text{ in } D'(\Omega). \qquad (5.2.11)$$

Proof: Define $p_{\varepsilon'} \in H_0^1(\Omega_{\varepsilon'})$ to be the solution of (5.2.4). We write,

$$\begin{aligned}\int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'}.\nabla u_{\varepsilon'}\, dx &= \int_{\Omega_{\varepsilon'}} \nabla p_{\varepsilon'}.\nabla u_{\varepsilon'}\, dx - \int_{\Omega_{\varepsilon'}} (\nabla p_{\varepsilon'} - B\nabla u_{\varepsilon'}).\nabla u_{\varepsilon'}\, dx \\ &= \int_{\Omega_{\varepsilon'}} \nabla p_{\varepsilon'}.\nabla u_{\varepsilon'}\, dx = \int_\Omega \chi_{\varepsilon'} f_{\varepsilon'} \widetilde{p_{\varepsilon'}}\, dx\end{aligned}$$

where we have used the fact that $u_{\varepsilon'}$ and $p_{\varepsilon'}$ are solutions of (5.1.3) and (5.2.4) respectively. Therefore, by integration by parts and using Proposition 5.2.1,

$$\begin{aligned}\lim_{\varepsilon' \to 0} \int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'}.\nabla u_{\varepsilon'}\, dx &= \int_\Omega fp\, dx \\ &= < -\Delta u + u\mu, p > \\ &= < u, -\Delta p + p\mu > \\ &= < u, -\text{div}(B\nabla u) + u\mu_B > .\end{aligned}$$

From this, we get (5.2.10), after an integration by parts.

Let $\phi \in D(\Omega)$. Set $\eta_{\varepsilon'}$ as in Proposition 5.2.1, then

$$\begin{aligned}\int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'}.\nabla u_{\varepsilon'}\, \phi\, dx &= \int_{\Omega_{\varepsilon'}} \nabla p_{\varepsilon'}.\nabla u_{\varepsilon'}\, \phi\, dx - \int_{\Omega_{\varepsilon'}} \eta_{\varepsilon'}.\nabla u_{\varepsilon'}\, \phi\, dx \\ &\equiv I_{\varepsilon'} + J_{\varepsilon'}.\end{aligned}$$

On one hand, it can be shown that (cf. arguments for convergence of $L_{\varepsilon''}$ in Proposition 5.2.1)

$$
\begin{aligned}
\lim_{\varepsilon' \to 0} I_{\varepsilon'} &= \lim_{\varepsilon' \to 0} \int_{\Omega_{\varepsilon'}} \nabla p_{\varepsilon'}.\nabla u_{\varepsilon'}\, \phi\, dx \\
&= < -\Delta u + u\mu, p\,\phi > - \int_{\Omega} \nabla u.\nabla \phi\; p\, dx \\
&= \int_{\Omega} \nabla u.\nabla p\; \phi\, dx + < u\mu, p\,\phi > .
\end{aligned}
$$

On the other hand, using the fact that $p_{\varepsilon'}$ solves (5.2.4) and using Proposition 5.2.1, we get,

$$
\begin{aligned}
\lim_{\varepsilon' \to 0} J_{\varepsilon'} &= \lim_{\varepsilon' \to 0} - \int_{\Omega_{\varepsilon'}} \eta_{\varepsilon'}.\nabla u_{\varepsilon'}\, \phi\, dx \\
&= \lim_{\varepsilon' \to 0} \int_{\Omega_{\varepsilon'}} \eta_{\varepsilon'}.\nabla \phi\; u_{\varepsilon'}\, dx \\
&= \int_{\Omega} \eta.\nabla \phi\; u\, dx \\
&= - \int_{\Omega} \eta.\nabla u \phi\, dx + < -\mathrm{div}\eta, u\phi > \\
&= - \int_{\Omega} \eta.\nabla u\; \phi\, dx + < u\mu_B - p\mu, u\,\phi > .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{\varepsilon' \to 0} \int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'}.\nabla u_{\varepsilon'}\, \phi\, dx &= \lim_{\varepsilon' \to 0} (I_{\varepsilon'} + J_{\varepsilon'}) \\
&= \int_{\Omega} \nabla u.\nabla p\; \phi\, dx + < u\mu, p\,\phi > \\
&\quad - \int_{\Omega} \eta.\nabla u\; \phi\, dx + < u\mu_B - p\mu, u\,\phi > \\
&= \int_{\Omega} (\nabla p - \eta).\nabla u\; \phi\, dx + < u^2\mu_B, \phi > \\
&= \int_{\Omega} B\nabla u.\nabla u\; \phi\, dx + < u^2\mu_B, \phi > .
\end{aligned}
$$

This holds for all $\phi \in D(\Omega)$. This proves (5.2.11). ■

Theorem 5.2.2 *Let μ_B be as defined in (5.2.3). Then,*

$$
< \mu_B, \phi > = \lim_{\varepsilon' \to 0} \int_{\Omega_{\varepsilon'}} B\nabla w_{\varepsilon'}.\nabla w_{\varepsilon'}\; \phi\, dx \text{ for all } \phi \in D(\Omega). \tag{5.2.12}
$$

Proof: Let $\phi \in D(\Omega)$. We have,

$$\begin{aligned}\int_{\Omega_{\varepsilon'}} B\nabla w_{\varepsilon'}.\nabla w_{\varepsilon'}\ \phi\, dx &= \int_{\Omega_{\varepsilon'}} \nabla\psi_{\varepsilon'}.\nabla w_{\varepsilon'}\ \phi\, dx - \int_{\Omega_{\varepsilon'}} (\nabla\psi_{\varepsilon'} - B\nabla w_{\varepsilon'}).\nabla w_{\varepsilon'}\ \phi\, dx \\ &= -\int_{\Omega_{\varepsilon'}} \nabla w_{\varepsilon'}.\nabla\phi\ \psi_{\varepsilon'}\, dx + < -\Delta w_{\varepsilon'}, \widetilde{\psi_{\varepsilon'}}\phi > \\ &\quad + \int_{\Omega_{\varepsilon'}} (\nabla\psi_{\varepsilon'} - B\nabla w_{\varepsilon'}).\nabla\phi\ w_{\varepsilon'}\, dx.\end{aligned}$$

Passing to the limit is easy now and we get,

$$\begin{aligned}\lim_{\varepsilon'\to 0}\int_{\Omega_{\varepsilon'}} B\nabla w_{\varepsilon'}.\nabla w_{\varepsilon'}\ \phi\, dx &= < \mu, \psi\phi > + \int_{\Omega} \nabla\psi.\nabla\phi\, dx \\ &= < \mu, \psi\phi > + < -\Delta\psi, \phi > \\ &= < \mu_B, \phi > .\end{aligned}$$

This completes the proof. ■

Corollary 5.2.1 *μ_B is a positive measure.* ■

Proof: Theorem 5.2.2 implies that μ_B is a positive distribution. Hence, by Riesz Representation Theorem it is a positive measure. ■

We now prove a result on the partial uniqueness of μ_B.

Theorem 5.2.3 *Suppose that μ_0 and μ_1 are measures. Let $f \in H^{-1}(\Omega)$ and let u_ε solve the Dirichlet problem (5.1.3) and let u solve (5.1.4) for this data. Then, $\widetilde{u_\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ (cf. [12]). Suppose that for every $f \in H^{-1}(\Omega)$,*

$$\begin{aligned} B\nabla u_\varepsilon.\nabla u_\varepsilon &\longrightarrow B\nabla u.\nabla u + u^2\mu_0 \text{ in } D'(\Omega) \text{ and,} \\ B\nabla u_\varepsilon.\nabla u_\varepsilon &\longrightarrow B\nabla u.\nabla u + u^2\mu_1 \text{ in } D'(\Omega).\end{aligned}$$

Then $\mu_0 = \mu_1$ in $D'(\Omega)$.

Proof: Let $v \in H_0^1(\Omega)$ be arbitrary. Then $\Delta v \in H^{-1}(\Omega)$. Since, $\mu \in W^{-1,\infty}(\Omega)$, we also have $v\mu \in H^{-1}(\Omega)$. Thus we are allowed to take $f \equiv -\Delta v + v\mu$ in the hypothesis. Now, let u_ε be the solution of (5.1.3) with right hand side $Q_\varepsilon^* f$. Then

$\widetilde{u_\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ where u solves

$$\begin{aligned} -\Delta u + u\mu &= -\Delta v + v\mu \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Since μ is a positive measure, the solution to the above equation is unique and therefore, $u \equiv v$. Now, from the hypothesis of the theorem, we conclude that,

$$v^2\mu_0 = v^2\mu_1 \text{ in } D'(\Omega).$$

But $v \in H_0^1(\Omega)$ was arbitrary. For any $\omega \subset\subset \Omega$, we choose $v \in D(\Omega)$ such that $v \equiv 1$ on ω. Then, $\mu_0(\phi) = \mu_1(\phi)$ for any $\phi \in D(\Omega)$ with supp$\phi \subset \omega$. That is, $\mu_{0|_\omega} = \mu_{1|_\omega}$ or $\mu_0 = \mu_1$ in $D'(\omega)$. As this holds, for all $\omega \subset\subset \Omega$, we have $\mu_0 = \mu_1$ in $D'(\Omega)$. This ends the proof. ■

The proof of Theorem 5.2.1 becomes simpler, when a strong corrector result of Cioranescu and Murat (cf. [12]) holds. The corrector result is as follows:

Proposition 5.2.2 *Let $f \in L^2(\Omega)$ and let u_ε be the solution of (5.1.3) and let u be the solution of (5.1.4). Then, $\widetilde{u_\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Further assume that u is $C_0^1(\Omega)$, then*

$$\widetilde{u_\varepsilon} - uw_\varepsilon \to 0 \textit{ strongly in } H_0^1(\Omega). \blacksquare \tag{5.2.13}$$

When this holds we can give a proof of Theorem 5.2.1 using Theorem 5.2.2 as follows.

Alternate Proof of Theorem 5.2.1: Set $r_{\varepsilon'} = \widetilde{u_{\varepsilon'}} - uw_{\varepsilon'}$. By Proposition 5.2.2, $r_{\varepsilon'} \to 0$ strongly in $H_0^1(\Omega)$. Therefore,

$$\begin{aligned} I_{\varepsilon'} \equiv \int_{\Omega_{\varepsilon'}} B\nabla u_{\varepsilon'}.\nabla u_{\varepsilon'}\,dx &= \int_\Omega B\nabla\widetilde{u_{\varepsilon'}}.\nabla\widetilde{u_{\varepsilon'}}\,dx \\ &= \int_\Omega B\nabla(uw_{\varepsilon'}).\nabla(uw_{\varepsilon'})\,dx + o(1). \end{aligned}$$

Now, since $\nabla w_{\varepsilon'} \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and B and u are bounded functions on Ω,

$$\begin{aligned} \int_\Omega B\nabla(uw_{\varepsilon'}).\nabla(uw_{\varepsilon'})\,dx &= \int_\Omega B\nabla w_{\varepsilon'}.\nabla w_{\varepsilon'}u^2\,dx \\ &\quad + \int_\Omega B\nabla u.\nabla u w_{\varepsilon'}^2\,dx + o(1) \end{aligned}$$

Note that, in Theorem 5.2.2, it is possible to take $\phi = u$ and the same proof works. Therefore, using Theorem 5.2.2 and the strong convergence of $w_{\varepsilon'}$ to 1 in $L^2(\Omega)$, we conclude that,

$$\lim_{\varepsilon' \to 0} I_{\varepsilon'} = < \mu_B, u^2 > + \int_\Omega B\nabla u.\nabla u \, dx.$$

This proves (5.2.10). The proof of (5.2.11) is similar. ■

5.3 The Homogenized Problem

Case $a_\varepsilon = O(c_\varepsilon)$: As remarked in Section 2.1, the function F^2 defined there and the space U_{ad} which corresponds to the particular choice of U_{ad}^ε satisfy the hypotheses of Lemma 2.1.1. We also remark that, by Rellich's compactness theorem, $w_\varepsilon \longrightarrow 1$ strongly in $L^2(\Omega)$; so, passing to the limit in the identity, $\chi_\varepsilon w_\varepsilon = w_\varepsilon$ we conclude that χ is the constant function which takes value the 1 in Ω. From Theorem 5.2.1 it is possible to conclude that the function F^1 defined below satisfies the remaining hypothesis of Lemma 2.1.1.

$$F^1(\theta) = \frac{1}{2}\int_\Omega B\nabla u.\nabla u \, dx + \int_\Omega u^2 \, d\mu_B,$$

where $u = u(\theta)$ solves the Dirichlet problem,

$$\left.\begin{aligned} -\Delta u + \mu u &= g + \theta \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}\right\}$$

Thus, the limiting optimal control problem is: minimize the cost functional

$$J(\theta) = \frac{1}{2}\int_\Omega B\nabla u.\nabla u \, dx + \int_\Omega u^2 \, d\mu_B + \frac{N}{2}\int_\Omega \theta^2 \, dx$$

over θ in U_{ad}, where $u = u(\theta)$ is as given above.

Case $a_\varepsilon << c_\varepsilon$: The same conclusions as in the previous case but both μ and μ_B are identically zero.

Case $a_\varepsilon >> c_\varepsilon$: We have seen that $\int_{\Omega_\varepsilon} B\nabla u_\varepsilon.\nabla u_\varepsilon \, dx \longrightarrow 0$. Thus, F^1 is the identically zero function. So, the limiting optimal control problem is: minimize the functional

$$J(\theta) = \frac{1}{2}\int_\Omega \frac{\theta^2}{\chi}\,dx \tag{5.3.1}$$

over $\theta \in U_{ad}$ where U_{ad} is one of (2.1.10), provided we assume that $\chi^{-1} \in L^\infty$. That is, the limiting optimal control is the projection of 0 onto U_{ad} in the space $L^2(\Omega, \frac{dx}{\chi})$.

Part II

Flow in porous media

Chapter 6

Correctors for a flow in a partially fissured medium

6.1 Introduction

A fissured medium consists of a porous and permeable *matrix* interlaced, on a fine scale, by a system of highly permeable *fissures*. Fluid flow in such a medium takes place, primarily, through the fissures. The fissured medium is said to be *totally fissured* if the matrix is broken up into disjoint cells by the fissures. In this case, there is no direct flow through the matrix but only an exchange of fluids between the cells and the surrounding fissures. If, on the other hand, the matrix is connected there is a global flow through the matrix as well. This is the *partially fissured case* and this is the one we will consider.

The exact microscopic model for flow in a fissured medium, written as a classical interface problem, is both analytically and numerically intractable. But, by modelling the flow on two separate scales, one microscopic and the other macroscopic, the approximate global behaviour of the flow can be obtained from a knowledge of the flow in a typical cell and the flow in a homogenized problem. Such a model for flow in a partially fissured medium was considered by Douglas, Peszyńyska and Showalter [20] assuming the diffusion operator to be linear; and later, by Clark and

Showalter [14], assuming the diffusion operator to be quasi-linear. The models were homogenized in the framework of two-scale convergence, while assuming weak monotonicity conditions on the quasi-linear operator. Though, by this, it was shown that the first two terms in the asymptotic expansion of the flow approximate the flow of the exact micro-model, the approximation was only in a weak sense. By assuming that the quasi-linear operator is strongly monotone, we show that the approximation is strong. This is a *corrector result.*

Such a corrector result is proved for the homogenization of quasi-linear equations

$$-\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)\right) = f \tag{6.1.1}$$

by Dal Maso and Defranceschi [17] under some strong monotonicity conditions on the function a. Later, the proof of the corrector result was greatly simplified using the two-scale convergence method by Allaire [1]. The results of these papers provide the inspiration for the result proved in this chapter.

The plan of this chapter is as follows. In Section 6.2, we describe the micro-model for flow in a partially fissured medium as considered in [14]. In Section 6.3, we recall the results on the homogenization of this model, obtained by Clark and Showalter in [14], under weak monotonicity of the diffusion operator. In Section 6.4, we obtain corrector results, under strong monotonicity conditions.

The results of this chapter appeared in Rajesh [33].

6.2 The Micro-Model

The flow domain Ω is a bounded open set in $\mathbb{R}^n$. It is made up of fissures Ω_1^ε and a matrix Ω_2^ε, both having a periodic micro structure. The micro structure is obtained by tesellating $\mathbb{R}^n$ with the cells εY where $Y = [0,1]^n$; the region occupied by the fissure and the matrix in the cell Y are denoted by Y^1 and Y^2, respectively. The micro-structure on Ω is obtained by restriction of the micro-structure on $\mathbb{R}^n$ to Ω. It is assumed that Ω_ε^1 and Ω_ε^2 are connected.

$\chi_j(y)$ will denote the characteristic function of Y_j (j=1, 2) extended Y-periodically to all of R^N. Then, $\chi^j_\varepsilon(\frac{x}{\varepsilon})$ is clearly the characteristic function of Ω^j_ε; this will simply be denoted by χ^j_ε. $\Gamma^\varepsilon_{1,2} = \partial\Omega^\varepsilon_1 \bigcap \partial\Omega^\varepsilon_2 \bigcap \Omega$ will denote the interface of Ω^ε_1 with Ω^ε_2 which is interior to Ω and, $\Gamma_{1,2} = \partial Y_1 \bigcap \partial Y_2 \bigcap Y$ will denote the corresponding interface in the reference cell. We also set $\Omega^\varepsilon_3 \equiv \Omega^\varepsilon_2$, $Y_3 \equiv Y_2$, and $\chi_3 \equiv \chi_2$, to be used to simplify notation at times.

Let $\mu_j : R^N \times R^N \longrightarrow R^N$ $(j = 1, 2, 3)$ be Carathéodory functions Y-periodic in the ~~second~~ first variable for which there exist positive constants k, C, c_0 and $1 < p < \infty$ such that for every ξ, $\eta \in R^N$ and a.e. $y \in Y$

$$|\mu_j(y\,,\,\xi)| \;\leq\; C|\,\xi\,|^{p-1} + k \tag{6.2.1}$$

$$(\mu_j(y\,,\,\xi) - \mu_j(y\,,\,\eta))\,.\,(\xi - \eta)) \;\geq\; 0 \tag{6.2.2}$$

$$\mu_j(y\,,\,\xi)\,.\,\xi \;\geq\; c_0|\,\xi\,|^p - k. \tag{6.2.3}$$

Let $c_j \in C_\sharp(Y)$ $(j = 1, 2, 3)$ be continuous Y-periodic functions on R^N such that

$$0 < c_0 \leq c_j \leq C. \tag{6.2.4}$$

The exact microscopic model for diffusion in a partially fissured medium is given by the system

$$c_1(\frac{x}{\varepsilon})\frac{\partial u^\varepsilon_1}{\partial t} - \operatorname{div}\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u^\varepsilon_1\right) \;=\; 0 \quad \text{in } \Omega^\varepsilon_1 \tag{6.2.5}$$

$$c_2(\frac{x}{\varepsilon})\frac{\partial u^\varepsilon_2}{\partial t} - \operatorname{div}\mu_2\left(\frac{x}{\varepsilon}\,,\,\nabla u^\varepsilon_2\right) \;=\; 0 \quad \text{in } \Omega^\varepsilon_2 \tag{6.2.6}$$

$$c_3(\frac{x}{\varepsilon})\frac{\partial u^\varepsilon_3}{\partial t} - \varepsilon\operatorname{div}\mu_3\left(\frac{x}{\varepsilon}\,,\,\varepsilon\nabla u^\varepsilon_3\right) \;=\; 0 \quad \text{in } \Omega^\varepsilon_2 \tag{6.2.7}$$

$$\alpha u^\varepsilon_2 + \beta u^\varepsilon_3 \;=\; u^\varepsilon_1 \quad \text{on } \Gamma^\varepsilon_{1,2} \tag{6.2.8}$$

$$\alpha\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u^\varepsilon_1\right)\,.\,\nu^\varepsilon_1 \;=\; \mu_2\left(\frac{x}{\varepsilon}\,,\,\nabla u^\varepsilon_2\right).\nu^\varepsilon_1 \tag{6.2.9}$$

$$\beta\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u^\varepsilon_1\right)\,.\,\nu^\varepsilon_1 \;=\; \varepsilon\mu_3\left(\frac{x}{\varepsilon}\,,\,\nabla\varepsilon u^\varepsilon_3\right).\nu^\varepsilon_1 \tag{6.2.10}$$

where the last two conditions hold on $\Gamma^\varepsilon_{1,2}$. We have the homogeneous Neumann condition on the external boundary

$$\mu_1\left(\frac{x}{\varepsilon}, \nabla u_1^\varepsilon\right).\nu = 0 \text{ on } \partial\Omega_1^\varepsilon \cap \partial\Omega \quad (6.2.11)$$

$$\mu_2\left(\frac{x}{\varepsilon}, \nabla u_2^\varepsilon\right).\nu = 0 \text{ on } \partial\Omega_2^\varepsilon \cap \partial\Omega \quad (6.2.12)$$

$$\mu_3\left(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon\right).\nu = 0 \text{ on } \partial\Omega_2^\varepsilon \cap \partial\Omega \quad (6.2.13)$$

where ν denotes the outward normal on $\partial\Omega$.

The system is completed by the initial conditions

$$u_j^\varepsilon(0\,,\,.) = u_j^0 \in L^2(\Omega),\ 1 \leq j \leq 3. \quad (6.2.14)$$

$u_1^\varepsilon(x,t)$ is the flow in the fissures Ω_1^ε with the flux given by $-\mu_1\left(\frac{x}{\varepsilon}, \nabla u_1^\varepsilon\right)$. The flow in the matrix has two components: $u_2^\varepsilon(x,t)$ with the flux $-\mu_2\left(\frac{x}{\varepsilon}, \nabla u_2^\varepsilon\right)$, is the usual flow through the matrix and; the slow scale flow $u_3^\varepsilon(x,t)$ with flux $-\varepsilon\mu_3\left(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon\right)$, leading to local storage in the matrix. The "total flow " in the matrix is $\alpha u_2^\varepsilon + \beta u_3^\varepsilon$, where $\alpha + \beta = 1$ with $\alpha \geq 0, \beta > 0$. (6.2.8) represents the continuity of flow across the interface and (6.2.9), (6.2.10) determine the partition of flux across the interface.

We now describe the variational formulation needed to study the well posedness of the Cauchy problem. The *state space* is the Hilbert space

$$H_\varepsilon \equiv L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon) \times L^2(\Omega_2^\varepsilon)\ (= L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon)^2)$$

equipped with the inner product

$$([u_1,\, u_2,\, u_3], [\phi_1,\, \phi_2,\, \phi_3])_{H_\varepsilon} = \sum_{j=1}^{3} \int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})\, u_j(x)\, \phi_j(x)\, dx.$$

Define the *energy space*

$$B_\varepsilon \equiv H_\varepsilon \cap \{[\overrightarrow{u}] \in W^{1,p}(\Omega_1^\varepsilon) \times W^{1,p}(\Omega_2^\varepsilon)^2 : u_1 = \alpha u_2 + \beta u_3 \text{ on } \Gamma_{1,2}^\varepsilon\}$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$. B_ε is a Banach space with the norm

$$\|\,[u_1,\, u_2,\, u_3]\|_{B_\varepsilon} \equiv \sum_{j=1}^{3} \|\,\chi_j^\varepsilon\, u_j\,\|_{L^2(\Omega)} + \sum_{j=1}^{3} \|\,\chi_j^\varepsilon\,\nabla\, u_j\,\|_{L^p(\Omega)}.$$

Define the operator $A_\varepsilon : B_\varepsilon \longrightarrow B_\varepsilon'$ (where B_ε' denotes the dual of B_ε) by,

$$A_\varepsilon([u_1, u_2, u_3])([\phi_1, \phi_2, \phi_3]) \equiv \sum_{j=1}^{2} \int_{\Omega_j^\varepsilon} \mu_j(\tfrac{x}{\varepsilon}, \nabla u_j) \,.\, \nabla\phi_j \, dx + \int_{\Omega_2^\varepsilon} \mu_3(\tfrac{x}{\varepsilon}, \varepsilon\nabla u_3) \,.\, \varepsilon\nabla\phi_3 \, dx$$

for $[u_1, u_2, u_3]$, $[\phi_1, \phi_2, \phi_3] \in B_\varepsilon$.

Let $V_\varepsilon \equiv \{\overrightarrow{u^\varepsilon} \in L^p([0,T]; B_\varepsilon) : (\overrightarrow{u^\varepsilon})' \in L^q([0,T]; B_\varepsilon')\}$, q being $p/(p-1)$.

For $\varepsilon > 0$, the Cauchy problem is equivalent to finding a solution $\overrightarrow{u^\varepsilon} \in V_\varepsilon$ to the problem

$$\frac{d\overrightarrow{u^\varepsilon}}{dt} + A_\varepsilon \overrightarrow{u^\varepsilon} = 0 \text{ in } L^q([0,T]; B_\varepsilon') \tag{6.2.15}$$

$$\overrightarrow{u^\varepsilon}(0) = \overrightarrow{u^0} \text{ in } H_\varepsilon \tag{6.2.16}$$

and this problem is well-posed, thanks to the conditions (6.2.1)-(6.2.3) (cf. Showalter [34]). We end with an identity(cf. [14]),

$$\frac{1}{2}\left\|\overrightarrow{u^\varepsilon}(T)\right\|_{H_\varepsilon}^2 - \frac{1}{2}\left\|\overrightarrow{u^\varepsilon}(0)\right\|_{H_\varepsilon}^2 + \int_0^T A_\varepsilon(\overrightarrow{u^\varepsilon})(\overrightarrow{u^\varepsilon})dt = 0. \tag{6.2.17}$$

6.3 Homogenization

The micro-model presented in the previous section was homogenized in [14], using two-scale convergence; we recall the main results.

In this case, the definition of two-scale convergence (cf. [1], [14]) is the following.

Definition 6.3.1 *A function, $\psi(t,x,y) \in L^q([0,T] \times \Omega, C_\sharp(Y))$, which is Y-periodic in y and satisfies*

$$\lim_{\varepsilon \longrightarrow 0} \int_0^T \int_\Omega \psi\left(t, x, \frac{x}{\varepsilon}\right)^q dx\, dt = \int_0^T \int_\Omega \int_Y \psi(t,x,y)^q \, dy\, dx\, dt$$

is called an admissible *test function.* ■

Definition 6.3.2 *A sequence f^ε in $L^p([0,T] \times \Omega))$ two-scale converges to a function $f(t,x,y) \in L^p([0,T] \times \Omega \times Y)$ if for any admissible test function $\psi(t,x,y)$,*

$$\lim_{\varepsilon \longrightarrow 0} \int_0^T \int_\Omega f^\varepsilon(t,x)\, \psi\left(t, x, \frac{x}{\varepsilon}\right) dx\, dt = \int_0^T \int_\Omega \int_Y f(t,x,y)\, \psi(t,x,y)\, dy\, dx\, dt$$

We write $f^\varepsilon \xrightarrow{2-s} f$. ■

Remark 6.3.1 *The space of admissible functions used in the definition of two-scale convergence differs from the one used in Chapter 3. But this is justified by Remark 3.2.3. Two-scale convergence is also obtainable for sequences in L^p spaces, $1 < p < \infty$ (cf. Allaire [1]).* ■

Proposition 6.3.1 *[14] Let $\overrightarrow{u^\varepsilon}$ be the solution of the Cauchy problem (6.2.5)-(6.2.14). The following estimate holds*

$$\sum_{j=1}^{2} \|\chi_j^\varepsilon \nabla u_j^\varepsilon\|_{p,\Omega_T}^p + \|\chi_2^\varepsilon \varepsilon \nabla u_3^\varepsilon\|_{p,\Omega_T}^p \leq \frac{C}{2c} \sum_{j=1}^{3} \|u_j^0\|_{2,\Omega}^2 . \blacksquare \tag{6.3.1}$$

Proposition 6.3.2 *[14] Let $\overrightarrow{u^\varepsilon}$ be the solution of the Cauchy problem (6.2.5)-(6.2.14). There exist functions u_j in $L^p([0,T]; W^{1,p}(\Omega))$, $j = 1,2$ and functions U_j in $L^p([0,T] \times \Omega; W_\sharp^{1,p}(Y_j)/R)$, $j = 1,2,3$ such that, for a subsequence of $\overrightarrow{u^\varepsilon}$, (to be indexed by ε again) the following hold:*

$$\begin{aligned}
\chi_j^\varepsilon u_j^\varepsilon &\xrightarrow{2-s} \chi_j(y) u_j(t,x), \quad j = 1,2, \\
\chi_2^\varepsilon u_3^\varepsilon &\xrightarrow{2-s} \chi_2(y) U_3(t,x,y), \\
\chi_j^\varepsilon \nabla u_j^\varepsilon &\xrightarrow{2-s} \chi_j(y)(\nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)), \quad j = 1,2, \\
\varepsilon \chi_2^\varepsilon \nabla u_3^\varepsilon &\xrightarrow{2-s} \chi_2(y) \nabla_y U_3(t,x,y), \\
\chi_j^\varepsilon \mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon) &\xrightarrow{2-s} \chi_j(y) \mu_j(y, \nabla_x u_j + \nabla_y U_j), \quad j = 1,2, \\
\chi_2^\varepsilon \mu_3(\frac{x}{\varepsilon}, \varepsilon \nabla u_3^\varepsilon) &\xrightarrow{2-s} \chi_2(y) \mu_3(y, \nabla_y U_3), \\
\chi_j^\varepsilon u_j^\varepsilon(T,x) &\xrightarrow{2-s} \chi_j(y) u_j(T,x), \quad j = 1,2, \\
\chi_2^\varepsilon u_3^\varepsilon(T,x) &\xrightarrow{2-s} \chi_2(y) U_3(T,x,y) \text{ and,} \\
u_1(t,x) &= \alpha u_2(t,x) + \beta U_3(t,x,y) \text{ for all } y \in \Gamma_{1,2}. \quad \blacksquare
\end{aligned}$$

Proposition 6.3.3 *[14] The functions u_1, u_2, U_1, U_2, U_3 satisfy the homogenized sys-*

tem

$$
\begin{aligned}
&-\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j} c_j(y)u_j\frac{\partial\phi_j}{\partial t}\,dy\,dx\,dt-\int_0^T\int_\Omega\int_{Y_2} c_3(y)U_3\frac{\partial\Phi_3}{\partial t}\,dy\,dx\,dt\\
&-\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)u_j^0\,\phi_j(0,x)\,dy\,dx-\int_\Omega\int_{Y_2} c_3(y)u_3^0\,\Phi_3(0,x,y)\,dy\,dx\\
&+\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\mu_j(y,\nabla_x u_j+\nabla_y U_j).(\nabla_x\phi_j+\nabla_y\Phi_j)\,dy\,dx\,dt\\
&+\int_0^T\int_\Omega\int_{Y_2}\mu_3(y,\nabla_y U_3).(\nabla_y\Phi_3)\,dy\,dx\,dt=0
\end{aligned}
\tag{6.3.2}
$$

for all

$$
\begin{aligned}
\phi_j(t,x) &\in L^p([0,T];W^{1,p}(\Omega)),\ j=1,2\\
\Phi_j(t,x,y) &\in L^p([0,T]\times\Omega;W_\sharp^{1,p}(Y_j)),\ j=1,2,3
\end{aligned}
$$

satisfying

$$
\begin{aligned}
\frac{\partial\phi_j}{\partial t} &\in L^q([0,T];W^{-1,q}(\Omega)),\ j=1,2\\
\frac{\partial\Phi_j}{\partial t} &\in L^q([0,T]\times\Omega;(W_\sharp^{1,p}(Y_j))'),\ j=1,2,3
\end{aligned}
$$

$$\beta\Phi_3(t,x,y)=\phi_1(t,x)-\alpha\phi_2(t,x)\ \textit{for all}\ y\in\Gamma_{1,2}\ \textit{and,}$$

$$\phi_1(T,x)=\phi_2(T,x)=\Phi_3(T,x,y)=0.\quad\blacksquare$$

The strong form of the homogenized problem has the following description. The *state space* is $H\equiv L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega\times Y_2)$ equipped with the scalar product

$$
\begin{aligned}
(\vec{\psi},\vec{\phi})_H &= \sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)\psi_j(x)\phi_j(x)\,dy\,dx\\
&\quad+\int_\Omega\int_{Y_2} c_3(y)\Psi_3(x,y)\Phi_3(x,y)\,dy\,dx
\end{aligned}
$$

for every $\vec{\psi}=[\psi_1,\psi_2,\Psi_3]$, $\vec{\phi}=[\phi_1,\phi_2,\Phi_3]\in H$. Define the *energy space*,

$$
\begin{aligned}
B\equiv\{[\phi_1,\phi_2,\Phi_3]\in\ & H\cap W^{1,p}(\Omega)\times W^{1,p}(\Omega)\times L^2(\Omega;W_\sharp^{1,p}(Y_2)/R)\\
&:\ \beta\Phi_3(x,y)=\phi_1(x)-\alpha\phi_2(x,y)\text{ for all }y\in\Gamma_{1,2}\}
\end{aligned}
$$

and the corresponding *evolution space* $V\equiv L^p([0,T];B)$.

Proposition 6.3.4 *[14] $\overrightarrow{u} = [u_1, u_2, U_3] \in V$ and is the solution of the strong homogenized system,*

$$\left(\int_{Y_1} c_1(y)\,dy\right)\frac{\partial u_1}{\partial t}(t,x) + \frac{1}{\beta}\frac{\partial}{\partial t}\left(\int_{Y_2} c_3(y)U_3(t,x,y)\,dy\right) = \operatorname{div}_x\left(\int_{Y_1} \mu_1(y, \nabla_x u_1 + \nabla_y U_1)\,dy\right) \quad (6.3.3)$$

$$\left(\int_{Y_2} c_2(y)\,dy\right)\frac{\partial u_2}{\partial t}(t,x) - \frac{\alpha}{\beta}\frac{\partial}{\partial t}\left(\int_{Y_2} c_3(y)U_3(t,x,y)\,dy\right) = \operatorname{div}_x\left(\int_{Y_2} \mu_2(y, \nabla_x u_2 + \nabla_y U_2)\,dy\right) \quad (6.3.4)$$

$$c_3(y)\frac{\partial U_3(t,x,y)}{\partial t} - \operatorname{div}_y \mu_3(y, \nabla U_3(t,x,y)) = 0 \quad (6.3.5)$$

where $U_3(t,x,y)$ and $\mu_3(y, \nabla_y U_3(t,x,y)).\nu$ are Y-periodic and,

$$\beta U_3(t,x,y) = u_1(t,x) - \alpha u_2(t,x) \ \textit{for}\ y \in \Gamma_{1,2} \quad (6.3.6)$$

with boundary conditions

$$\int_{Y_1} \mu_1(y, \nabla_x u_1 + \nabla_y U_1)\,dy.\nu = 0 \ \textit{on}\ \partial\Omega \quad (6.3.7)$$

$$\int_{Y_2} \mu_2(y, \nabla_x u_2 + \nabla_y U_2)\,dy.\nu = 0 \ \textit{on}\ \partial\Omega \quad (6.3.8)$$

and initial conditions

$$u_j(0,x) = u_j^0(x)\ j = 1,2;\ U_3(0,x,y) = u_3^0(x). \quad (6.3.9)$$

The functions $U_j(t,x,y)$ solve the cell problems,

$$\operatorname{div}_y \mu_j(y, \nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)) = 0 \ \textit{for}\ y \in Y_j \quad (6.3.10)$$

$$\mu_j(y, \nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)).\nu = 0 \ \textit{on}\ \Gamma_{1,2}\ \textit{and} \quad (6.3.11)$$

Y-periodic on $\Gamma_{2,2}$, for $j = 1,2$. In the above, t, x are treated as parameters and the cell equations are solved. ■

For $\xi \in R^N$, define the following functions;

$$\lambda_j(\xi) = \int_{Y_j} \mu_j(y, \xi + \nabla_y V_j^\xi(y)\, dy, \; j = 1, 2 \tag{6.3.12}$$

where V_j^ξ is the Y-periodic solution of

$$\operatorname{div}_y \mu_j(y, \xi + \nabla_y V_j^\xi(y)) = 0 \text{ in } Y_j \tag{6.3.13}$$

$$\mu_j(y, \xi + \nabla_y V_j^\xi(y)).\nu = 0 \text{ on } \Gamma_{1,2} \tag{6.3.14}$$

Then, because of (6.3.10), (6.3.11), the right hand sides in (6.3.3), (6.3.4) can be replaced by the functions $\operatorname{div}_x \lambda_1(\nabla_x u_1(t,x))$ and $\operatorname{div}_x \lambda_2(\nabla_x u_2(t,x))$ respectively. Also the left hand sides of (6.3.7), (6.3.8) can be replaced by $\lambda_1(\nabla_x u_1).\nu$ and $\lambda_2(\nabla_x u_2).\nu$ respectively.

Remark 6.3.2 *We note that the functions λ_j can be interpreted as the integrands in the Γ − limit of the functionals*

$$F_{j,\varepsilon}(\nabla v) = \int_\Omega \chi_j^\varepsilon \mu_j(\frac{x}{\varepsilon}, \nabla v)\, dx\,.$$

In fact, $\Gamma - \lim F_{j,\varepsilon}(\nabla v) = \int_\Omega \lambda_j(\nabla v)\, dx$ (cf. Dal Maso [16]). Further, the functions λ_j, $j = 1, 2$ satisfy conditions (6.2.1)-(6.2.3) for the same p but maybe for different constants $\tilde{k}, \tilde{C}, \tilde{c}_0$ (cf. [17], [10]). ■

Proposition 6.3.5 *[14] The following* energy identity *holds*

$$\begin{aligned}
\frac{1}{2}\sum_{j=1}^{2} \int_\Omega\int_{Y_j} c_j(y)|u_j(T,x)|^2\, dy\, dx + \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)|U_3(T,x,y)|^2\, dy\, dx \\
-\frac{1}{2}\sum_{j=1}^{2} \int_\Omega\int_{Y_j} c_j(y)|u_j^0(x)|^2\, dy\, dx - \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)|u_3^0(x)|^2\, dy\, dx \\
+\sum_{j=1}^{2}\int_0^T \int_\Omega\int_{Y_j} \mu_j(y, \nabla_x u_j + \nabla_y U_j).(\nabla_x u_j + \nabla_y U_j)\, dy\, dx\, dt \\
+\int_0^T \int_\Omega\int_{Y_2} \mu_3(y, \nabla_y U_3).\nabla_y U_3\, dy\, dx\, dt = 0.
\end{aligned}$$ ■

6.4 Correctors

We now prove corrector results for the gradient of flows under stronger hypotheses on μ_j's than (6.2.1)-(6.2.3). Let $k_1, k_2 > 0$ be constants and assume now that the μ_j's are Carathéodory functions, Y-periodic in the second variable, satisfying for $\xi, \eta \in R^N$ with $|\xi| + |\eta| > 0$ and a.e. $y \in Y$:

$$\mu_j(y,0) = 0, \tag{6.4.1}$$

$$|\mu_j(y,\xi) - \mu_j(y,\eta)| \leq k_1(|\xi| + |\eta|)^{p-2}|\xi - \eta|, \tag{6.4.2}$$

$$(\mu_j(y,\xi) - \mu_j(y,\eta)).(\xi - \eta) \geq k_2(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2. \tag{6.4.3}$$

Remark 6.4.1 *Note that (6.4.1) and (6.4.2) imply*

$$|\mu_j(y,\xi)| \leq k_1|\xi|^{p-1} \tag{6.4.4}$$

and, (6.4.1) and (6.4.3) imply

$$\mu_j(y,\xi).\xi \geq k_2|\xi|^p. \tag{6.4.5}$$

Thus, the new hypotheses are indeed stronger than the original hypotheses on μ_j's. Moreover,

$$(\mu_j(y,\xi) - \mu_j(y,\eta)).(\xi - \eta) \geq k_2|\xi - \eta|^p \quad \text{if } p \geq 2 \tag{6.4.6}$$

$$|\mu_j(y,\xi) - \mu_j(y,\eta)| \leq k_1|\xi - \eta|^{p-1} \quad \text{if } 1 < p < 2. \tag{6.4.7}$$

These inequalities follow from (6.4.3) and (6.4.2) and triangle inequality in R^N. ■

Remark 6.4.2 *An example of μ_j satisfying (6.4.1)- (6.4.3) is $\mu_j = |\xi|^{p-2}\xi$, i.e. the corresponding diffusion operator is the p-Laplacian. Let Γ, γ be positive constants. The following class of functions, $f \in C^0(\overline{\Omega} \times R^N; R^N) \cap C^1(\Omega \times R^N \setminus \{0\}; R^N)$, which satisfy condition (6.4.1) and*

$$\sum_{j,j=1}^{N} |\frac{\partial f_j}{\partial \eta_i}|(x,\eta) \leq \Gamma|\eta|^{p-2}$$

$$\sum_{j,j=1}^{N} |\frac{\partial f_j}{\partial \eta_i}|(x,\eta)\xi_i\xi_j \geq \gamma|\eta|^{p-2}|\xi|^2$$

for all $x \in \Omega, \eta \in R^N \setminus \{0\}$ *and* $\xi \in R^N$, *also satisfy (6.4.1)-(6.4.3) (cf. Damascelli [18]).* ■

Let $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$ be the solution of the Cauchy problem (6.2.5)- (6.2.14) and let $[u_1, u_2, U_1, U_2, U_3]$ be as in Section 6.3. We will denote $[0,T] \times \Omega$ by Ω_T. Define the sequence of functions

$$\xi_j(t,x,y) \equiv \chi_j(y)(\nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)),\ j = 1,2, \quad (6.4.8)$$

$$\xi_3(t,x,y) \equiv \chi_2(y)\nabla_y U_3(t,x,y) \quad (6.4.9)$$

and let,

$$\xi_j^\varepsilon(t,x) \equiv \xi_j(t,x,\frac{x}{\varepsilon}),\ j = 1,2,3. \quad (6.4.10)$$

The main theorems of this Chapter are the following:

Theorem 6.4.1 *Let* ξ_j^ε*'s be as above and assume that the functions* $\nabla_y U_j$, $j = 1,2,3$ *are admissible (cf. Definition 6.3.1), then*

$$\overline{\lim}_{\varepsilon\to 0} \left\| \chi_j(\frac{x}{\varepsilon}) \left(\nabla u_j^\varepsilon(t,x) - \xi_j^\varepsilon(t,x)\right) \right\|_{p,\Omega_T} \longrightarrow 0,\ j{=}1,2,$$

$$\overline{\lim}_{\varepsilon\to 0} \left\| \chi_2(\frac{x}{\varepsilon}) \left(\varepsilon\nabla u_3^\varepsilon(t,x) - \xi_3^\varepsilon(t,x)\right) \right\|_{p,\Omega_T} \longrightarrow 0. ■$$

Theorem 6.4.2 *Under the same assumptions as in Theorem 6.4.1*

$$\overline{\lim}_{\varepsilon\to 0} \left\| \chi_j(\frac{x}{\varepsilon}) \left(\mu_j\left(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon\right) - \mu_j\left(\frac{x}{\varepsilon}, \xi_j^\varepsilon(t,x)\right)\right) \right\|_{q,\Omega_T} \longrightarrow 0,\ j{=}1,2,$$

$$\overline{\lim}_{\varepsilon\to 0} \left\| \chi_2(\frac{x}{\varepsilon}) \left(\mu_3\left(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon\right) - \mu_3\left(\frac{x}{\varepsilon}, \xi_3^\varepsilon(t,x)\right)\right) \right\|_{q,\Omega_T} \longrightarrow 0. ■$$

Remark 6.4.3 *Theorem 6.4.1 shows that* $\chi_j(\frac{x}{\varepsilon})\nabla_x(u_j(t,x) + \varepsilon U_j(t,x,\frac{x}{\varepsilon}))$ *strongly approximates* $\chi(\frac{x}{\varepsilon})\nabla u_j^\varepsilon(t,x)$ *for* $j = 1,2$ *in* $L^p([0,T]\times\Omega)$, *whereas Proposition 6.3.2 only implies that these two sequences have the same two-scale limit and hence, the same weak limit in* L^P. *Similarly, for the third component of the flow. Theorem 6.4.2 is about a strong approximation for the flux terms. The utility of the corrector results* *lie in the fact that the approximations involve the homogenized Cauchy problem and cell problems which are computationally simpler compared to the original Cauchy problem. In this context, it is desirable to get order of ε estimates for the corrector results and, this is still open.* ■

We first prove a few lemmas yielding some limits and estimates required in proving Theorems 6.4.1 and 6.4.2.

Henceforth, M will denote a generic constant which does not depend on ε, but probably on p, k_1, k_2, c_0, C, and the L^2 norm of the initial vector $\overrightarrow{u^0}$. Let $0 < \kappa < 1$ be a constant and $\Phi_j(t,x,y)$ be admissible test functions such that

$$\sum_{j=1}^{3} \|\nabla_y U_j - \Phi_j\|^p_{p,[0,T]\times\Omega\times Y_j} \leq \kappa.$$

Note that,

$$\Phi_j(t,x,\frac{x}{\varepsilon}) \overset{2-s}{\longrightarrow} \Phi_j(t,x,y)$$

for j=1,2,3. Define the functions:

$$\eta^\varepsilon_j(t,x) = \chi_j(\frac{x}{\varepsilon})(\nabla_x u_j(t,x) + \Phi_j(t,x,\frac{x}{\varepsilon})),\ j=1,2 \tag{6.4.11}$$

$$\eta^\varepsilon_3(t,x) = \chi_2(\frac{x}{\varepsilon})\Phi_3(t,x,\frac{x}{\varepsilon}). \tag{6.4.12}$$

Then we note that the functions $\eta^\varepsilon_j(t,x)$ and $\mu^\varepsilon_j(\frac{x}{\varepsilon}, \eta^\varepsilon_j(t,x))$ arise from admissible test functions and we have the following two-scale convergence (cf. [14]),

$$\begin{aligned}
\eta^\varepsilon_j &\overset{2-s}{\longrightarrow} \chi_j(y)(\nabla_x u_j(t,x) + \Phi_j(t,x,y)) \doteq \eta_j(t,x,y),\ j=1,2,\\
\eta^\varepsilon_3 &\overset{2-s}{\longrightarrow} \chi_2(y)\Phi_3(t,x,y) \doteq \eta_3(t,x,y),\\
\mu_j(\frac{x}{\varepsilon},\eta^\varepsilon_j) &\overset{2-s}{\longrightarrow} \chi_j(y)\,\mu_j(y\,,\,\eta_j(t,x,y)),\ j=1,2,3.
\end{aligned}$$

Lemma 6.4.1 *(cf. Lemma 3.1 [17]) Let $1 < p < 2$ and $\phi_1, \phi_2 \in L^p(\Omega_T)^N$. Then,*

$$\begin{aligned}
\|\phi_1 - \phi_2\|^p_{p,\Omega_T} \leq& \left[\int_0^T \int_\Omega |\phi_1 - \phi_2|^2 (|\phi_1| + |\phi_2|)^{p-2}\chi\, dx\, dt\right]^{\frac{p}{2}}\\
&\times \left[\int_0^T \int_\Omega (|\phi_1| + |\phi_2|)^p\, dx\, dt\right]^{\frac{2-p}{2}}
\end{aligned}$$

where χ denotes the characteristic function of the set

$$\{(t,x) \in [0,T] \times \Omega : |\phi_1|(t,x) + |\phi_2|(t,x) > 0\}.$$

Proof: Multiply and divide the integrand in left hand side by $(|\phi_1| + |\phi_2|)^{(2-p)p/2}$ and apply Hölder's inequality to get the result. ■

Lemma 6.4.2

$$\sum_{j=1}^{2} \|\chi_j(y)(\nabla_x u_j + \nabla_y U_j)\|_p^p + \|\chi_2(y)\nabla_y U_3\|_p^p \leq \frac{C}{2k_2} \sum_{j=1}^{3} \|u_j^0\|_{2,\Omega}^2$$

Proof: Follows from the energy identity (Proposition 6.3.5) and (6.4.5). ■

Lemma 6.4.3 *Let $\xi_i, \eta_i, \xi_i^\varepsilon, \eta_i^\varepsilon, i = 1,2,3$ be functions as defined above. Then,*

$$\overline{\lim}_{\varepsilon\to 0} \int_0^T \int_{\Omega_i^\varepsilon} \left(\mu_i(\frac{x}{\varepsilon}, \nabla u_i^\varepsilon) - \mu_i(\frac{x}{\varepsilon}, \eta_i^\varepsilon)\right).(\nabla u_i^\varepsilon - \eta_i^\varepsilon)\, dx\, dt$$
$$\leq \textstyle\sum_{j=1}^3 \int_0^T \int_\Omega \int_{Y_j} (\mu_j(y,\xi_j) - \mu_j(y,\eta_j)).(\xi_j - \eta_j)\, dy\, dx\, dt$$

for i=1,2 and

$$\overline{\lim}_{\varepsilon\to 0} \int_0^T \int_{\Omega_2^\varepsilon} \left(\mu_3(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon) - \mu_3(\frac{x}{\varepsilon}, \eta_3^\varepsilon)\right).(\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon)\, dx\, dt$$
$$\leq \textstyle\sum_{j=1}^3 \int_0^T \int_\Omega \int_{Y_j} (\mu_j(y,\xi_j) - \mu_j(y,\eta_j)).(\xi_j - \eta_j)\, dy\, dx\, dt\,.$$

Proof: Denote the integrals appearing in the left hand sides of the above relations by $l_1^\varepsilon, l_2^\varepsilon$ and l_3^ε respectively. Then for i=1,2,3, using (6.2.17), we obtain,

$$\begin{aligned}
l_i^\varepsilon &\leq \sum_{j=1}^3 l_j^\varepsilon \\
&= \frac{1}{2}\sum_{j=1}^3 \int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^0(x)|^2\, dx - \frac{1}{2}\sum_{j=1}^3 \int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(T,x)|^2\, dx \\
&\quad - \sum_{j=1}^2 \int_0^T \int_{\Omega_j^\varepsilon} \mu_j(\frac{x}{\varepsilon}, \eta_j^\varepsilon).(\nabla u_j^\varepsilon - \eta_j^\varepsilon)\, dx\, dt \\
&\quad - \int_0^T \int_{\Omega_j^\varepsilon} \mu_3(\frac{x}{\varepsilon}, \eta_3^\varepsilon).(\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon)\, dx\, dt \\
&\quad - \sum_{j=1}^2 \int_0^T \int_{\Omega_j^\varepsilon} \mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon).\eta_j^\varepsilon\, dx\, dt - \int_0^T \int_{\Omega_2^\varepsilon} \mu_3(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon).\eta_3^\varepsilon\, dx\, dt
\end{aligned}$$

We now use the two-scale convergence properties of various functions discussed so far to pass to the limit. We get,

$$\begin{aligned}
\overline{\lim}_{\varepsilon\to 0}\sum_{j=1}^{3} l_j^\varepsilon &= \frac{1}{2}\sum_{j=1}^{3}\int_\Omega\int_{Y_j} c_j(y)|u_j^0(x)|^2\,dy\,dx \\
&\quad -\underline{\lim}_{\varepsilon\to 0}\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(T,x)|^2\,dx \\
&\quad -\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\mu_j(y,\eta_j).(\xi_j-\eta_j)\,dy\,dx\,dt \\
&\quad -\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\mu_j(y,\xi_j).\eta_j\,dy\,dx\,dt
\end{aligned}$$

The right hand side can be written as

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{3}\int_\Omega\int_{Y_j} c_j(y)|u_j^0(x)|^2\,dy\,dx - \underline{\lim}_{\varepsilon\to 0}\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(x,T)|^2\,dx \\
&\qquad +\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\left(\mu_j(y,\xi_j)-\mu_j(y,\eta_j)\right).(\xi_j-\eta_j)\,dy\,dx\,dt \\
&\qquad -\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\mu_j(y,\xi_j).\xi_j\,dy\,dx\,dt
\end{aligned}$$

which, using Proposition 6.3.5 to replace the last expression, is nothing but,

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)|u_j(T,x)|^2\,dy\,dx + \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)|U_3(T,x,y)|^2\,dy\,dx \\
&\qquad -\underline{\lim}_{\varepsilon\to 0}\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(x,T)|^2\,dx \\
&\qquad +\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\left(\mu_j(y,\xi_j)-\mu_j(y,\eta_j)\right).(\xi_j-\eta_j)\,dy\,dx\,dt
\end{aligned}$$

However, by standard arguments,

$$\begin{aligned}
&\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)|u_j(T,x)|^2\,dy\,dx + \int_\Omega\int_{Y_2} c_3(y)|U_3(T,x,y)|^2\,dy\,dx \\
&\qquad \le \underline{\lim}_{\varepsilon\to 0}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(x,T)|^2\,dx
\end{aligned}$$

This completes the proof. ■

Lemma 6.4.4 *Let ξ_j, η_j, κ be as before. Then,*

$$\sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} (\mu_j(y,\xi_j) - \mu_j(y,\eta_j)) .(\xi_j - \eta_j)\, dy\, dx\, dt \le M\kappa^{\delta(p)}$$

where

$$\delta(p) = \begin{cases} 1 & \text{if } 1 < p < 2, \\ \frac{2}{p} & \text{if } p \ge 2. \end{cases}$$

Proof: Let the left hand side of the estimate be denoted by S.

Case 1: $1 < p < 2$. Using (6.4.7) we get,

$$\begin{aligned} S &\le \sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} |(\mu_j(y,\xi_j) - \mu_j(y,\eta_j))||(\xi_j - \eta_j)|\, dy\, dx\, dt \\ &\le k_1 \sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} |\xi_j - \eta_j|^p\, dy\, dx\, dt \\ &\le M\kappa \end{aligned}$$

Case 2: $2 \le p$. Using (6.4.2) and Hölder's inequality we get,

$$\begin{aligned} S &\le \sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} |\mu_j(y,\xi_j) - \mu_j(y,\eta_j)||\xi_j - \eta_j|\, dy\, dx\, dt \\ &\le k_1 \sum_{j=1}^{3} \int_0^T \int_\Omega \int_{Y_j} |\xi_j - \eta_j|^2 (|\xi_j| + |\eta_j|)^{p-2}\, dy\, dx\, dt \\ &\le k_1 \sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^2 \left(\int_0^T \int_\Omega \int_{Y_j} (|\xi_j| + |\eta_j|)^p\, dy\, dx\, dt \right)^{\frac{p-2}{p}} \\ &\le k_1 \sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^2 \left(\|\xi_j\|_p + \|\eta_j\|_p\right)^{p-2} \\ &\le k_1 \left(\sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^p \right)^{\frac{2}{p}} \left(\sum_{j=1}^{3} (\|\xi_j\|_p + \|\eta_j\|_p)^p \right)^{\frac{p-2}{p}} \\ &\le k_1 \left(\sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^p \right)^{\frac{2}{p}} \left(\sum_{j=1}^{3} (2\|\xi_j\|_p + \|\xi_j - \eta_j\|_p)^p \right)^{\frac{p-2}{p}} \\ &\le k_1 2^{\frac{(p-2)(p-1)}{p}} \left(\sum_{j=1}^{3} \|\xi_j - \eta_j\|_p^p \right)^{\frac{2}{p}} \left(\sum_{j=1}^{3} \left(2^p \|\xi_j\|_p^p + \|\xi_j - \eta_j\|_p^p\right) \right)^{\frac{p-2}{p}} \end{aligned}$$

Therefore, by the estimate for the second term proved in Lemma 6.4.2, we get the result.■

Theorem 6.4.3

$$\overline{\lim}_{\varepsilon\to 0}\left\|\chi_j(\frac{x}{\varepsilon})\left(\nabla u_j^\varepsilon(t,x)-\eta_j^\varepsilon(t,x)\right)\right\|_{p,\Omega_T}^p \leq M\kappa^{r(p)},$$
$$\overline{\lim}_{\varepsilon\to 0}\left\|\chi_2(\frac{x}{\varepsilon})\left(\varepsilon\nabla u_3^\varepsilon(t,x)-\eta_3^\varepsilon(t,x)\right)\right\|_{p,\Omega_T}^p \leq M\kappa^{r(p)}$$

where

$$r(p)=\begin{cases}\frac{p}{2} & \text{if } 1<p<2,\\ \frac{2}{p} & \text{if } p\geq 2.\end{cases}$$

Proof: Case 1: $1<p<2$. We use Lemma 6.4.1 with the functions $\chi_j^\varepsilon\nabla u_j^\varepsilon$ and η_j^ε, $j=1,2$ to get,

$$\|\chi_j^\varepsilon\nabla u_j^\varepsilon-\eta_j^\varepsilon\|_{p,\Omega_T}^p$$
$$\leq (\textstyle\int_0^T\int_{\Omega_j^\varepsilon}|\nabla u_j^\varepsilon-\eta_j^\varepsilon|^2(|\nabla u_j^\varepsilon|+|\eta_j^\varepsilon|)^{p-2}\,dx\,dt)^{\frac{p}{2}}(\int_0^T\int_{\Omega_j^\varepsilon}(|\nabla u_j^\varepsilon|+|\eta_j^\varepsilon|)^p\,dxdt)^{\frac{2-p}{2}}$$

Therefore, using strong monotonicity (6.4.3), we get,

$$\|\chi_j^\varepsilon\nabla u_j^\varepsilon-\eta_j^\varepsilon\|_{p,\Omega_T}^p\leq k\left(\int_0^T\int_{\Omega_j^\varepsilon}\left(\mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon)\right).\left(\nabla u_j^\varepsilon-\eta_j^\varepsilon\right)\,dx\,dt\right)^{\frac{p}{2}}$$
$$\times(\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|_p^p+\|\eta_j^\varepsilon\|_p^p)^{\frac{2-p}{2}}$$

where $k=2^{\frac{(p-1)(2-p)}{2}}/k_2^{\frac{p}{2}}$. Similarly,

$$\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon\|_{p,\Omega_T}^p\leq k\left(\int_0^T\int_{\Omega_2^\varepsilon}\left(\mu_3(\frac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon)-\mu_3(\frac{x}{\varepsilon},\eta_3^\varepsilon)\right).(\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon)\,dx\,dt\right)^{\frac{p}{2}}$$
$$\times(\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|_p^p+\|\eta_3^\varepsilon\|_p^p)^{\frac{2-p}{2}}$$

Let,

$$S_1^\varepsilon = \sum_{j=1}^2\|\chi_j^\varepsilon\nabla u_j^\varepsilon-\eta_j^\varepsilon\|_{p,\Omega_T}^p+\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon\|_{p,\Omega_T}^p,$$
$$S_2^\varepsilon = \sum_{j=1}^2\int_0^T\int_{\Omega_j^\varepsilon}\left(\mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon)\right).(\nabla u_j^\varepsilon-\eta_j^\varepsilon)\,dx\,dt$$
$$+\int_0^T\int_{\Omega_2^\varepsilon}\left(\mu_3(\frac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon)-\mu_3(\frac{x}{\varepsilon},\eta_3^\varepsilon)\right).(\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon)\,dx\,dt \text{ and },$$
$$S_3^\varepsilon = \sum_{j=1}^2\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|_p^p+\|\eta_j^\varepsilon\|_p^p+\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|_p^p+\|\eta_3^\varepsilon\|_p^p.$$

Then, from the estimates for the individual terms in S_1^ε and a simple application of Hölder's inequality in $\mathbb{R}^3$, $S_1^\varepsilon \leq k(S_2^\varepsilon)^{\frac{p}{2}} \times (S_3^\varepsilon)^{\frac{2-p}{2}}$.

Note that η_j^ε arise from admissible test functions. Therefore,

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{j=1}^{3}\|\eta_j^\varepsilon\|_p^p &= \sum_{j=1}^{3}\|\eta_j\|^p_{p,[0,T]\times\Omega\times Y}\\ &\leq \sum_{j=1}^{3}2^{p-1}(\|\xi_j\|^p_{p,[0,T]\times\Omega\times Y}+\sum_{j=1}^{3}\|\eta_j-\xi_j\|^p_{p,[0,T]\times\Omega\times Y})\\ &\leq M\end{aligned}$$

where the last estimate follows from Lemma 6.4.2. Also by (6.2.17) and (6.4.5), we get,

$$\sum_{j=1}^{2}\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|_p^p+\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|_p^p \leq \frac{1}{2k_2}\left\|\overrightarrow{u^0}\right\|^2_{H_\varepsilon}\leq M$$

From this we conclude that, $\overline{\lim}_{\varepsilon\to 0}S_3^\varepsilon \leq M$. Therefore, taking lim sup as $\varepsilon \to 0$ and using Lemmas 6.4.3 and 6.4.4, we get

$$\overline{\lim}_{\varepsilon\to 0}S_1^\varepsilon \leq M\kappa^{\frac{p}{2}}.$$

This concludes the proof in this case.

Case 2: $2 \leq p$. From (6.4.6), we get,

$$|\nabla u_j^\varepsilon-\eta_j^\varepsilon|^p \leq \tfrac{1}{k_2}\left(\mu_j(\tfrac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\tfrac{x}{\varepsilon},\eta_j^\varepsilon)\right).(\nabla u_j^\varepsilon-\eta_j^\varepsilon)$$

Therefore, by integrating with respect to t in $[0,T]$ and x in Ω_j^ε, we get,

$$\left\|\chi_j^\varepsilon\nabla u_j^\varepsilon-\eta_j^\varepsilon\right\|^p_{p,\Omega_T}\leq \tfrac{1}{k_2}\int_0^T\int_{\Omega_j^\varepsilon}(\mu_j(\tfrac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\tfrac{x}{\varepsilon},\eta_j^\varepsilon)).(\nabla u_j^\varepsilon-\eta_j^\varepsilon)\,dx\,dt$$

Similarly,

$$\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon\|^p_{p,\Omega_T}\leq \tfrac{1}{k_2}\int_0^T\int_{\Omega_2^\varepsilon}(\mu_3(\tfrac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon)-\mu_3(\tfrac{x}{\varepsilon},\eta_3^\varepsilon)).(\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon)\,dx\,dt$$

We note that if S_1^ε and S_2^ε are defined as in the previous case, then $S_1^\varepsilon \leq \frac{1}{k_2}S_2^\varepsilon$. Passing to the limit, as before, we reach our conclusions. ■

Theorem 6.4.4

$$\overline{\lim}_{\varepsilon\to 0}\left\|\chi_j^\varepsilon\mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon)\right\|_{q,\Omega_T}^q \leq M\kappa^{s(p)}$$

$$\overline{\lim}_{\varepsilon\to 0}\left\|\chi_2^\varepsilon\mu_3(\frac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon)-\mu_3(\frac{x}{\varepsilon},\eta_3^\varepsilon)\right\|_{q,\Omega_T}^q \leq M\kappa^{s(p)}$$

where

$$s(p)=\begin{cases}\frac{p}{2} & \text{if } 1<p<2,\\ \frac{2}{p-1} & \text{if } p\geq 2.\end{cases}$$

Proof: We will prove only the first of these estimates, the other is proved similarly.

If $1<p<2$, by (6.4.2) and triangle inequality in R^N, we get,

$$\int_0^T\int_{\Omega_j^\varepsilon}|\mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon)|^q\,dx\,dt \leq k_1\int_0^T\int_{\Omega_j^\varepsilon}|\nabla u_j^\varepsilon-\eta_j^\varepsilon|^{q(p-1)}\,dx\,dt$$

Since $q(p-1)=p$, using the Theorem 6.4.3, the estimate follows easily. Let $2\leq p$. Then,

$$\int_0^T\int_{\Omega_j^\varepsilon}|\mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon)|^q\,dx\,dt$$
$$\leq k_1\int_0^T\int_{\Omega_j^\varepsilon}|\nabla u_j^\varepsilon-\eta_j^\varepsilon|^q\left(|\nabla u_j^\varepsilon|+|\eta_j^\varepsilon|\right)^{(p-2)q}\,dx\,dt$$

The right hand side, by Hölder's inequality,

$$\leq k_1 2^{p-1}\left(\int_0^T\int_{\Omega_j^\varepsilon}|\nabla u_j^\varepsilon-\eta_j^\varepsilon|^p\,dx\,dt\right)^{\frac{1}{p-1}}\left(\int_0^T\int_{\Omega_j^\varepsilon}(|\nabla u_j^\varepsilon|^p+|\eta_j^\varepsilon|^p)\,dx\,dt\right)^{\frac{p-2}{p-1}}$$
$$\leq M\left\|\chi_j^\varepsilon\nabla u_j^\varepsilon-\eta_j^\varepsilon\right\|_{p,\Omega_T}^{\frac{p}{p-1}}.$$

So, again using Theorem 6.4.3, we get the desired result. ■

Proof of Theorems 6.4.1 and 6.4.2: Since, $\nabla_y U_j$'s are assumed to be admissible test functions, we can take $\Phi_j\equiv\nabla_y U_j$. Thus, κ can be taken arbitrarily small and therefore, Theorem 6.4.1 follows from Theorem 6.4.3. Similarly, Theorem 6.4.2 follows from Theorem 6.4.4. ■

Remark 6.4.4 *The functions $\nabla_y U_j(t,x,y)$ will be admissible if we have C^1 regularity of U_j in the variable y. Even if the functions $\nabla_y U_j$ are not admissible, Theorems 6.4.3 and 6.4.4 are corrector results in their own right.* ■

Bibliography

[1] Allaire, G. Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23** (1992), 1482-1518.

[2] Allaire, G. and Briane, M. Multi-scale convergence and reiterated homogenization, *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1994), no. 2, 297-342.

[3] Allaire, G. and Murat, M. Homogenization of the Neumann problem with non-isolated holes, *Asymptotic Anal.* **7** (1993), no. 2, 81-95.

[4] Bakhvalov, N.S. and Panasenko, G.P. Homogenization: Averaging Processes in Periodic Media, Mathematics and its Applications **v. 36**, *Kluwer* , Dordrecht, 1989.

[5] Bensoussan, A. , Boccardo, L. , Dall'Aglio, A. and Murat, F. H-convergence for quasi-linear elliptic equations under natural hypothesis on the correctors, *Publications du Laboratoire d'analyse Numérique*, **R94023**.

[6] Bensoussan, A. , Lions, J.L. and Papanicolaou, G. Asymptotic Analysis for Periodic Structures, *North Holland*, Amsterdam, 1978.

[7] Briane, M. , Damlamian, A. and Donato, P. H_0-convergence for perforated domains, *Publications du Laboratoire d'analyse Numérique*, **R93031**.

[8] Casado- Díaz, J. Existence of a sequence satisfying Cioranescu-Murat conditions in the homogenization of Dirichlet problems in perforated domains, *Rend. Mat. Appl.(7)* **16** (1996), no. 3, 387-413.

[9] Casado- Díaz J. Private communication.

[10] Chiadò Piat, V. , Dal Maso, G. and Defranceschi A. G-Convergence of monotone operators, *Ann. Inst. H. Poincaré, Anal. Nonlinéaire* 7 (1996), no. 6, 123-160.

[11] Cioranescu, D. and Murat, F. Un terme étrange venu d'ailleurs, In *Nonlinear Partial Differential Equations and their Applications, College de France Seminar, Vol II and III,* ed. by H. Brézis -J. L. Lions. , Research Notes in Math. 60 and 70, Pitman, London, 1982, 98-138 and 154-178.

[12] Cioranescu, D. and Murat, F. A strange term coming from nowhere, *Topics in Mathematical Modelling of Composite Materials*, 45-93, Progr. Nonlinear Partial Differential Equations Appl. **31**, Birkhäuser Boston, Boston, MA, 1997.

[13] Cioranescu, D. and Saint Jean Paulin, J. Homogenization in open sets with holes, *J. Math. Anal. and Appl.* **71** (1979), no. 2, 590-607.

[14] Clark, G.W. and Showalter, R.E. Two-scale convergence of a model for flow in a partially fissured medium, *Electronic Journal of Differential Equations* **Vol. 1999** (1999), no. 2, 1-20.

[15] Conca, C. , Planchard, J. and Vanninathan, M. Fluids and Periodic Structures, RAM **38**, *John Wiley & Sons*, Masson, 1995.

[16] Dal Maso, G. Introduction to Γ− Convergence, **PNLDE 8**, *Birkhaüser*, Boston, 1993.

[17] Dal Maso, G. and Defranceschi, A. Correctors for the homogenization of monotone operators, *Differential and Integral Equations* **3** (1990), no. 6, 1151-1166.

[18] Damascelli, L. Comparison theorems for some quasi-linear elliptic operators and applications to symmetry and monotonicity results, *Ann. Inst. H. Poincaré, Anal. Nonlinéaire* **15** (1998), no. 4, 493-516.

[19] Donato, P. Alcune osservazioni sulla convergenza debole di funzioni non uniformemente oscillanti, *Ricerche di Mathematica* **32** (1983), no. 2, 203-219.

[20] Douglas, Jr. J. , Peszyńska, M. and Showalter, R.E. Single phase flow in partially fissured media, *Transport in Porous Media* **28** (1995), 285-306.

[21] Hornung, U. (Editor) Homogenization and Porous Media, **IAM 6**, *Springer*, 1997.

[22] Jikov, V.V. , Kozlov, S.M. and Oleinik, O.A. Homogenization of Differential Operators and Integral Functionals, *Springer Verlag*, Berlin Heidelberg, 1994.

[23] Kesavan, S. and Rajesh, M. Homogenization of periodic optimal control problems via multi-scale convergence, *Proc. Indian Acad. Sci.(Math. Sci.)* **108** (1998), no. 2, 189-207.

[24] Kesavan, S. and Saint Jean Paulin, J. Homogenization of an optimal control problem, *SIAM J. Control. Optim.* **35** (1997), no. 5, 1557-1573.

[25] Kesavan, S. and Saint Jean Paulin, J. Optimal control on perforated domains, *J. Math. Anal. Appl.* **229** (1999), 563-586.

[26] Kesavan, S. and Saint Jean Paulin, J. Low cost control problems, *Manuscript in preparation.*

[27] Kesavan, S. and Vanninathan, M. L'homogénéisation d'un problème de contrôle optimal, *C. R. Acad. Sci. Paris Ser. A-B* **285** (1977), 441-444.

[28] Lions, J.L. Optimal Control of Systems Governed by Partial Differential Equations, *Springer-Verlag*, Berlin, 1971.

[29] Murat, F. H-convergence, *Séminaire d'analyse fonctionelle et numérique* (1978), Univ. d'Alger., mimeographed notes.

[30] Murat, F. Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **5** (1978), no. 4, 489-567.

[31] Nguetseng, G. A general convergence result of a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20** (1989), 608-623.

[32] Oleinik, O.A., Shamaev, A.S. and Yosifian, G.A. Mathematical Problems in Elasticity and Homogenization, *North Holland*, Amsterdam, 1992.

[33] Rajesh, M. Correctors for flow in a partially fissured medium, *Electron. J. Diff. Eqns.* **Vol. 1999** (1999), no. 27, 1-15.

[34] Showalter, R.E. Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, *Mathematical Surveys and Monographs*, **vol 49** AMS, 1997.

[35] Sanchez-Palencia, E. Non-homogeneous Media and Vibration Theory, Lecture Notes in Physics **127**, *Springer-Verlag*, Berlin, 1980.

[36] Tartar, L. Cours Peccot au Collège de France, 1977.